Image Processing for Physicists

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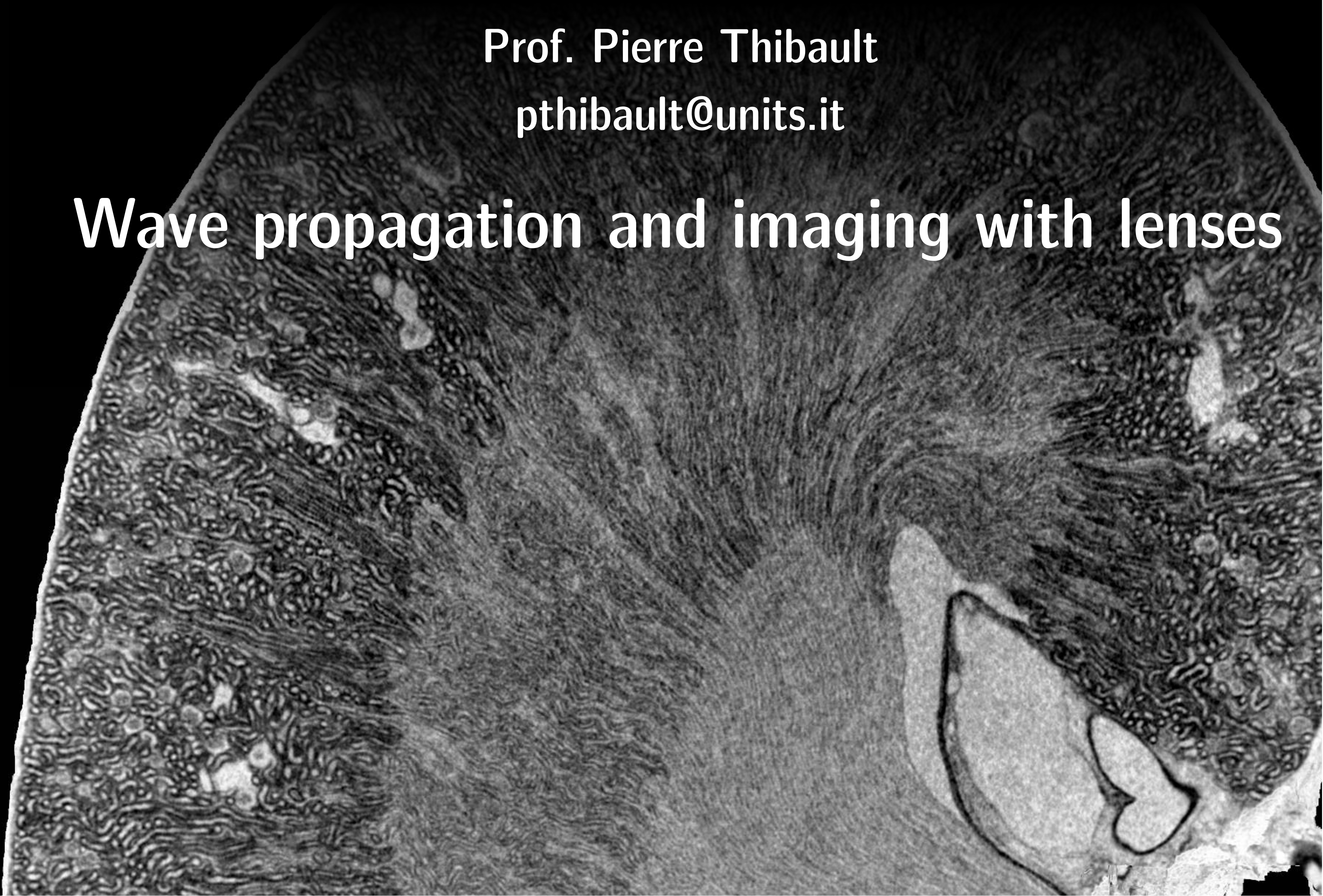
Wave propagation and imaging with lenses



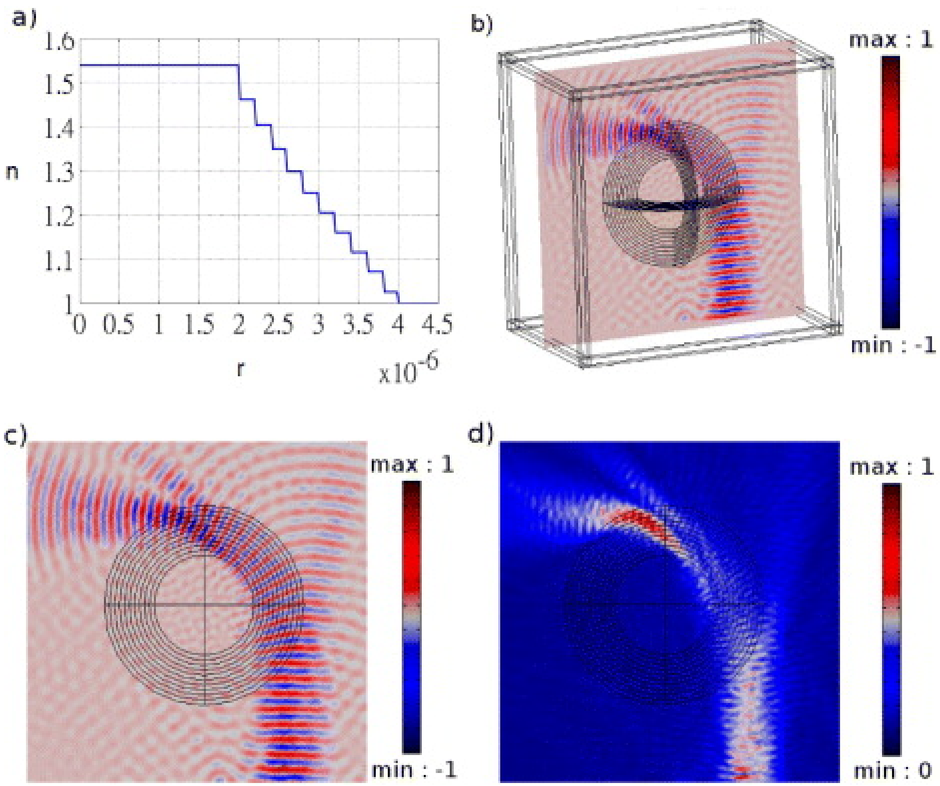
Overview

- Propagation modelization
- Wave propagation:
  - Near-field regime
  - Far-field regime

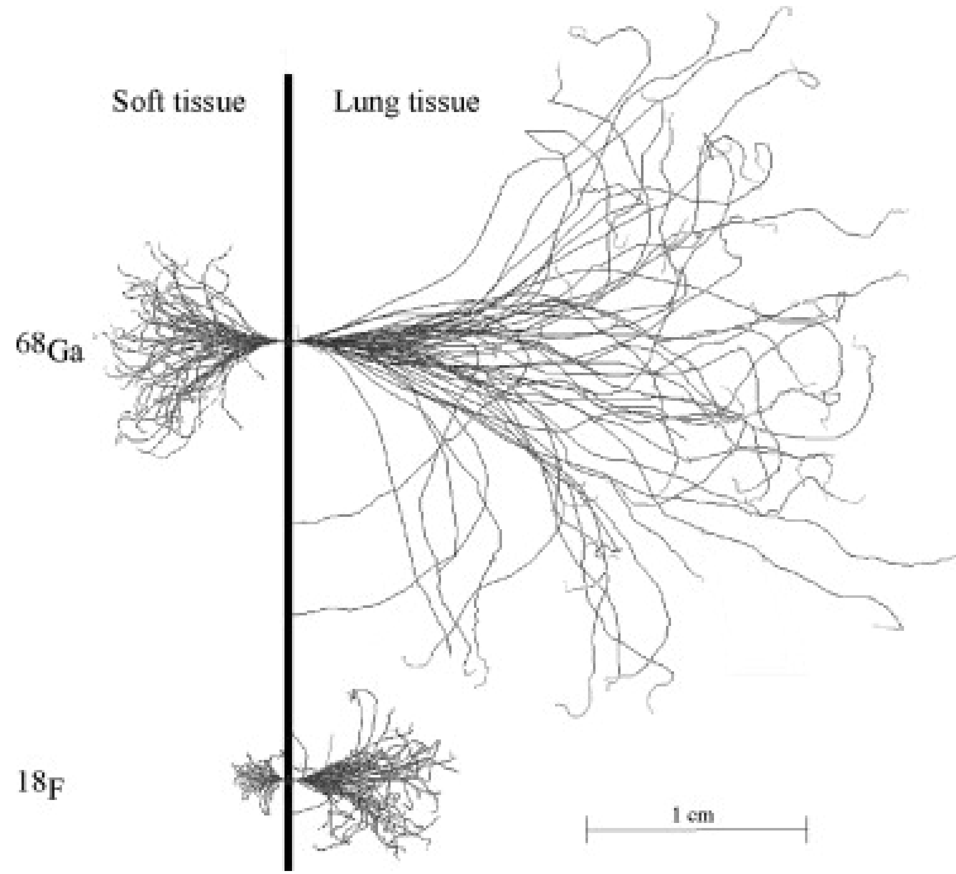
Propagation modeling

- Motivations:

  1. Validation



Finite element simulation of an electro-magnetic field in a dielectric



Monte Carlo simulation of positrons trajectories resulting from ^{68}Ga and ^{18}F decay.

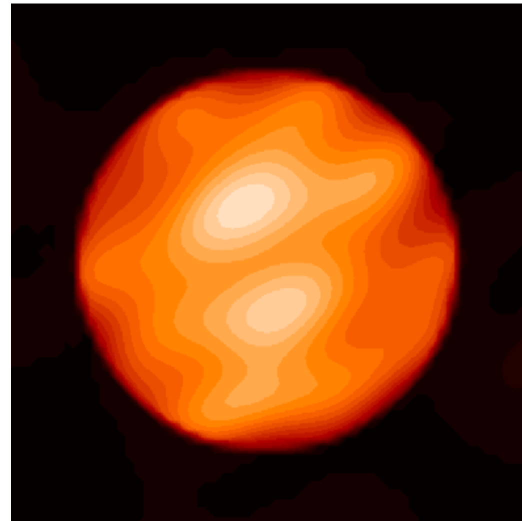
sources: T.M. Chang *et al.* New J. Phys. (2012)
A. Sanchez-Crespo, Appl. Rad. Isotopes (2012)

Propagation modeling

- Motivations:

  2. Inversion

Image reconstruction from sound wave propagation (ultrasonography)



The surface of Betelgeuse reconstructed from interferometric data (IOTA)

sources: wikipedia
Haubois *et al. Astronom. & Astrophys.* (2009)

Propagation modeling

- Particles
  - Model particle tracks (rays) through different media
  - Model may include: refraction, force fields, particle decay and interactions
  - Not included: diffraction
- Wave
  - Model the interaction of a field with a medium
  - Can be very complicated $\rightarrow$ approximations are needed

Propagation modeling

Starting point: Helmholtz equation

- for EM field: neglect polarization (scalar wave approximation) comes from Maxwell's equations
- for electron wave, assume high energy electrons

↳ comes from Schrödinger

result: Helmholtz:

$$\nabla^2 \psi - \frac{n^2}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

fixed frequency (monochromatic)

