Nov. 7

Lemma $E = X \oplus Y$ X and Y closed

then the two maps $P(x+y) = x$

$Q(x+y) = y$

are projection $P: E \to X$

$Q: E \to Y$

($P^2 = P$ P continuous , same for Q).

$Q = 1 - P$

Pf For $x \in X$ $Px = P(x+0) = x$

For $z = x + y \in E$ $Pz = x$. $\forall z \in E$

$P^2 z = PPz = Px = \underset{= Pz}{x} \Rightarrow P^2 z = Pz$ $\forall z$

$\Rightarrow P^2 = P$

We need to prove $P \in \mathcal{L}(E)$

$E = X \oplus Y$

$\phi : X \times Y \longrightarrow E$

$\|(x,y)\| = \|x\|_E + \|y\|_E$

$(x,y) \longrightarrow x+y$

It is a continuous map because $\|x+y\|_E \le \|x\|_E + \|y\|_E = \|(x,y)\|$

ϕ is continuous and bijective

$\phi^{-1} : E \longrightarrow X \times Y$

Suppose I need to show that ϕ^{-1} continuous.

But since X and Y are closed in E, they are B-spaces

$\Rightarrow$ $X \times Y$ is a B-space.

$\Rightarrow$ ϕ^{-1} is continuous.

$\phi^{-1} : E \to \overbrace{X}^{X} \times Y$

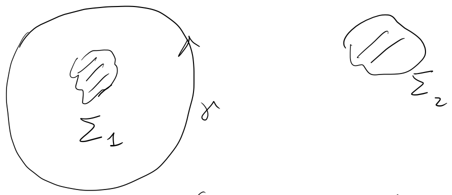
$\phi^{-1} z = (\underset{\substack{\| \\ x}}{Pz}, \underset{\substack{\| \\ y}}{Qz})$

$\|x\|_E \le \|(Pz, Pz)\| \le C \|z\|_E$

$\|Pz\|_E \le C \|z\|_E$

X B space

$A \in \mathcal{L}(X)$ $\qquad \sigma(A) = \Sigma_1 \dot\cup \Sigma_2$



$\gamma : [0,1] \to \rho(A)$

$$P = -\frac{1}{2\pi i}\int_\gamma R_A(z)\,dz = -\frac{1}{2\pi i}\int_0^1 R_A(\gamma(t))\,\gamma'(t)\,dt$$

P is a projector $\qquad R_A(z) = (A-z)^{-1}$

$$\|P\|_{\mathcal{L}(X)} = \left\| -\frac{1}{2\pi i}\int_0^1 R_A(\gamma(t))\,\gamma'(t)\,dt\right\|_{\mathcal{L}(X)}$$

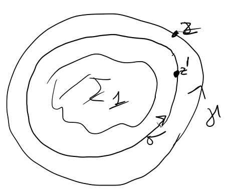
$$= \frac{1}{2\pi}\left\|\int_0^1 R_A(\gamma(t))\,\gamma'(t)\,dt\right\|_{\mathcal{L}(X)}$$

$$\leq \frac{1}{2\pi}\int_0^1 \|R_A(\gamma(t))\|_{\mathcal{L}(X)}\,|\gamma'(t)|\,dt < +\infty$$

How $\quad R_A \in C^\omega(\rho(A), \mathcal{L}(X)) \xrightarrow{\|\ \|_{\mathcal{L}(X)}} \mathbb{R}$

