14th November

E B space

E' $\quad \sigma(E',E)$ is the weakest topology in E' s.t. the maps associated to the $x \in E$

$$f \longrightarrow f(x) \quad \text{are continuous}$$

$$f(x) = \langle x, f \rangle_{E\times E'}$$

$$\langle x, \ \rangle_{E\times E'} : E' \longrightarrow \mathbb{R}$$

$\forall f_0 \in E'$ a basis of neigh $\quad x_1, \dots, x_n \in E$, $\varepsilon > 0$

$$V_{f_0}(x_1, \dots, x_n, \varepsilon) = \{ f : |f(x_j) - f_0(x_j)| < \varepsilon \quad \forall j = 1, \dots, n \}$$

The following holds

Lemma If (Y, τ) is a topological space, then a map $F : (Y,\tau) \longrightarrow (E', \sigma(E',E))$ is continuous iff

$$\begin{tikzcd}
(Y,\tau) \arrow[r, "F"] \arrow[dr, dashed] & (E', \sigma(E',E)) \arrow[d, "\langle x, \cdot \rangle_{E\times E'}"] \\
& \mathbb{R}
\end{tikzcd}$$

for any $x \in E$ the map $y \longrightarrow \langle x, F(y) \rangle_{E\times E'}$ is continuous.

Lemma

If $\{f_n\}$ is a sequence ~~in~~ in E'

1) $f_n \rightharpoonup f$ in $\sigma(E',E) \iff f_n(x) \to f(x) \ \forall x \in E$.

2) $f_n \to f$ strongly in $E' \implies f_n \rightharpoonup f$ $\sigma(E',E)$

3) If $f_n \rightharpoonup f$ in $\sigma(E',E)$ then $\exists\, c > 0$ s.t.

$\|f_n\|_{E'} \le c$. Furthermore

$$\|f\|_{E'} \le \liminf_{n \to +\infty} \|f_n\|_{E'}$$

Example Remember that we proved that

if $C \subseteq E$ is convex,

C is closed strongly $\iff$ C is closed in $\sigma(E,E')$

The above equivalence is not true any more in E' with $\sigma(E',E)$ topology

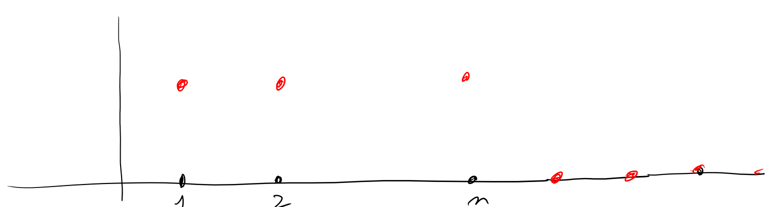
One can prove C convex strongly closed in E' with C not closed in $\sigma(E',E)$

$c_0(\mathbb{N}) = \{ f : \mathbb{N}\to\mathbb{R} : \lim_{n\to+\infty} f(n)=0\}$

$c_0(\mathbb{N})$ is closed subspace of $\ell^\infty(\mathbb{N}) = \{ f:\mathbb{N}\to\mathbb{R} : \sup_{n\in\mathbb{N}}|f(n)|<\infty\}$

$\ell^\infty(\mathbb{N}) = (\ell^1(\mathbb{N}))'$

$f_n \in c_0(\mathbb{N})$ $\underline{f_n(m)} = \begin{cases} 1 & \text{if } m\le n \\ 0 & m > n \end{cases}$



$\langle f_n, g\rangle_{\ell^\infty(\mathbb{N})\times\ell^1(\mathbb{N})} = \sum_{m=1}^{\infty} f_n(m)\, g(m) =$

$g\in\ell^1(\mathbb{N})$

$= \sum_{m=1}^{n} g(m)$

$\lim_{n\to+\infty}\langle f_n, g\rangle_{\ell^\infty\times\ell^1} = \lim_{n\to+\infty}\sum_{m=1}^{n} g(m) = \sum_{m=1}^{\infty} g(m)\in\mathbb{R}$

$= \langle 1, g\rangle_{\ell^\infty\times\ell^1}$

$\Rightarrow$ $f_n \to 1$ in $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$

$1\notin c_0(\mathbb{N})$ because $\lim_{n\to\infty} 1 = 1 \neq 0$

$\Rightarrow$ $c_0(\mathbb{N})$ is not closed for $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$

So therefore keep in mind that the $(E', \sigma(E',E))$ is different from $(E', \sigma(E',E''))$

Theorem If $\phi : E' \to \mathbb{R}$ is linear and continuous for $\sigma(E', E)$ topology then $\exists\ x \in E$ s.t. $\phi(f) = f(x)$ $\forall f \in E'$.

$$f \longrightarrow f(x)$$

Theorem (Banach Alaoglu)

Let E be a B-space and consider the dual E'. Then $\overline{D_{E'}(0,1)} = \{f \in E' : \|f\|_{E'} \le 1\}$ is compact for $\sigma(E', E)$.

$X_x = \mathbb{R}$

Pf $\quad \phi : E' \longrightarrow \mathbb{R}^E = \prod_{x \in E} X_x \qquad \{\cancel{\mathbb{R}}\}_{x \in E}$

$= \prod_{x \in E} \mathbb{R}$

$\mathbb{R}^E$ = {is the set of functions $f : E \longrightarrow \mathbb{R}$}

E' = is the set of continuous linear maps $f : E \longrightarrow \mathbb{R}$

$\phi(f) = f$

$E' \xrightarrow{\phi} \underset{E'}{\phi(E')} \subseteq \mathbb{R}^E$

we want to show that ϕ is an homeomorphism

$$(E', \sigma(E', E)) \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} (E', \text{ with the topology as a subspace of } \mathbb{R}^E)$$

$$\begin{array}{ccc} f & \xrightarrow{\quad\phi\quad} & f \\ (E', \sigma(E', E)) & \longrightarrow & \mathbb{R}^E \\ & & f \downarrow \\ & \dashrightarrow & f(x) \end{array}$$

is continuous.

Fix $x \in E$

$f \in \mathbb{R}^E \longrightarrow f(x) \in \mathbb{R}$

is continuous map from $\mathbb{R}^E \longrightarrow \mathbb{R}$

ϕ is continuous iff

But indeed we know that for any $x \in E$ the map

$f \in E' \longrightarrow f(x) \in \mathbb{R}$ is continuous for $(E', \sigma(E', E))$

Therefore $\phi : E'_{\sigma(E', E)} \longrightarrow \mathbb{R}^E$ is continuous

$x \in E$

$$\begin{array}{ccc} f & \longrightarrow & f \\ \phi^{-1} : (E', \text{ with the product top}) & \longrightarrow & (E', \sigma(E', E)) \\ & & \downarrow \quad f \downarrow \\ f \longrightarrow f(x) & & \mathbb{R} \quad f(x) \end{array}$$

But the topolog