15th November

Reflexive spaces

Def E B space is reflexive if $J : E \to E''$ is an isomorphism.

Theorem (Goldstine) If E is a B-space then $J D_E(0,1) \subseteq D_{E''}(0,1)$ and $J D_E(0,1)$ is dense in $D_{E''}(0,1)$ for the $\sigma(E'', E')$ topology.

$\Rightarrow$ JE dense in E'' for $\sigma(E'', E')$

Lemma E B-space, $f_1, \dots, f_n \in E'$, $a_1, \dots, a_n \in \mathbb{R}$.

Then the following are equivalent

1) $\forall \varepsilon > 0 \ \exists x_\varepsilon \in E$, $\|x_\varepsilon\|_E \leq 1$ s.t. $|f_j(x_\varepsilon) - a_j| < \varepsilon \ \forall j = 1, \dots, n$

2) $\forall b_1, \dots, b_n \in \mathbb{R}$ we have
$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left\| \sum_{j=1}^n b_j f_j \right\|_{E'}$$

Pf 1) $\Rightarrow$ 2)
$$\left| \sum_{j=1}^n b_j a_j \right| = \lim_{\varepsilon \to 0} \left| \sum_{j=1}^n b_j f_j(x_\varepsilon) \right| \leq \left\| \sum_{j=1}^n b_j f_j \right\|_{E'}$$

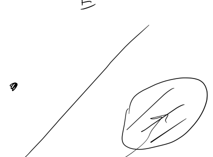
2) $\Rightarrow$ 1) $F = (f_1, \dots, f_n) : E \to \mathbb{R}^n$

Claim 1) means that $\underset{\mathbb{R}^n}{a} := (a_1, \dots, a_n)$ is $a \in \overline{F(D_E(0,1))}$

suppose by contradiction that $\exists a$ for which 2) is true but $a \notin \overline{F(D_E(0,1))}$

$\{a\}$ and $\overline{F(D_E(0,1))}$ are closed convex subsets of $\mathbb{R}^n$ compact and disjoint.

Therefore



there exists a hyperplane in $\mathbb{R}^n$ which separates them strictly.

$\exists b_1, \dots, b_n$ $\qquad b_1 x_1 + \dots + b_n x_n = \alpha$

$(f_1(x), \dots, f_n(x))$, $x \in D_E(0,1)$

$$\sum_{j=1}^n b_j f_j(x) < \alpha < \sum_{j=1}^n b_j a_j \quad *$$

$\forall x \in D_E(0,1)$

In particular for $x = 0$ in E we get
$$0 < \alpha < \sum_{j=1}^n b_j a_j = \left| \sum_{j=1}^n b_j a_j \right|$$

$$* \Rightarrow \left| \sum_{j=1}^n b_j f_j(x) \right| < \alpha < \left| \sum_{j=1}^n b_j a_j \right|$$

$$\left\| \sum_{j=1}^n b_j f_j \right\|_{E'} \leq \alpha < \left| \sum_{j=1}^n b_j a_j \right|$$

Back to Goldstine

We have to show that $J D_E(0,1)$ is dense in $D_{E''}(0,1)$ for the $\sigma(E'', E')$ topology.

We need to show that $\forall\, \xi \in D_{E''}(0,1)$ and for any neigh. V of ξ in the $\sigma(E'',E')$ top

$\exists \quad x \in D_E(0,1)$ s.t. $Jx \in V$.

We can take V of the form. $\varepsilon > 0$, $f_1, \dots, f_n \in E'$

$$V = \{ \eta \in E'' : |\langle \eta - \xi, f_j \rangle_{E'' \times E'}| < \varepsilon,\ j = 1, \dots, n \}$$

We need to find $x \in D_E(0,1)$ s.t.

$$|\langle Jx - \xi, f_j \rangle_{E'' \times E'}| = |\langle Jx, f_j \rangle_{E'' \times E'} - \langle \xi, f_j \rangle_{E'' \times E'}| =$$

$$= |f_j(x) - \langle \xi, f_j \rangle_{E'' \times E'}| < \varepsilon \quad \forall\ j = 1, \dots, n.$$