$\gamma \in C^1([0,1], \rho(A))$

$P^2 = P$



$\mathrm{Ind}(\sigma, z) = 0$

$\mathrm{Ind}(\gamma, z') = 1$

$P^2 = P \quad P$

$$= -\frac{1}{2\pi i}\int_\sigma R_A(z')\,dz' \quad \frac{(-1)}{2\pi i}\int_\gamma R_A(z)\,dz$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma\int_\gamma R_A(z')R_A(z)\,dz'\,dz =$$

$$R_A(z')R_A(z) = (A-z')^{-1}(A-z)^{-1} = \frac{1}{z'-z}\left(\frac{1}{A-z'} - \frac{1}{A-z}\right)$$
$$= \frac{1}{z'-z}\left(R_A(z') - R_A(z)\right)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma\int_\gamma \frac{1}{z'-z} R_A(z')\,dz'\,dz - \frac{1}{(2\pi i)^2}\int_\sigma\int_\gamma \frac{1}{z'-z} R_A(z)\,dz'\,dz$$

$$= \frac{1}{2\pi i}\int_\sigma dz'\,R_A(z')\underbrace{\left(\frac{1}{2\pi i}\int_\gamma \frac{1}{z'-z}\,dz\right)}_{\mathrm{Ind}(\gamma,z') = 1} -$$

$$+ \frac{1}{2\pi i}\int_\gamma dz\,R_A(z)\underbrace{\left(\frac{1}{2\pi i}\int_\sigma \frac{1}{z'-z}\,dz'\right)}_{\mathrm{Ind}(\sigma,z) = 0} = P$$

Def Given E a TVS ~~the~~ weak $\sigma(E,E')$ topology is a topology with a subbasis of seminorms $\{|f(\cdot)|\}_{f\in E'}$

Or equivalently is the weakest (coarsest) topology on E s.t. $f: E \to \mathbb{R}$ is continuous for $\forall f \in E'$

coarse

Exercise For any $x_0 \in E$ a basis of neigh. for the $\sigma(E,E')$ topology is of the form

$$V_{x_0}(f_1, \dots, f_n, \varepsilon) = \{x \in E : |\underbrace{f_j(x-x_0)}_{\langle f_j, x-x_0\rangle_{E'\times E}}| < \varepsilon \text{ for } j = 1, \dots, n\}$$

where $\varepsilon > 0$ and $f_1, \dots, f_n \in E'$

If E is a B-space

Pr $\sigma(E,E')$ is a weaker topology than the ~~topo~~ original topology of E.

If $\dim E = +\infty$, then $D_E(0,1)$ is not open for the $\sigma(E,E')$

<u>Lemma</u> Let E a B-space. It is Hausdorff for the $\sigma(E,E')$ topology.

<u>Pf</u> Given only $x_0 \neq x_1$ in E

$\{x_0\}$ $\{x_1\}$ are compact convex ~~set~~ disjoint subspaces of E

By H-B theorem $\exists$ $f \in E'$ s.t. and $\alpha \in \mathbb{R}$

s.t. $f(x_0) < \alpha < f(x_1)$ $f^{-1}(\alpha)$

$f(x) = \alpha$

$\Rightarrow$ $f^{-1}((-\infty,\alpha))$ is a neigh of x_0 in $\sigma(E,E')$

$f^{-1}((\alpha,+\infty))$ " x_1 in $\sigma(E,E')$

When $\lim_{n\to+\infty} x_n = x$ in E for $\sigma(E,E')$ we will write $x_n \rightharpoonup x$ ($x_n \to x$ convergence in in the strong topology)

Lemma E a B space $\{x_n\}$ a sequence in E. $\cancel{L^1(\mathbb{R})}$ $\ell^\infty(\mathbb{N})$

1) $x_n \rightharpoonup x \iff f(x_n) \to f(x)\ \forall f \in E'$

2) $x_n \to x$ strongly $\Rightarrow x_n \rightharpoonup x$ in $\sigma(E,E')$

$$\boxed{3)\quad x_n \rightharpoonup x \Rightarrow \{\|x_n\|_E\} \text{ is boundes} \quad \|x\|_E \le \liminf_{n\to+\infty} \|x_n\|_E \quad (\text{Fathou-Lemma})}$$

4) If $x_n \rightharpoonup x$ and $f_n \to f \Rightarrow f_n(x_n) \to f(x)$.