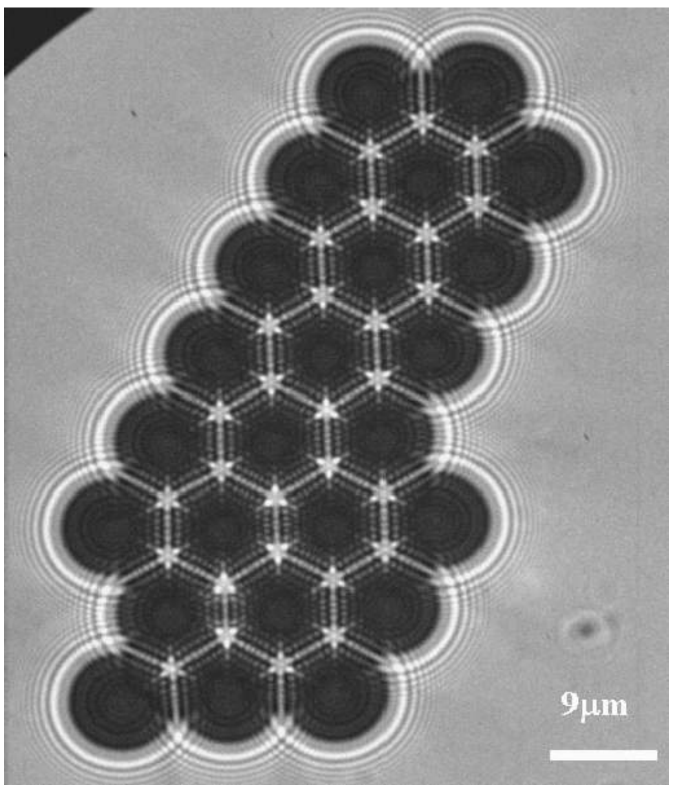
consider solutions of the form $\psi(\vec{r}, t) = \psi(\vec{r}) e^{i\omega t}$

$k = \frac{2\pi}{\lambda}$ (wavenumber) $\quad \nabla^2 \psi + k^2 n^2 \psi = 0 \qquad k^2 = \frac{\omega^2}{c^2}$

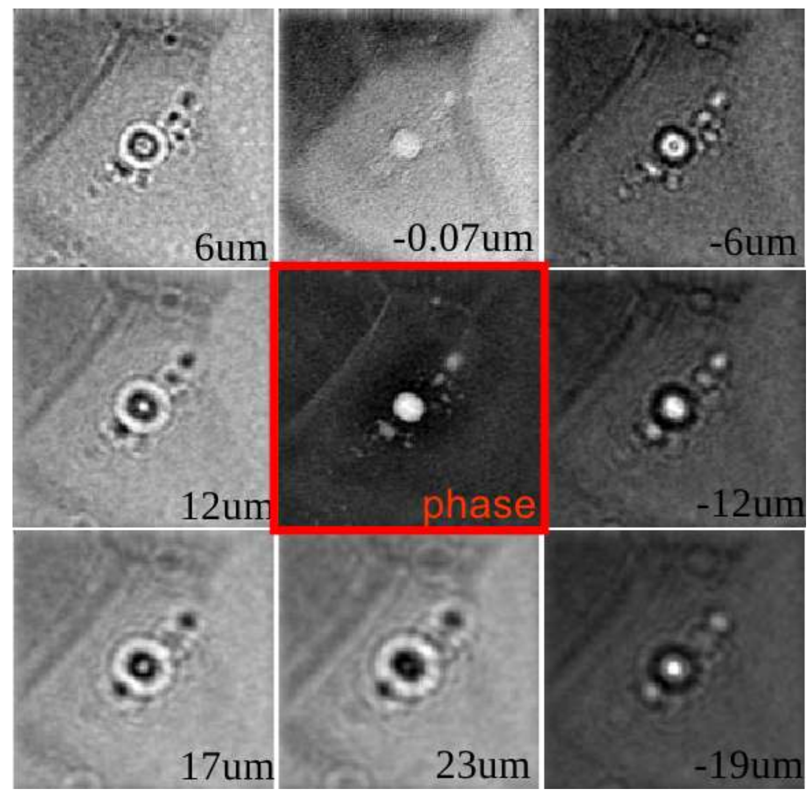
Propagation modeling

- Useful to:
  - better understand optical systems
  - understand diffraction, holography, phase contrast, interferometry, ...

X-ray hologram



TEM through-focus series



sources: Mayo *et al.* Opt. Express (2003)
http://www.christophtkoch.com/Vorlesung/

The physics of propagation

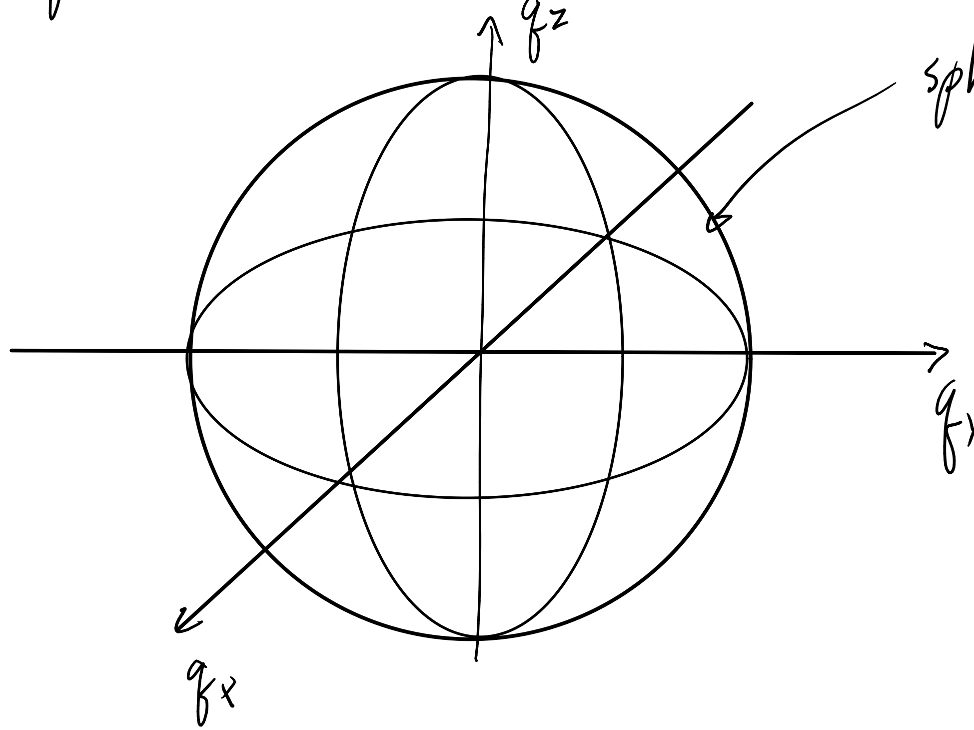
Free space propagation: $n = 1$

General solution: superposition of plane waves

$$\psi(\vec{r}) = \sum_{\vec{q}} A_{\vec{q}} \, e^{i\vec{q}\cdot\vec{r}}$$

sum is over $\vec{q}$ such that $q^2 = k^2$

$q_x^2 + q_y^2 + q_z^2 = k^2$ surface of a sphere



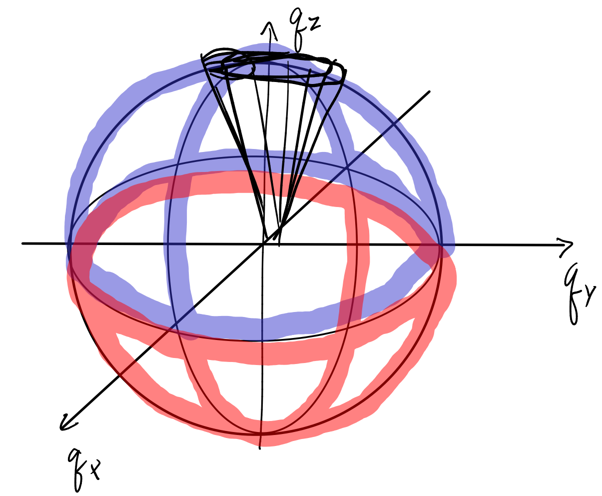
sphere of radius k

"Ewald sphere"

The physics of propagation

Angular spectrum representation

$$q_z = \pm\sqrt{k^2 - q_x^2 - q_y^2}$$



$$\psi(\vec{r}) = \sum_{q_x q_y} A^{+}_{q_x q_y}\, e^{i\left(q_x x + q_y y + \sqrt{k^2 - q_x^2 - q_y^2}\, z\right)}$$

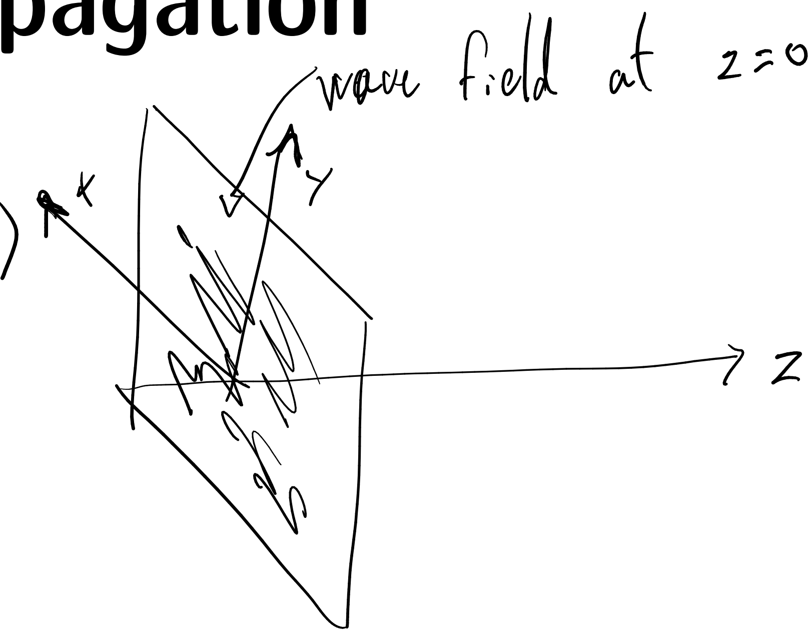
$$\cancel{+ \sum_{q_x q_y} A^{-}_{q_x q_y}\, e^{i\left(q_x x + q_y y - \sqrt{k^2 - q_x^2 - q_y^2}\, z\right)}}$$

we consider only propagation along positive z

$$\psi(x,y,z) = \underbrace{\sum_{q_x q_y} A_{q_x q_y}\, e^{i(q_x x + q_y y)}}_{\text{2D Fourier transform!}}\, e^{i\sqrt{k^2 - q_x^2 - q_y^2}\, z}$$

Forward propagation

Plane $z=0$:



$$\psi(x,y,z=0) = \sum_{q_x q_y} A_{q_x q_y} e^{i(q_x x + q_y y)}$$

$$A_{q_x q_y} = \mathcal{F}\{\psi(x,y,z=0)\}$$

Recipe: $$\psi(\overset{(x,y)}{\vec{r}_\perp}; z) = \mathcal{F}^{-1}\left\{\mathcal{F}\{\psi(\vec{r}_\perp; z=0)\} \exp\left(iz\sqrt{k^2 - \underset{(q_x,q_y)}{q_\perp^2}}\right)\right\}$$

In Practice: $$\psi(x,y,z) = \Psi(x,y,z)\, e^{ikz}$$

$$\boxed{\Psi(\vec{r}_\perp; z) = \mathcal{F}^{-1}\left\{\mathcal{F}\{\Psi(\vec{r}_\perp; z=0)\} \underbrace{\exp\left(iz\left[\sqrt{k^2-q^2} - k\right]\right)}\right\}}$$