Set $a_j = \langle \xi, f_j \rangle_{E'' \times E'}$ and let $b_1, \dots, b_n \in \mathbb{R}$ be arbitrary numbers ($\xi \in D_{E''}(0,1)$)

$$\left| \sum_{j=1}^{n} b_j a_j \right| = \left| \langle \xi, \sum_{j=1}^{n} b_j f_j \rangle_{E'' \times E'} \right| \leq \|\xi\|_{E''} \left\| \sum_{j=1}^{n} b_j f_j \right\|_{E'}$$

$$\leq \left\| \sum_{j=1}^{n} b_j f_j \right\|_{E'} \qquad \forall\ b_1, \dots, b_n$$

$(f_1, \dots, f_n)$
$(a_1, \dots, a_n)$

$$\Downarrow$$

$\forall \varepsilon > 0 \ \exists \ x_\varepsilon \in D_E(0,1)$ s.t.

$$|f_j(x_\varepsilon) - a_j| < \varepsilon$$

$$\Updownarrow$$

$$|f_j(x_\varepsilon) - \langle \xi, f_j \rangle_{E'' \times E'}| < \varepsilon$$

$$f_j(x_\varepsilon) = \langle J x_\varepsilon, f_j \rangle_{E'' \times E'}$$

$J x_\varepsilon \in V$

Thm (Kakutani) E is reflexive ~~iff~~ $\iff$ $\overline{D_E(0,1)} = \{x \in E : \|x\|_E \le 1\}$ is compact for $\sigma(E, E')$ topology.

Pf $\Longrightarrow$

E reflexive $\Longrightarrow$ $J : E \to E''$ is an ~~isomorphism~~ isometry for the strong topologies $\quad J\,\overline{D_E(0,1)} = \overline{D_{E''}(0,1)}$

On the other hand $\overline{D_{E''}(0,1)}$ is compact for $\sigma(E'', E')$ by Banach - Alaoglu.

$$\boxed{J^{-1} : (E'', \sigma(E'', E')) \longrightarrow (E, \sigma(E, E'))} \; *$$

and we need to show that it is continuous.

($J^{-1} : E'' \longrightarrow E$ continuous strongly

$\iff J^{-1} : (E'', \sigma(E'', E''')) \longrightarrow (E, \sigma(E, E'))$)

J^{-1} is continuous in the sense of $*$ if

$\forall\, f \in E'$ the map $E'' \ni \xi \longrightarrow f(J^{-1}\xi)$ is continuous.

$\xi \longrightarrow \langle J^{-1}\xi, f \rangle_{E \times E'} \qquad J^{-1}\xi = x \in E$

$$\langle J^{-1}\xi, f \rangle_{E \times E'} = \langle x, f \rangle_{E \times E'} =$$

$$= \langle Jx, f \rangle_{E'' \times E'} = \langle \xi, f \rangle_{E'' \times E'}$$

$$\xi \longrightarrow f(J^{-1}\xi) = \langle \xi, f \rangle_{E'' \times E'} \text{ is continuous } \forall\, f \in E'$$

But the topology $(E'', \sigma(E'', E'))$ has been defined s.t.

Hence $J^{-1} : (E'', \sigma(E'', E')) \longrightarrow (E, \sigma(E, E'))$ is continuous.

$$J^{-1}\,\overline{D_{E''}(0,1)} = \overline{D_E(0,1)}$$

This is compact in

Let $\overline{D_E(0,1)} = \{ x \in E : \|x\|_E \le 1 \}$ be compact for $\sigma(E,E')$. We want to show that $J : E \to E''$ is an isomorphism (for strong topologies)

We know that

$$\begin{tikzcd}
J : (E, \sigma(E,E')) \arrow[r, "\text{continuous}"] \arrow[dr, "\text{continuous}"'] & (E'', \sigma(E'',E''')) \arrow[d, "\text{continuous}"] \\
& (E'', \sigma(E'',E'))
\end{tikzcd}$$