Pf only of 3

$x_n \rightharpoonup x \overset{1)}{\iff} f(x_n) \to f(x)\quad \forall f \in E'$

$\Rightarrow \{|\langle x_n, f\rangle_{E\times E'}|\}$ is a founded sequence ~~for any $f\in E'$~~

$$\langle x_n, f\rangle_{E\times E'} = \langle Jx_n, f\rangle_{E''\times E'}$$

By Banach - Steinhaus $\|Jx_n\|_{E''} = \|x_n\|_E \le C$ $\forall n$

for an appropriate $C \geq 0$

$\cdot\ \|x\|_E \le \liminf_{n\to+\infty} \|x_n\|_E$

Let $\lim_{k\to+\infty} \|x_{n_k}\| = \liminf_{n\to+\infty} \|x_n\|_E$ for some subsequence

$$\lim_{k\to+\infty} f(x_{n_k}) = \lim_{n\to+\infty} f(x_n) = f(x)$$

$$|f(x)| = \lim_{k\to+\infty} |f(x_{n_k})| \le \lim_{k\to+\infty} \|f\|_{E'} \|x_{n_k}\|_E$$

$$|f(x)| \le \|f\|_{E'} \lim_{k\to+\infty} \|x_{n_k}\|_E \qquad \forall f \in E'$$

$$\Rightarrow \|x\| \le \lim_{k\to+\infty} \|x_{n_k}\|_E$$

$\exists\ f \in E'$ $\|f\|_{E'} = 1$ st

$f(x) = \|x\|_E$

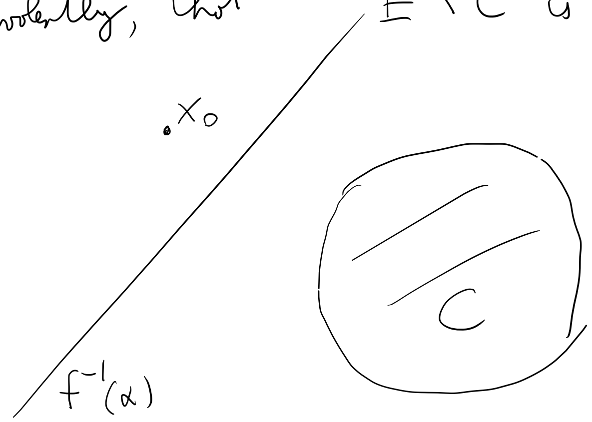
Theorem E B-space $C \subseteq E$ C convex.

Then C is closed for the strong topology

$$\Updownarrow$$

C closed for $\sigma(E,E')$ topology.

Pf Let C convex be strongly closed. We need to prove that it is closed for $\sigma(E,E')$ or, equivalently, that $E \setminus C$ is $\sigma(E,E')$ open



We need to show that any $x_0 \in E \setminus C$ has a neigh. U for the $\sigma(E,E')$ s.t. $U \cap C = \phi$.

$\{x_0\}$ is a compact convex ~~subspace~~ $\Big\}$ for the strong topologie
C closed convex

By 2° geom H-B theorem $\exists\, f \in E'$ and $\alpha \in \mathbb{R}$

s.t. $f(x_0) < \alpha < f(x) \quad \forall\, x \in C$

$U = f^{-1}((-\infty, \alpha))$

Lemma Let E B-space $\dim E = +\infty$ and let U be open $\neq \emptyset$ in $\sigma(E, E')$. Then U contains a line.

Pf Let $x_0 \in U$. $\exists\ \varepsilon > 0$ $f_1, \dots, f_n \in E'$ s.t.

$$V = V_{x_0}(f_1, \dots, f_n, \varepsilon) = \{x : |f_j(x - x_0)| < \varepsilon \ \forall j = 1, \dots n\} \subseteq U$$

$$F : E \longrightarrow \mathbb{R}^n$$

$$F(x) = (f_1(x), \dots, f_n(x))$$

$\ker F$ has finite codimension.

$\dim \ker F = +\infty$ $\ker F$ contains a line

$$x_0 + \ker F \subseteq V \subseteq U$$