Angular spectrum propagation method

Forward propagation

Discretization

Sampling: Δx : sampling pitch

Δu : Fourier-space sampling pitch

$$\Delta u = \frac{1}{N \Delta x}$$

$$e^{2\pi i u x} \leftrightarrow e^{2 i n k / N}$$

$x = n \Delta x$

$u = k \Delta u$

$u x = n k \Delta x \Delta u$

$\hookrightarrow \Delta x \Delta u = \frac{1}{N}$

$$\Delta u = \frac{1}{\underbrace{N \Delta x}}$$

$$q_x = 2\pi u = 2\pi k \Delta u$$

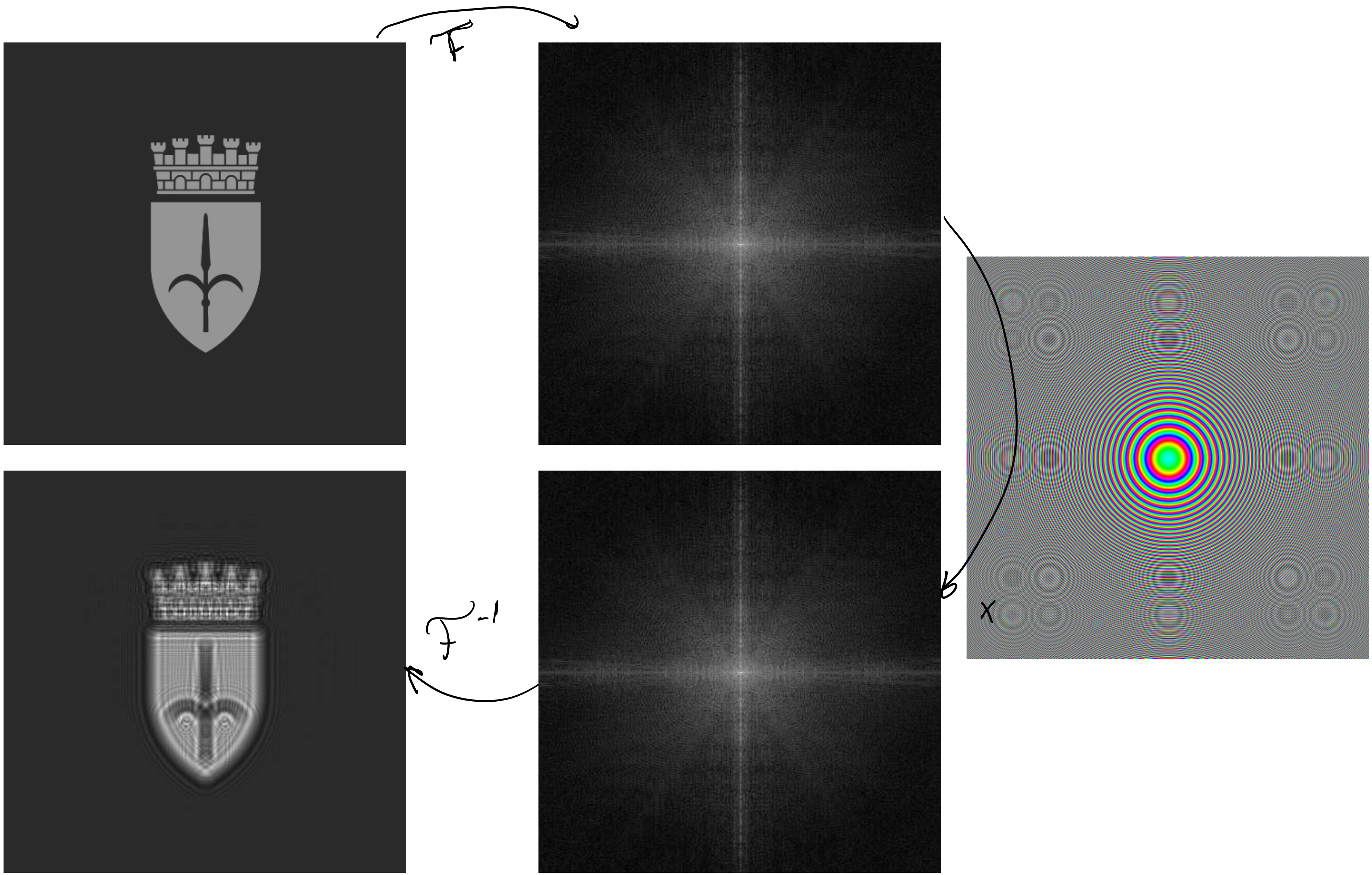
$$\sqrt{k^2 - q_x^2 - q_y^2} = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - (2\pi u)^2 - (2\pi v)^2} = \frac{2\pi}{\lambda}\sqrt{1 - \lambda^2 u^2 - \lambda^2 v^2} \leftarrow$$

$$= \frac{2\pi}{\lambda}\sqrt{1 - \lambda^2 \Delta u^2 (k^2 + l^2)}$$

$$\exp\left(iz\left[\sqrt{k^2 - q_\perp^2} - k\right]\right) = \underbrace{\exp\left(\frac{2\pi i z}{\lambda}\left[\sqrt{1 - \lambda^2(u^2 + v^2)} - 1\right]\right)}$$

Forward propagation

A numerical recipe



Near field, far field

Problem: as z increases, aliasing becomes a problem. There's a trick.

$$\frac{2\pi}{\lambda}\left[\sqrt{1-\lambda^2(u^2+v^2)}-1\right] \simeq \frac{2\pi}{\lambda}\left[\cancel{1}-\frac{1}{2}\lambda^2(u^2+v^2)-\cancel{1}\right]$$

← paraxial approximation

(small angle scattering)

$$= -\pi\lambda(u^2+v^2)$$

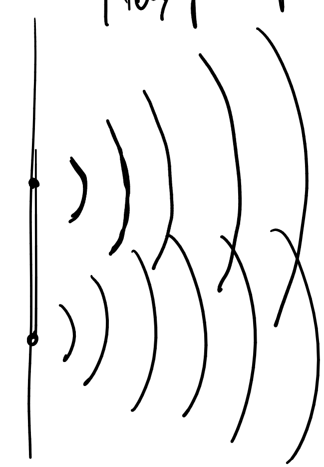
$$\Psi(r_\perp;z) = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\Psi(\vec{r}_\perp;z=0)\right\}\exp\left(-i\pi\lambda z(u^2+v^2)\right)\right\}$$

observation: this has the form of a convolution!

$$\Psi(\vec{r}_\perp;z) = \Psi(r_\perp;z=0) * P_z(\vec{r}_\perp)$$

← Fresnel-Huygens integral

Huygens principle



$$P_z(\vec{r}_\perp) = \mathcal{F}^{-1}\left\{\exp\left(-i\pi\lambda z(u^2+v^2)\right)\right\}$$

$$= -\frac{2\pi i}{\lambda z}\exp\left(\frac{i\pi r_\perp^2}{\lambda z}\right)$$

Near field, far field

$$\Psi(r_\perp; z) = \frac{-2\pi i}{\lambda z}\int d^2r'\, \Psi(\vec{r}\,'; z=0)\, \exp\left(\frac{i\pi(\vec{r}-\vec{r}\,')^2}{\lambda z}\right)$$

can insert a lens here

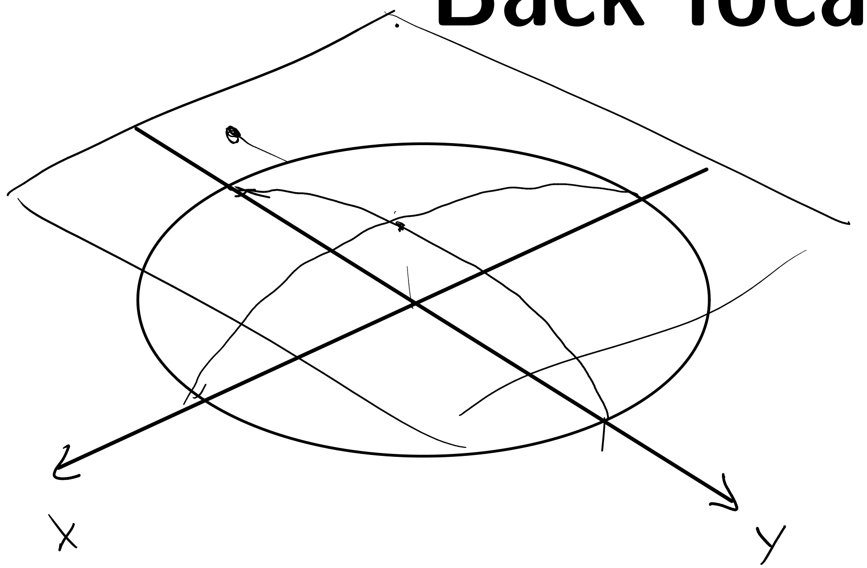
$$= \frac{-2\pi i}{\lambda z}\int d^2r'\, \Psi(\vec{r}\,'; z=0)\, \exp\left[\frac{i\pi}{\lambda z}\left[r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right]\right]$$

$$= \frac{-2\pi i}{\lambda z}\exp\left(\frac{i\pi r^2}{\lambda z}\right)\int d^2r'\, \Psi(r'; z=0)\, \exp\left(\frac{i\pi \vec{r}\,'^2}{\lambda z}\right)\exp\left(-2\pi i\frac{\overbrace{\vec{r}}^{\vec{u}}\cdot\vec{r}\,'}{\lambda z}\right)$$

$$\boxed{\Psi(\vec{r}_\perp; z) = -\frac{2\pi i}{\lambda z}\exp\left(\frac{i\pi r^2}{\lambda z}\right)\mathcal{F}\left\{\Psi(r; z=0)\exp\left(\frac{i\pi r^2}{\lambda z}\right)\right\}\left(\vec{u} = \frac{\vec{r}}{\lambda z}\right)}$$

observation: as $z\to\infty$ $\quad \left|\Psi(\vec{r}; z\to\infty)\right|^2 \propto \left|\mathcal{F}\left\{\Psi(r; z=0)\right\}\right|^2\left(\vec{u} = \frac{\vec{r}}{\lambda z}\right)$

Back focal plane of a lens

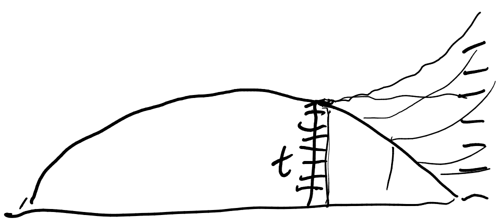


2D radial coordinate

$*$ thickness profile $t(r) = t_0 - \alpha r^2$

$*$ phase $\phi(\vec{r}_\perp) = \frac{2\pi\, t(\vec{r}_\perp)}{\lambda}(n-1)$

$= k(n-1)t = -\frac{2\pi\alpha}{\lambda}(n-1)r^2$



$*$ focal length $(n-1)\alpha = \frac{1}{2f} \Rightarrow \phi = \frac{-\pi\alpha r^2}{\lambda f}$

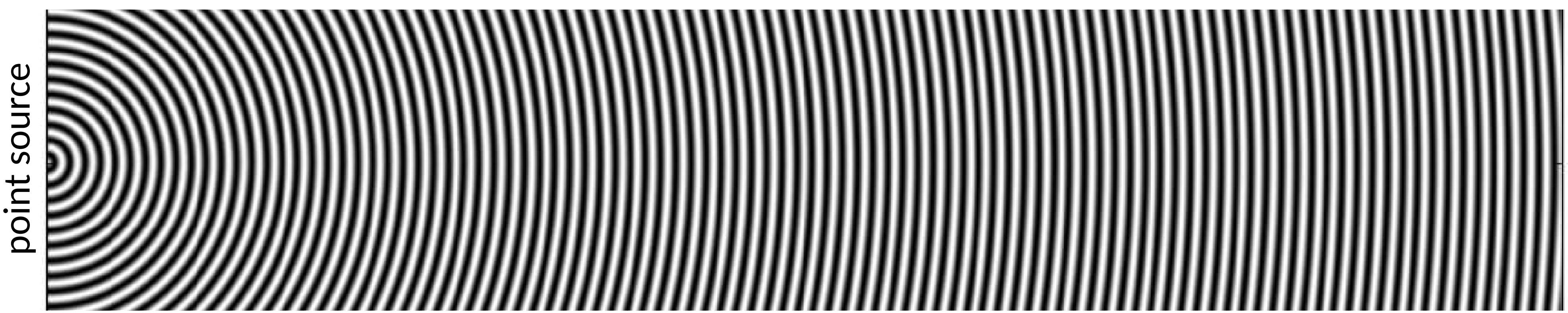
effect of a lens: $\Psi(r_\perp; z=0) \to \Psi \cdot \exp\left(\frac{-i\pi r^2}{\lambda f}\right)$

$$\Psi(\vec{r}_\perp; z) = \frac{-2\pi i}{\lambda z}\exp\left(\frac{i\pi r^2}{\lambda z}\right)\tilde{\mathcal{F}}\left\{\Psi(\vec{r}_\perp; z=0)\exp\left(\frac{i\pi r^2}{\lambda}\left(\frac{1}{z}-\frac{1}{f}\right)\right)\right\}\left(\vec{u} = \frac{\vec{r}}{\lambda z}\right)$$

special case: $z = f$: $\Psi(\vec{r}_\perp; z=f) = (\quad)\cdot$ F.T. of $\Psi(\vec{r}_\perp; z=0)$

$\Rightarrow$ a converging lens acts as a Fourier transform operator!

