8 Novembre

Calcoliamo tutte le derivate di alcune funzione

Esempi

$$(e^x)^{(n)} = e^x$$

$$(e^x)' = e^x$$

Supponendo $(e^x)^{(n-1)} = e^x$ si ha

$$(e^x)^{(n)} = \left((e^x)^{(n-1)} \right)' = (e^x)' = e^x$$

Concludiamo che $(e^x)^{(n)} = e^x \quad \forall \; n \in \mathbb{N}$

2) $(\sin x)^{(0)} = \sin(x)$

$(\sin x)' = \cos x$

$(\sin x)'' = (\cos x)' = -\sin x$

$(\sin x)''' = (-\sin x)' = -\cos x$

$(\sin x)^{(4)} = (-\cos x)' = \sin x$

$(\sin x)^{(5)} = (\sin x)' = \cos x$

$(\sin x)^{(6)} = (\sin x)^{(2)} = -\sin x$

$(\sin x)^{(7)} = (\sin x)^{(3)} = -\cos x$

$$\sin^{(n)} x \overset{?}{=} \sin^{(r)} x$$

$n = k4 + r \qquad r \in \{0, 1, 2, 3\}$

$$\sin^{(n)} x = \sin^{(4k+r)}(x) = \left(\sin^{(4k)}(x)\right)^{(r)} =$$

$$= \left(\sin^{(\overbrace{4+4\cdots+4}^{k \text{ volte}})}(x)\right)^{(r)} = \left(\left(\left(\sin^{(4)}(x)\right)^{\overbrace{(4)\cdots}^{k-1}}\right)^{(4)}\right)^{(r)}$$

$$= (\sin x)^{(r)} = \sin^{(r)} x$$

$\sin^{(0)} x \big|_{x=0} = 0$

$\sin' x \big|_{x=0} = \cos x \big|_{x=0} = 1$

$\sin'' x \big|_{x=0} = -\sin x \big|_{x=0} = 0$

$\sin''' x \big|_{x=0} = -\cos x \big|_{x=0} = -1$

$\sin^{(n)} x \big|_{x=0} = ?$

Esercizio: Se f è pari allora f' è dispari
" dispari " pari

$(x^n)' = n\, x^{n-1}$

Se f è dispari $f(-0) = -f(0)$
$\Updownarrow$
$f(0) = -f(0)$, $2f(0) = 0 \iff f(0) = 0$

Se n è pari
~~Se~~ $\sin^{(n)}(x)$ è una funzione dispari

o comunque $\sin^{(n)}(x)\big|_{x=0}$ $\begin{cases} \sin x \big|_{x=0} = 0 \\ -\sin x \big|_{x=0} = 0 \end{cases}$

$\sin^{(2n)}(0) = 0$

$\sin^{(2n+1)}(0)$ $\begin{cases} 1 & \text{quando } 2n+1 = k4+1 \\ -1 & \phantom{\text{quando }} 2n+1 = k4+3 \end{cases}$

$2n + \cancel{1} = 4k + \cancel{1}$ $\qquad$ $\cancel{2n = \overset{2}{\cancel{4}} k}$

$\boxed{n = 2k}$ cioè n pari

$2n + \cancel{1} = 4k + \overset{2}{\cancel{3}}$
$\Updownarrow$
$2n = 4k + 2 \iff \boxed{n = 2k+1}$ cioè n dispari

$\sin^{(2n+1)}(0) = \begin{cases} 1 & \text{se } n \text{ è pari} \\ -1 & \phantom{\text{se }} n \text{ dispari} \end{cases}$

Cioè

$\sin^{(2n+1)}(0) = (-1)^n \qquad \forall n$

Chiameremo polinomio di Taylor di ordine n rispetto a x_0 di una funzione $f(x)$ in un punto x_0 dove f ammette derivate fino all'ordine n

$$P_{n,x_0}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

$$P_{n+1,x_0}(x) = \sum_{j=0}^{n+1} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} =$$

Quando $x_0 = 0$ li chiamiamo anche polinomi di McLaurin.

$$= P_{n,x_0}(x) + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}$$

Polinomi di McLaurin di e^x

$$P_n(x) = \sum_{j=0}^{n} \frac{(e^x)^{(j)}(0)}{j!}x^j = \sum_{j=0}^{n} \frac{e^0}{j!}x^j = \sum_{j=0}^{n} \frac{x^j}{j!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Polinomi di McLaurin di $\sin x$

$$P_{2n+1}(x) = \sum_{j=0}^{2n+1} \frac{\sin^{(j)}(0)}{j!}x^j = \sum_{\substack{j=1 \\ j \text{ dispari}}}^{2n+1} \frac{\sin^{(j)}(0)}{j!}x^j =$$

j dispari è della forma

$j = 2k+1$ assume tutti i valori dispari da 1 a $2n+1$ facendo variare $k = 0, \dots, n$

$$= \sum_{k=0}^{n} \frac{\sin^{(2k+1)}(0)}{(2k+1)!}x^{2k+1}$$

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$P_{2n+2}(x) = P_{2n+1}(x) + \frac{\overbrace{\sin^{(2n+2)}(0)}^{0}}{(2n+2)!}x^{2n+2}$$