By Goldstine $J D_E(0,1)$ is dense in $D_{E''}(0,1)$ for $\sigma(E'',E')$

- $\overline{D_E(0,1)}$ $\sigma(E,E')$ compact $\Longrightarrow$ $J\overline{D_E(0,1)}$ is compact in $(E'', \sigma(E'',E'))$

$$J\overline{D_E(0,1)} \supseteq J D_E(0,1)$$

$$\Longrightarrow J\overline{D_E(0,1)} \supseteq D_{E''}(0,1)$$

$$J\{x \in E : \|x\|_E \le 1\} \supseteq D_{E''}(0,1)$$

$\Longrightarrow$ $J : E \to E''$ is surjective

$\Longrightarrow$ J is an isomorphism.

$0 \neq x'' \in E''$

$$\frac{x''}{2\|x''\|_{E''}} = Jx$$

$$x'' = J(2\|x''\|_{E''} x)$$

Lemma

1) E B space M is closed vector space in E.

Then E reflexive $\Rightarrow$ M reflexive

2) E reflexive $\Leftrightarrow$ E' reflexive

Lemma E refexive and Ω a closed bounded convex subspace of E. Then Ω is compact for $\sigma(E,E')$.

Pf . $\exists\, \lambda > 0$ st $\Omega \subseteq \lambda \overline{D_E(0,1)}$

$x \to \lambda x$ is continuous in E for $\sigma(E,E')$

$$\overline{D_E(0,1)} \longrightarrow \lambda \overline{D_E(0,1)} = \overline{D_E(0,\lambda)}$$

~~being~~ being the image of a compact set by means of a compact map $\overline{D_E(0,\lambda)}$ is $\sigma(E,E')$-compact

Furthermore Ω is closed for $\sigma(E,E')$

$\Omega \subseteq \overline{D_E(0,\lambda)} \Rightarrow \Omega$ is $\sigma(E,E')$ compact.

Corollary E reflexive $A \subseteq E$ convex and closed.

$\phi : A \to (-\infty, +\infty]$ lower semicontinuous, convex and such that $\lim\limits_{\substack{x\to\infty \\ x\in A}} \phi(x) = +\infty$

Then $\exists$ an absolute minimum $\bar{x} \in A$

Pf Consider $x_0 \in A$ and set $\lambda_0 = \phi(x_0)$

$K_0 = A \cap \phi^{-1}((-\infty, \lambda_0])$

$K_0 \ni x_0$, K_0 is convex , K_0 is closed

K_0 is bounded $\Rightarrow K_0$ is $\sigma(E,E')$ compact

$$\phi : A \longrightarrow (\cancel{-\infty}, +\infty]$$
$$K_0 \nearrow \phi$$

$\forall\ x \in A \setminus K_0$, $\phi(x) > \lambda_0 = \phi(x_0)$.

$\lambda_n \in \phi(K_0)$

$\lambda_n \xrightarrow{\text{decreasing}} \inf \phi(K_0)$

$\phi(x_n) = \lambda_n$ $\{x_n\}$ is a sequence in K_0

$K_n = \phi^{-1}((-\infty, \lambda_n]) \cap A$

$K_n \ni x_n$ K_n are convex ~~comp~~ set for $\sigma(E,E')$

We can assume λ_n is strictly decreasing.

$K = \bigcap_{n=1}^{\infty} K_n$. K is compact and it is not empty

K empty $\Rightarrow \exists\ m$ s.t. $\bigcap_{n=1}^{m} K_n$ is emp
$= K_m \ni x_m$

$x \in K \Rightarrow x$ point of minimum

$x \in K \Rightarrow x \in K_n \quad \forall n$

$\phi(x) \le \lambda_n \quad \forall n$, $\lambda_n \downarrow \inf \phi(K_0)$

$\phi(x) \le \inf \phi(K_0)$

$x \in K_0 \Rightarrow \phi(x) = \inf(K_0)$

$\Rightarrow x$ is a point of abs min.

ϕ strictly convex $\Rightarrow$ uniqueness of abs min.