Plane waves, point sources

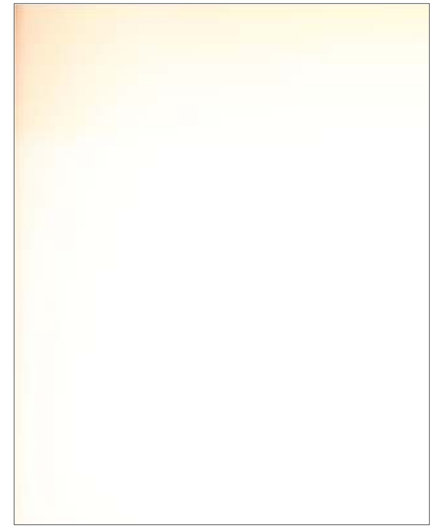


circular waves
evanescent waves
contact region

parabolic waves
near field
Fresnel region

plane waves
far field
Fraunhofer region

Why optical elements?

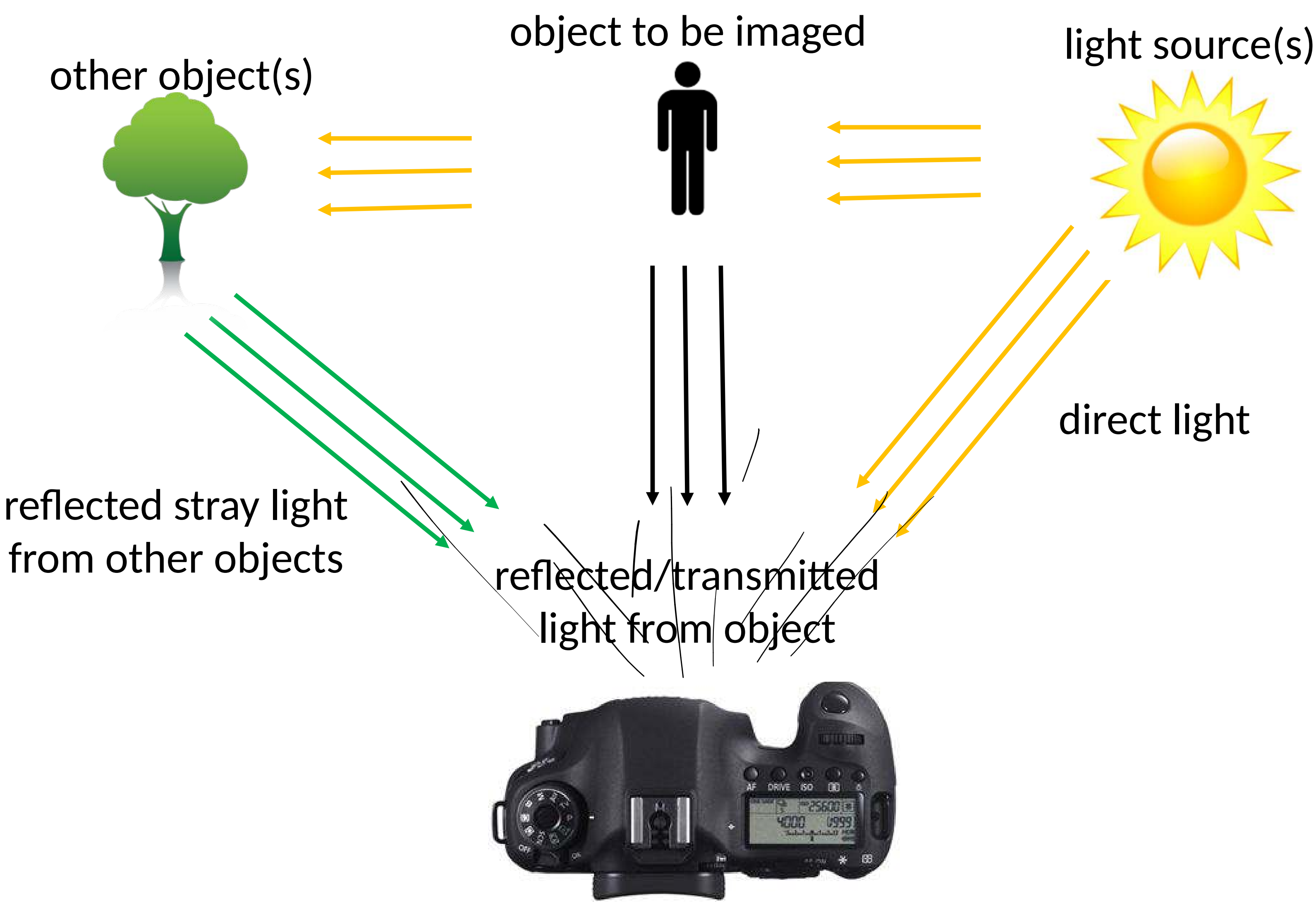


with objective lens

without objective lens

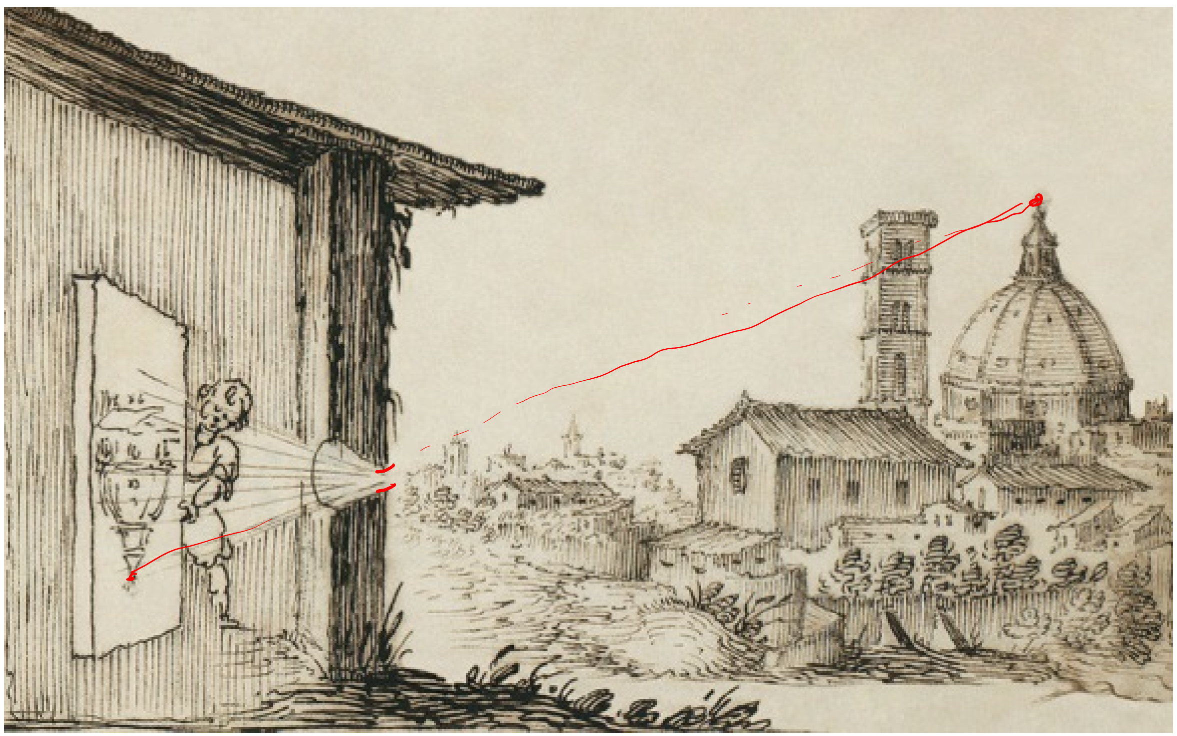
Why optical elements?

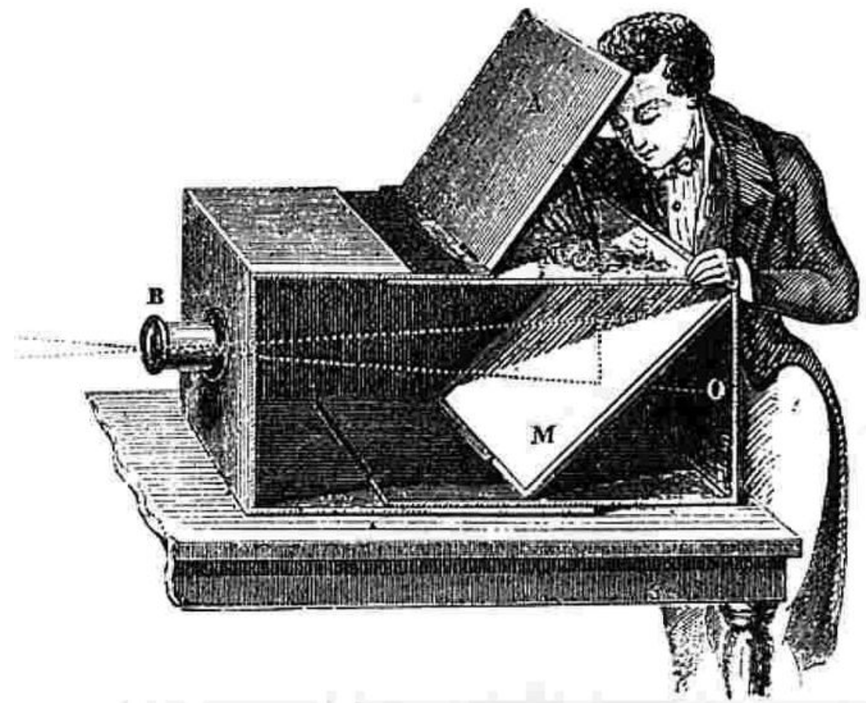
- Information from many sources overlaps in detector plane
- Need models to understand image forming systems