$$\lim_{x\to+\infty} \underbrace{(x+3)^2}_{\substack{\downarrow\\ +\infty}} \ \underbrace{\lg\left(\frac{2x^2}{2x^2+1}\right)}_{\substack{\downarrow\\ 0}}$$

$$\lim_{y\to+\infty} y^2 \lg\left(1+\frac{1}{y^2}\right)=1$$

$$(x+3)^2 \lg\left(\frac{2x^2}{2x^2+1}\right) =$$

$$= (x+3)^2 \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right) =$$

$$= x^2 \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right) + 6x \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right) + 9 \underbrace{\lg\left(\frac{1}{1+\frac{1}{2x^2}}\right)}_{\substack{\downarrow\\ 0}}$$

$$\frac{1}{2} \lim_{x\to+\infty} 2x^2 \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right) = \frac{1}{2} \lim_{x\to+\infty} 2x^2 \lg\left(1+\frac{1}{2x^2}\right)^{-1}$$

$$= -\frac{1}{2} \lim_{x\to+\infty} 2x^2 \lg\left(1+\frac{1}{2x^2}\right) =$$

$$= -\frac{1}{2} \lim_{x\to+\infty} \lg\left(1+\frac{1}{2x^2}\right)^{2x^2} \qquad y = 2x^2$$

$$= -\frac{1}{2} \lim_{y\to+\infty} \lg\left(1+\frac{1}{y}\right)^{y} = -\frac{1}{2} \lg e = -\frac{1}{2}$$

$$\lim_{x\to+\infty} 6x \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right) = 0$$

$$= 6 \lim_{x\to+\infty} \underbrace{\frac{1}{x}}_{\to 0} \overbrace{x^2 \lg\left(\frac{1}{1+\frac{1}{2x^2}}\right)}^{-\frac{1}{2}} = 0$$

$\cos^{(0)} x = \cos x$ $\qquad \cos^{(4)} x = (\sin x)' = \cos x$

$\cos' x = -\sin x$ $\qquad \cos^{(5)} x = (\cos)'(x) = -\sin x$

$\cos'' x = -\cos x$ $\qquad \cos^{(6)} x = \cos''(x) = -\cos x$

$\cos''' x = \sin x$ $\qquad \cos^{(7)} x = \cos'''(x) = \sin x$

$n = 4k + r$

$$\cos^{(n)}(x) = \cos^{(4k+r)}(x) = \left(\cos^{(4k)}(x)\right)^{(r)} =$$

$$= \left(\cos^{(\overbrace{4+\dots+4}^{k})}(x)\right)^{(r)} = \left(\Big(\underbrace{\left(\cos^{(4)}(x)\right)^{(4)}\dots}_{k-1}\Big)^{(4)}\right)^{(r)}$$

$$= \left(\cos x\right)^{(r)} = \cos^{(r)}(x)$$

$$\cos^{(n)}(x) = \left(\frac{d}{dx}\right)^n \cos(x) = \underbrace{\overbrace{\frac{d}{dx}\cdots}^{r}\overbrace{\cdots\frac{d}{dx}}^{4k}}_{n=4k+r} \cos(x)$$

$$= \left(\frac{d}{dx}\right)^r \left(\frac{d}{dx}\right)^{4k} \cos(x) = \left(\frac{d}{dx}\right)^r \overbrace{\left(\frac{d}{dx}\right)^4 \cdots \left(\frac{d}{dx}\right)^4}^{k} \cos x$$

$$= \left(\frac{d}{dx}\right)^r \cos x$$

$\cos^{(n)}(0) = ?$

$$\cos^{(2n+1)}(x)\Big|_{x=0} = \begin{cases} \sin x\,\big|_{x=0} = 0 \\ -\sin x\,\big|_{x=0} = 0 \end{cases}$$

$$\cos^{(2n)}(x)\Big|_{x=0} \begin{cases} \cos x\,\big|_{x=0} = 1 \\ -\cos x\,\big|_{x=0} = -1 \end{cases}$$

$2n = 4k + r$ $\qquad r \begin{cases} 0 \\ 2 \end{cases}$

Se n è pari $\quad r = 0$

dispari $\quad r = 2$

$n = 2k \qquad 2n = 4k \implies r = 0$

$n = 2k+1 \qquad 2n = 4k + 2 \implies r = 2$

$$\cos^{(2n)}(0) = \begin{cases} 1 & \text{se } n \text{ è pari} \\ -1 & \text{" dispari} \end{cases}$$

$$\cos^{(2n)}(0) = (-1)^n$$

$$P_{2n}(x) = \sum_{j=0}^{2n} \frac{\cos^{(j)}(0)}{j!} x^j = \sum_{\substack{j=0 \\ j \text{ pari}}}^{2n} \frac{\cos^{(j)}(0)}{j!} x^j =$$

$j = 2k$ assume tutti i valori pari da 0 a $2n$ quando si prende $k = 0, \dots, n$

$$= \sum_{k=0}^{n} \frac{\cos^{(2k)}(0)}{(2k)!} x^{2k} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} = P_{2n}(x)$$

$$P_{2n+1}(x) = P_{2n}(x) + \frac{\overbrace{\cos^{(2n+1)}(0)}^{0}}{(2n+1)!} x^{2n+1} = P_{2n}(x)$$