Pinhole camera model

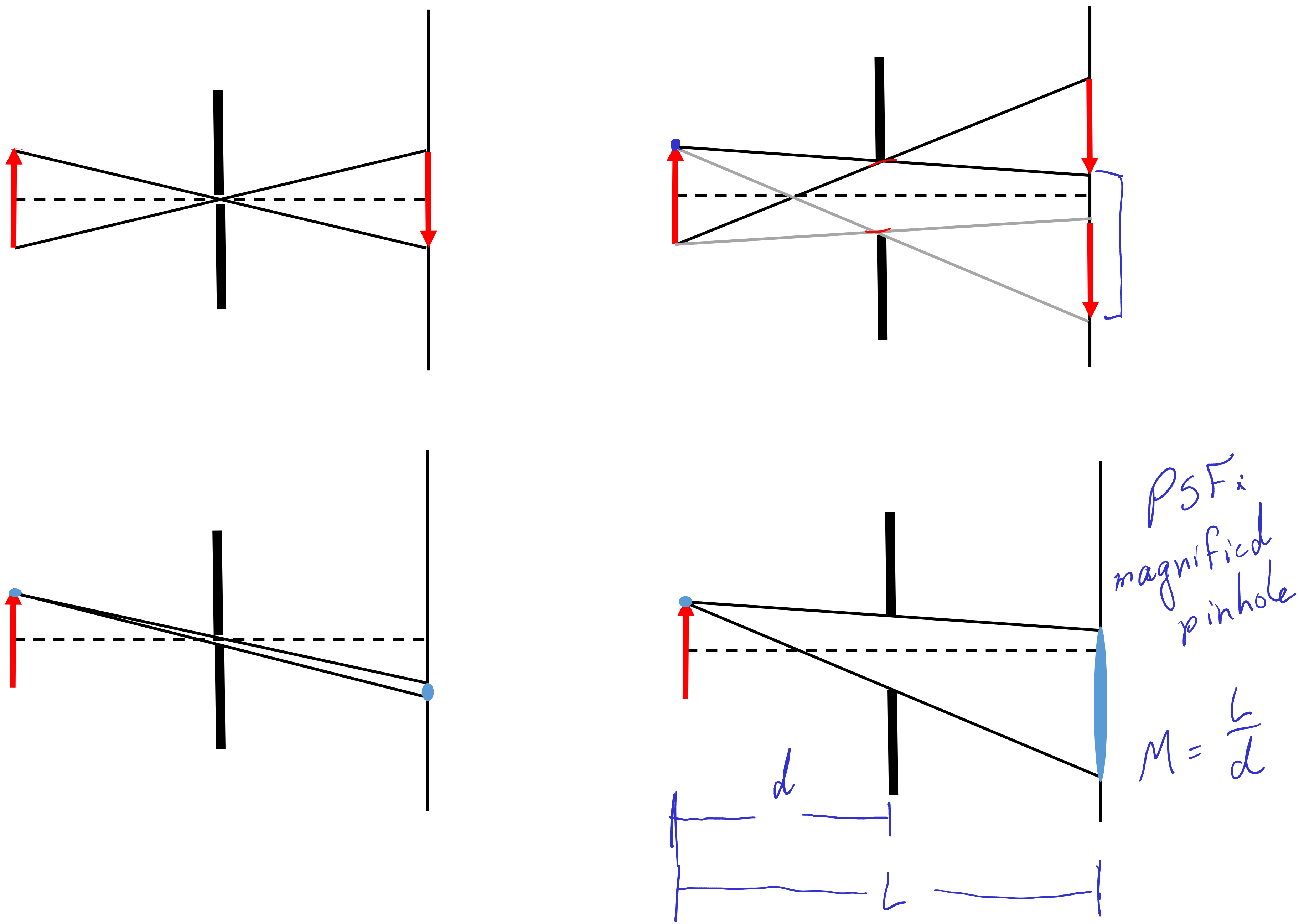
camera obscura





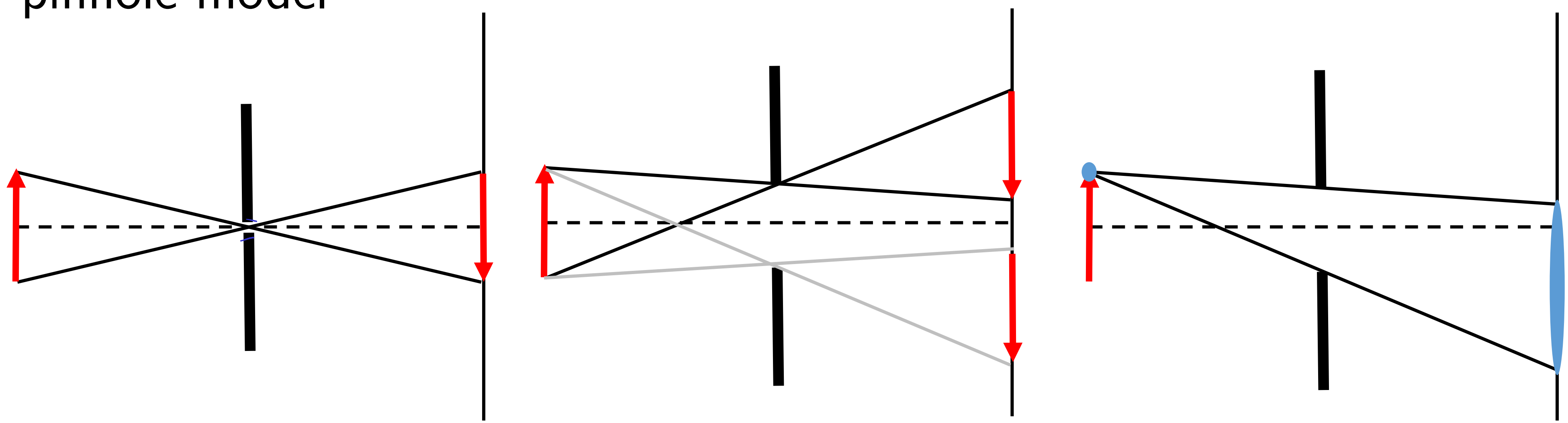
Pinhole camera model

PSF determined by aperture width



Projection model

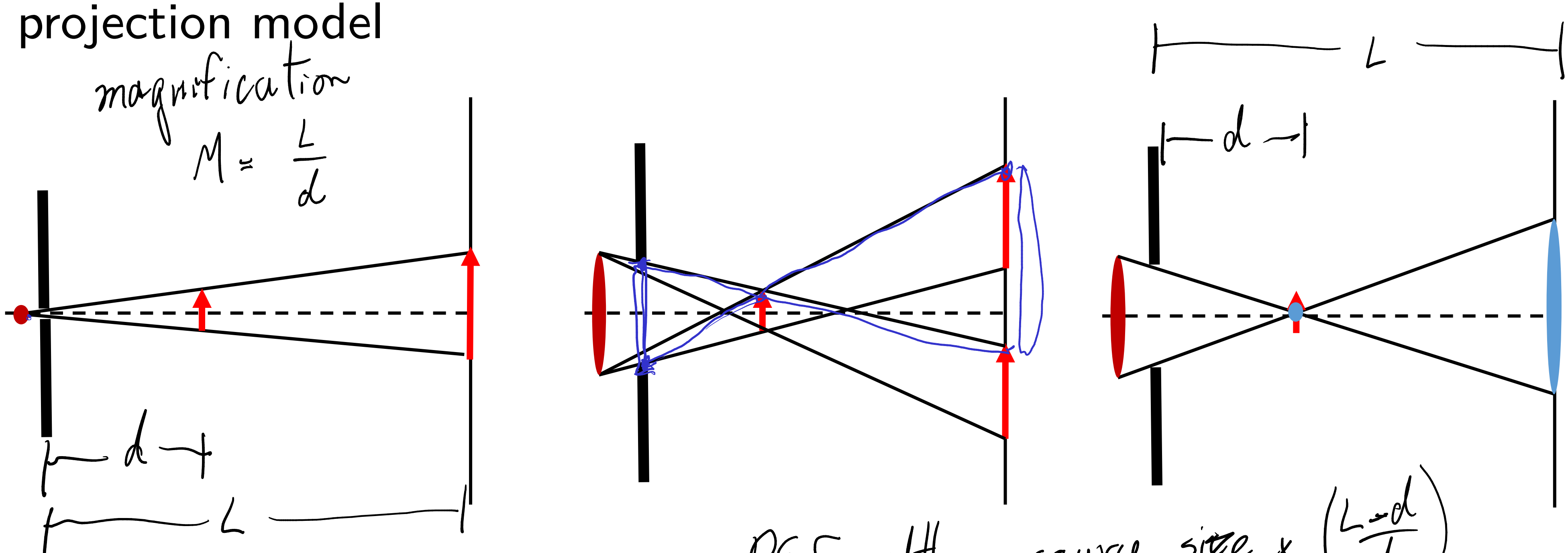
pinhole model



projection model

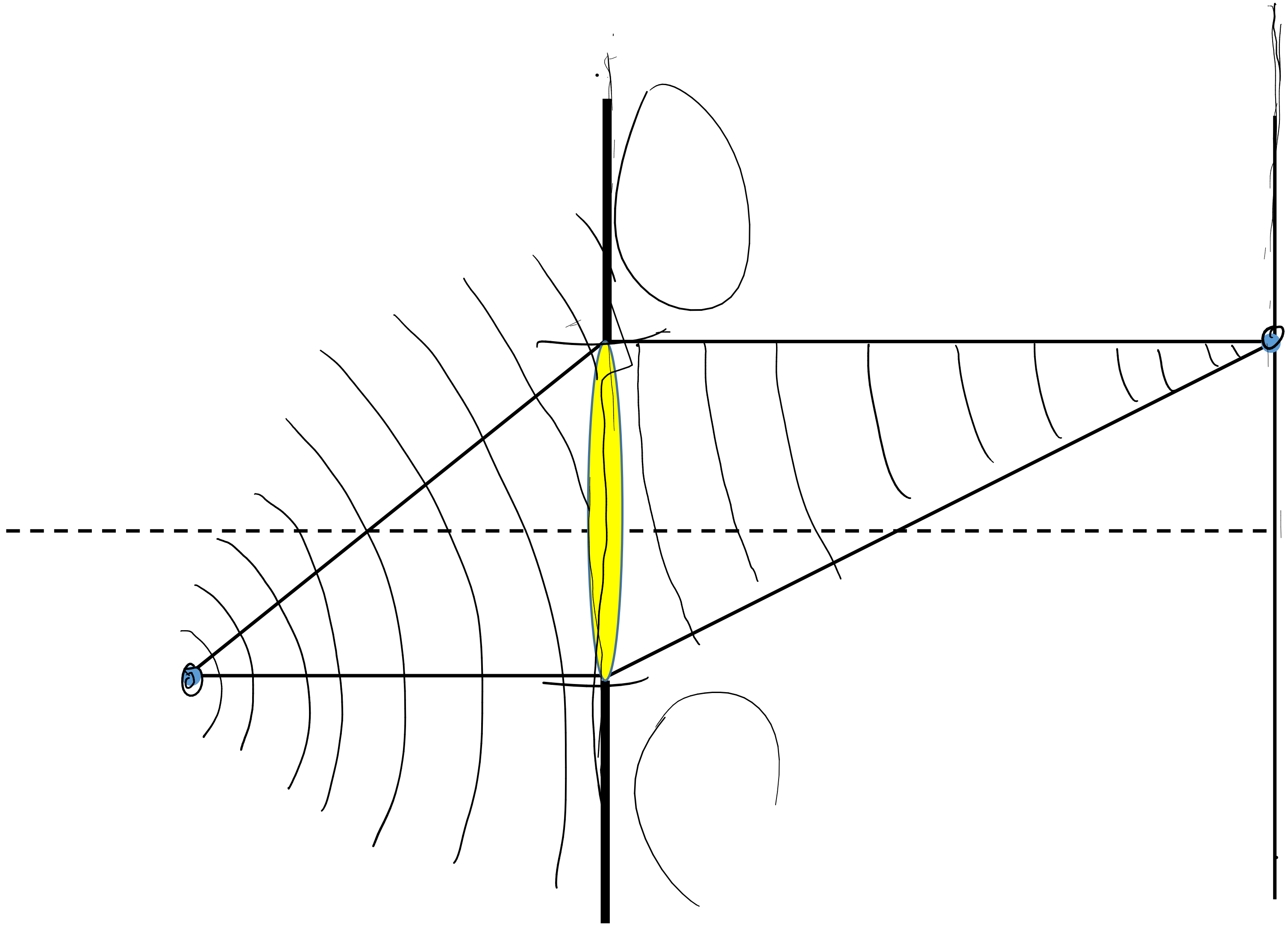
magnification

$$M = \frac{L}{d}$$



$$\text{PSF width} = \text{source size} \times \left(\frac{L-d}{d}\right)$$

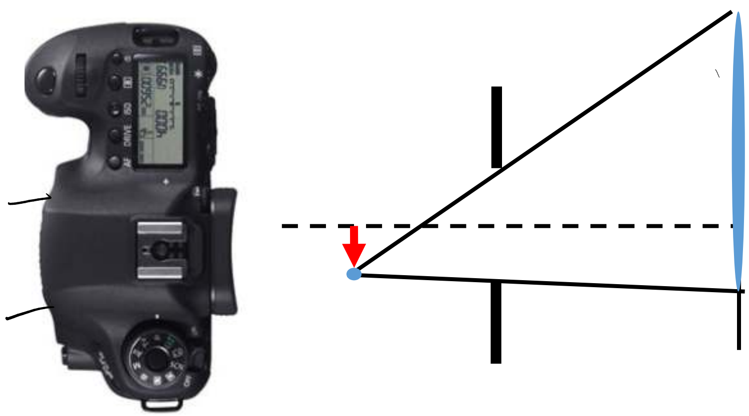
Lens camera model



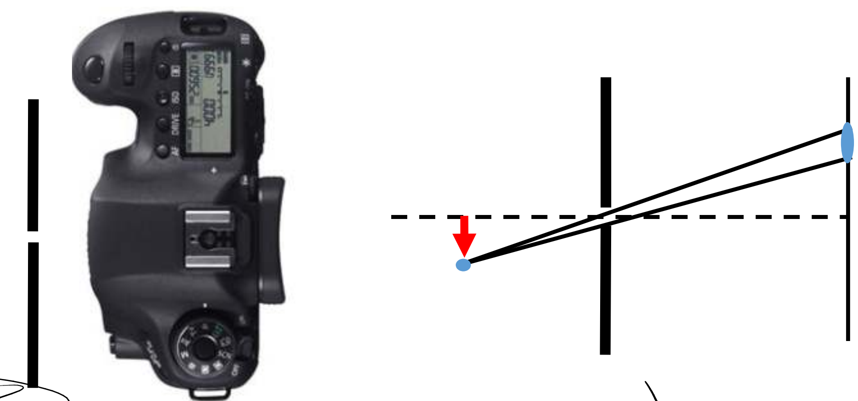
Result similar to small pinhole but without compromise on intensity

Lens camera model

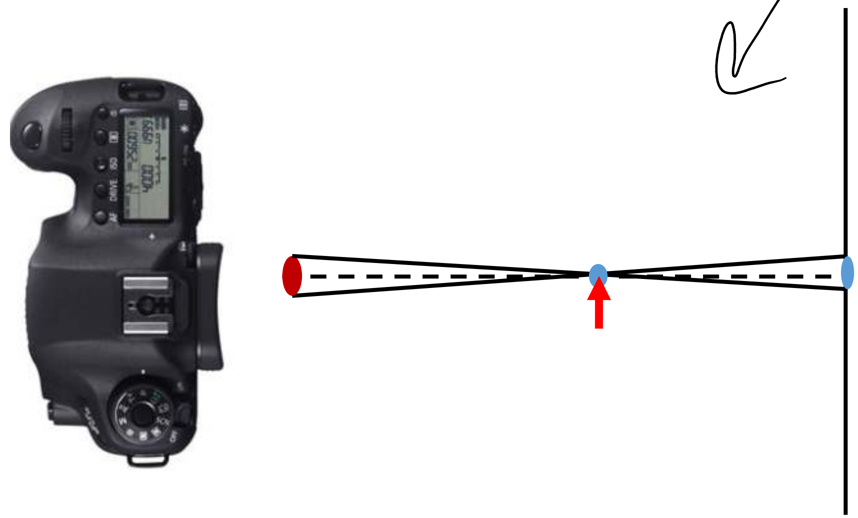
lensless model



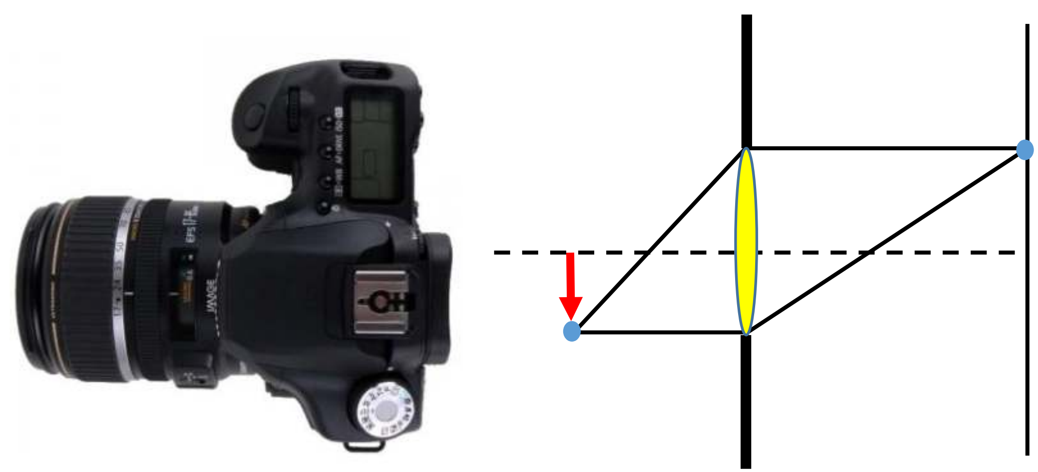
pinhole camera model



projection model



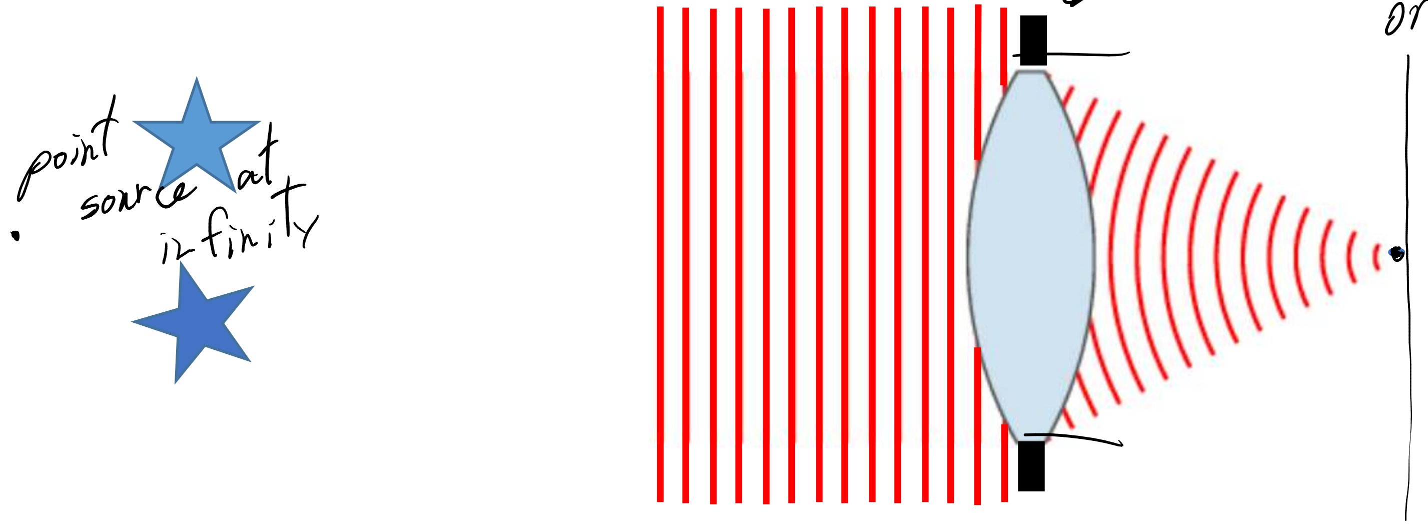
lens camera model



Diffraction-limited imaging systems

- Rayleigh criterion

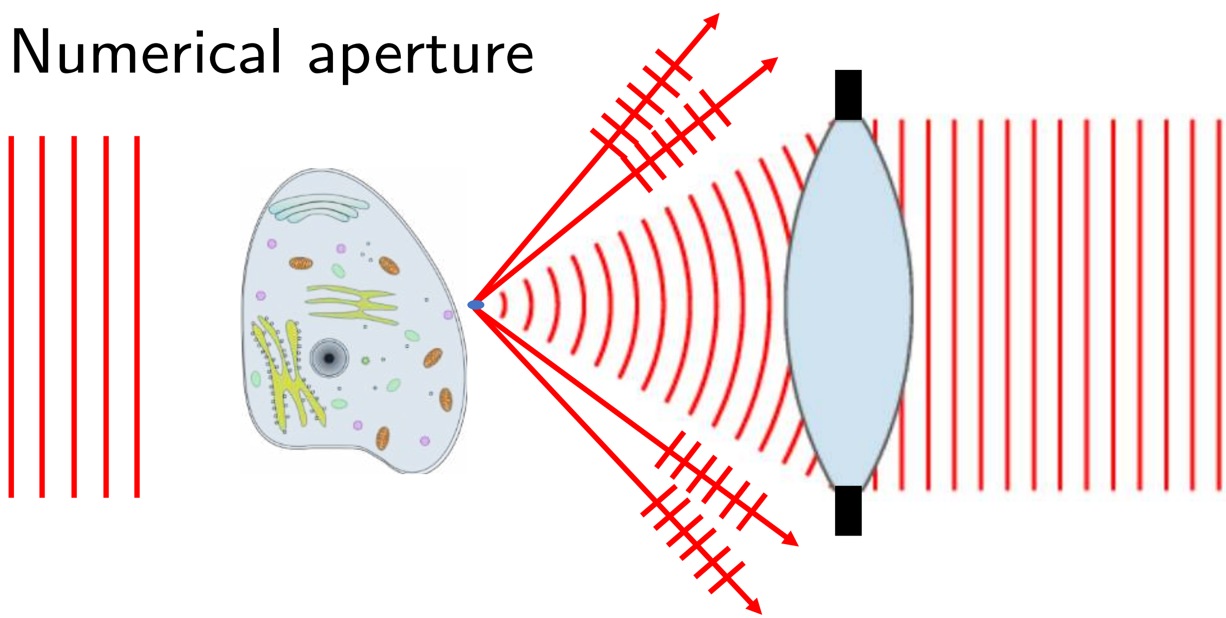
aperture = low pass filter on transmitted image



PSF : F.T. of a disc

Airy disc

- Numerical aperture



Inverse Fourier transform of disc of radius u_{max}:

$$\frac{J_1(2\pi r u_{max})}{r u_{max}}$$

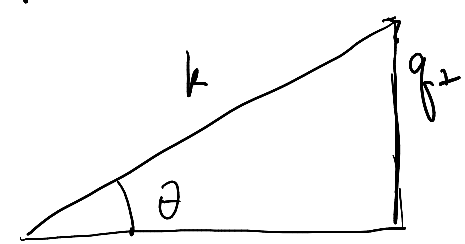
J_1: First Bessel function

Rayleigh criterion: resolution = distance to the first minimum from the origin:

$$u_{max} = \frac{q_{max}}{2\pi}$$

$$J_1(2\pi r_{min} u_{max}) = 0$$

$$q_{max} = k \sin\theta$$



$$3.83 = 2\pi r_{min} u_{max} \rightarrow r_{min} = \frac{3.83}{2\pi u_{max}}$$

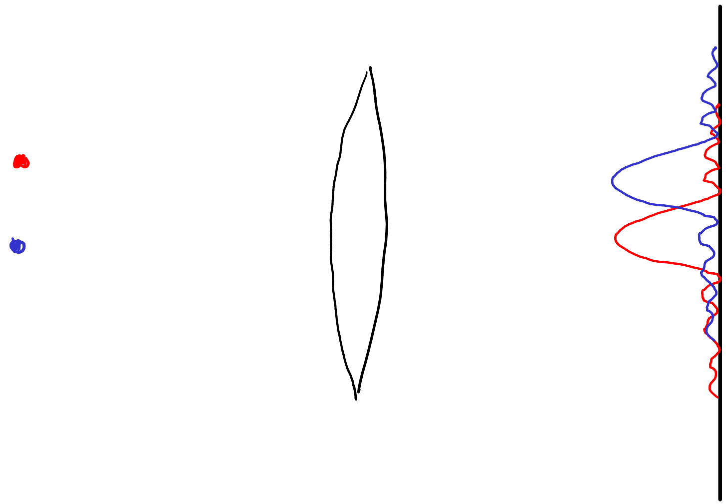
$$r_{min} = \frac{3.83}{\cancel{2\pi}} \frac{\cancel{2\pi}}{k\sin\theta} = \frac{3.83}{2\pi}\frac{\lambda}{\sin\theta}$$

$$\boxed{r_{min} = \frac{1.22\lambda}{2\sin\theta}}$$

$2\sin\theta = NA$

"numerical aperture"

Coherent vs incoherent imaging system



two images of point sources

1) if the two sources are perfectly coherent, then interference occurs:

$$I = |\psi_1 + \psi_2|^2$$

$$\hookrightarrow I = |PSF_{coh} * \psi|^2$$

$$|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}\{\psi_1^* \psi_2\}$$

$= 0$ if incoherent

2) if the two sources do not interfere (incoherent)

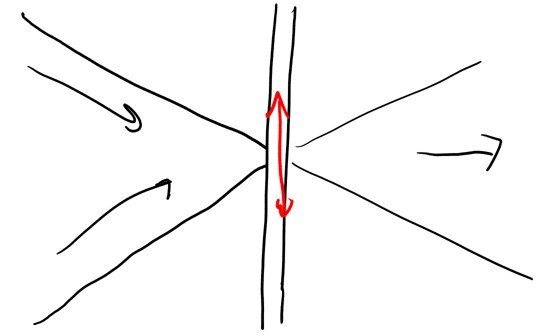
$$I = |\psi_1| + |\psi_2|^2$$

$\hookrightarrow$ In general:

$$I = PSF_{inc} * |\psi|^2$$

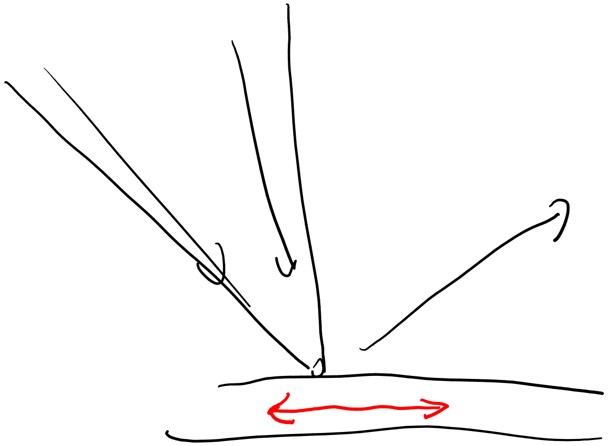
Scanning systems

Transmission



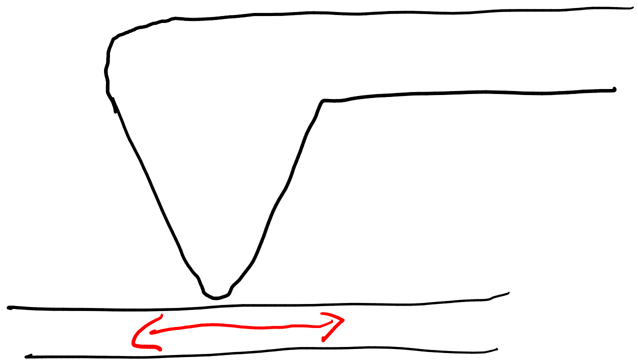
- **S**canning **T**ransmission **E**lectron **M**icroscopy
- **S**canning **T**ransmission **X**-ray **M**icroscopy
- …

Indirect (reflection, scattering, fluorescence, …)



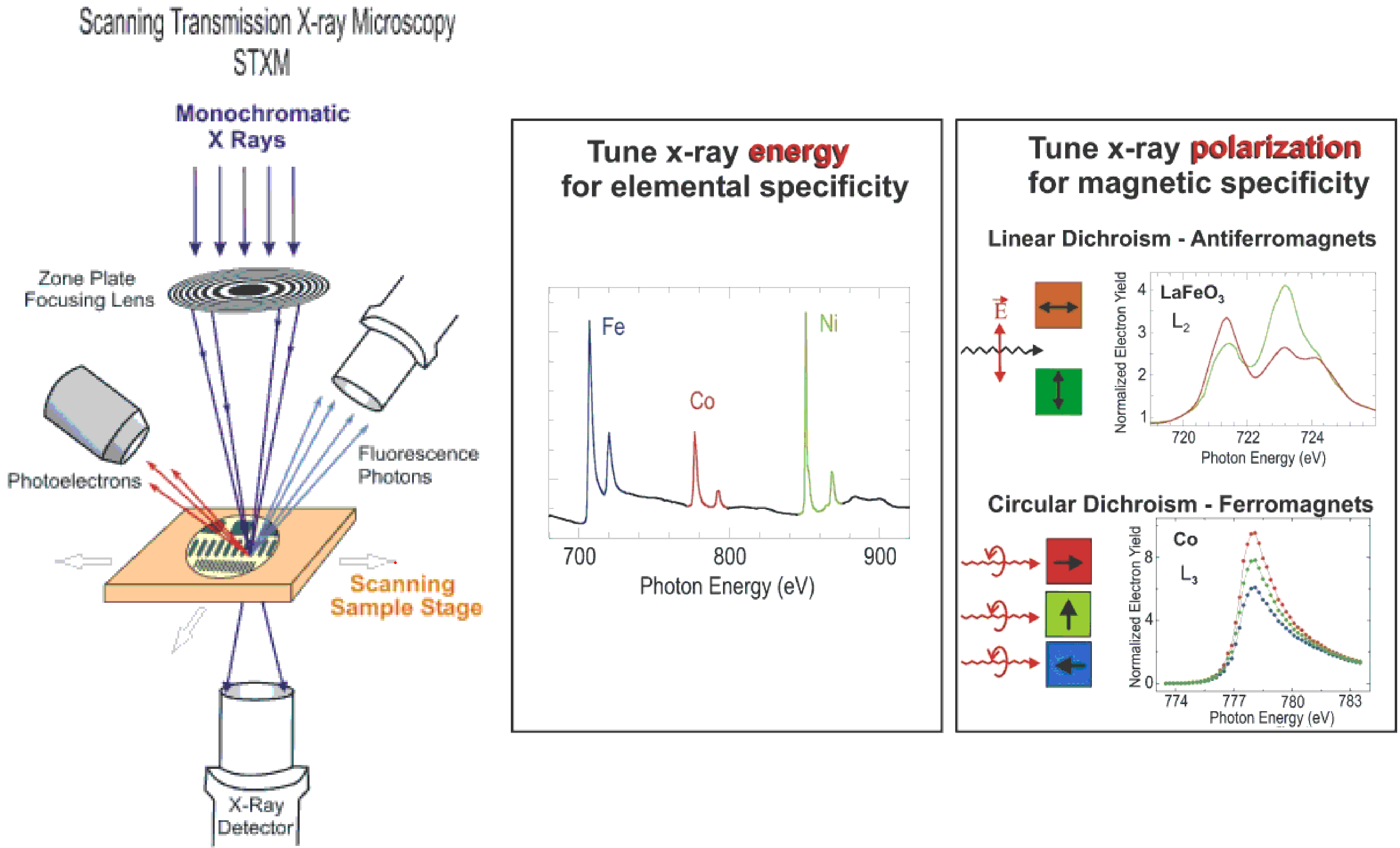
- **L**aser **S**canning **C**onfocal **M**icropsopy
- **S**canning **E**lectron **M**icroscopy
- **X**-ray **F**luorescence **M**icroscopy
- **P**hoto**E**mission **E**lectron **M**icroscopy
- ...

Physical probe



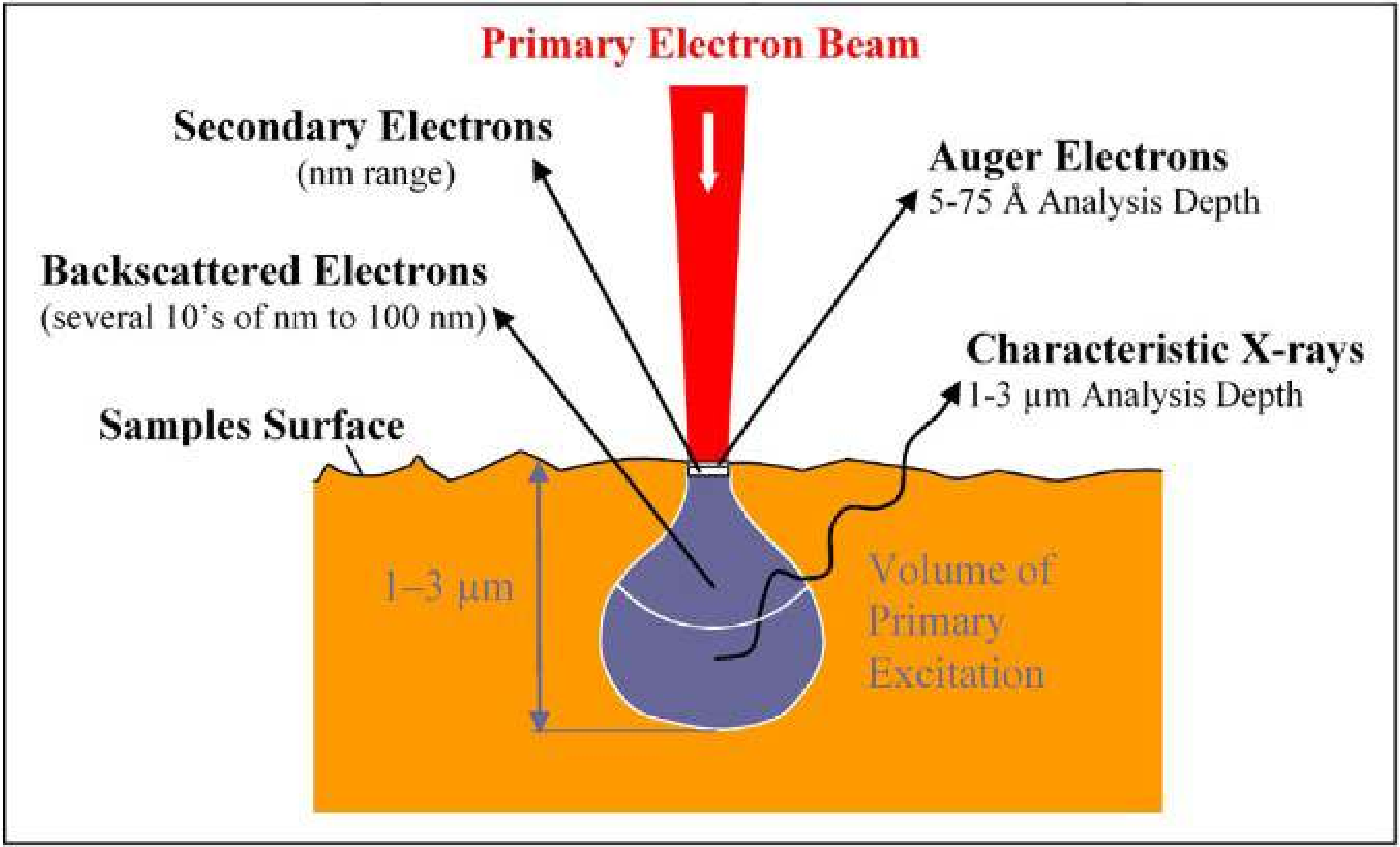
- **A**tomic **F**orce **M**icroscopy
- **S**canning **T**unneling **M**icroscopy
- ...

Scanning transmission X-ray microscopy

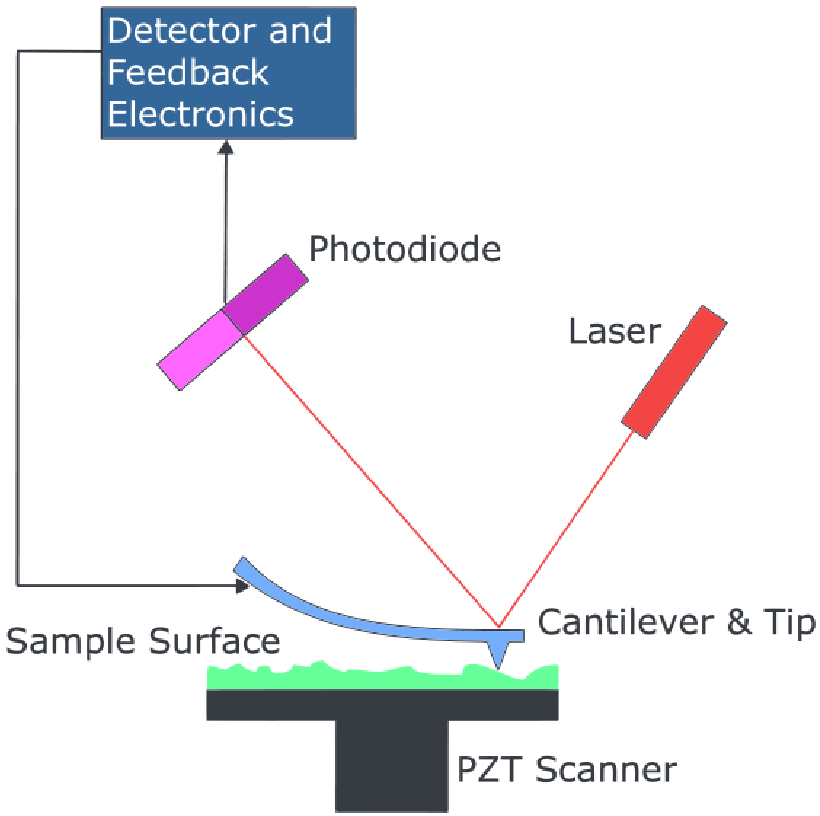


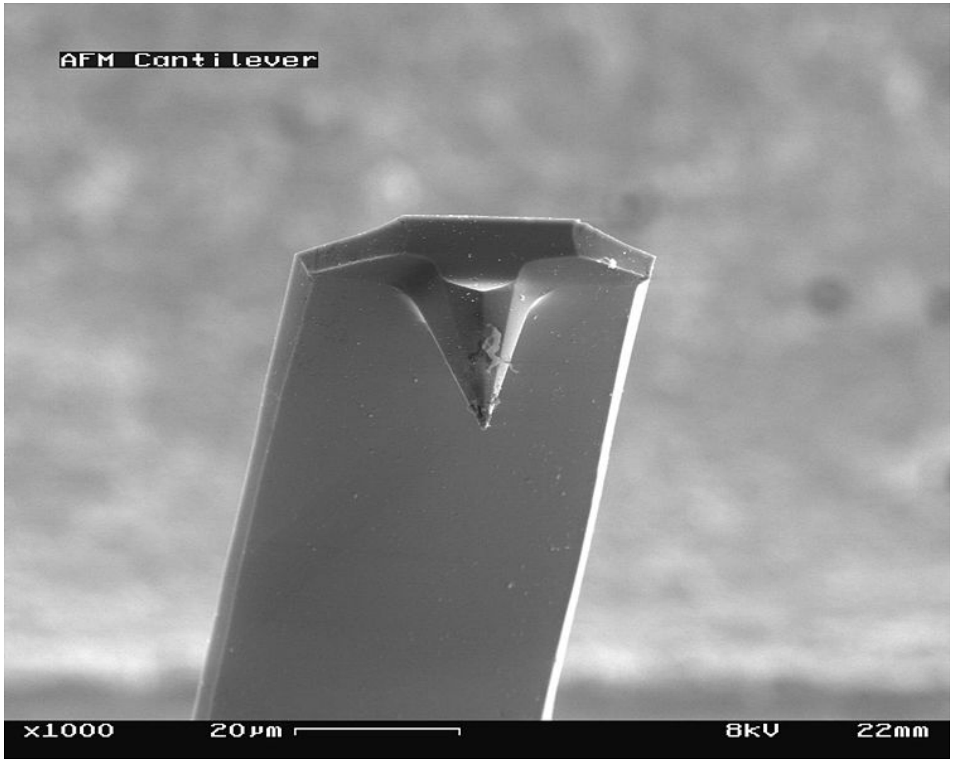
source: http://www-ssrl.slac.stanford.edu

Scanning electron microscopy



Atomic force microscopy

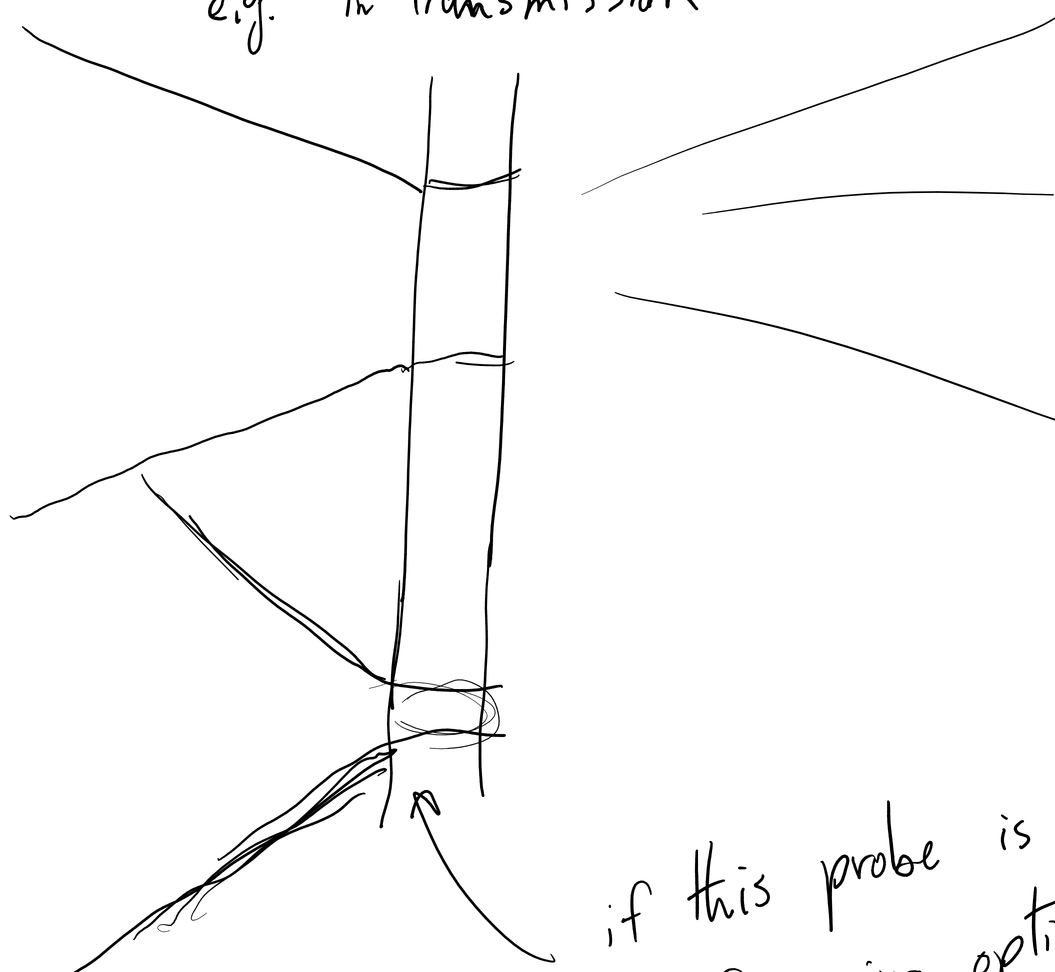




Resolution in scanning systems

Resolution mainly limited by probe size

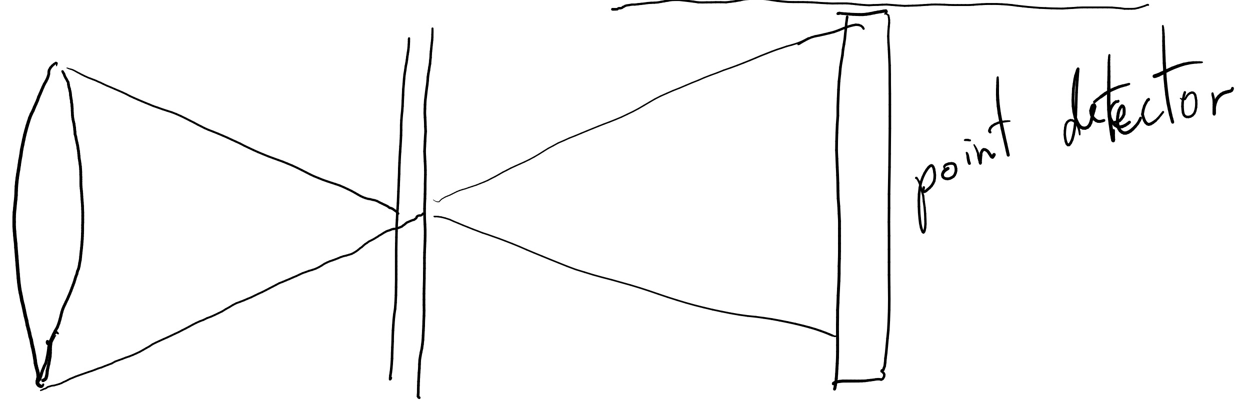
e.g. in transmission



if this probe is formed with focusing optics, then PSF is again related to numerical aperture

Scanning vs. full field systems

Transmission probe: the reciprocity theorem



is equivalent to