Quantum Foundations 3 – Collapse Models

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Modifying the Schrödinger equation

Another suggestion is that the collapse of the wave function is a real phenomenon. Cleary it cannot occur only during measurements; possibly, it is part of the quantum dynamics, but becomes dominant only during specific circumstances, such as measurements.

This means that the unitary dynamics as expressed (at the non relativistic level) by the Schrödinger equation is not exact; it is an approximation of a nonlinear (and stochastic) dynamics. This is not new in physics. The linearity of Newton's gravitational potential is a weak field approximation of a nonlinear dynamics as given by General Relativity.

One interesting question is how to modify the Schrödinger equation in a nonlinear sense. Apparently there infinite ways of doing it. But then one can show that this not the case. The reason is the following.

1. A **statistical mixture** is a a collection $\{(|\psi_n\rangle, p_n)\}_{n = 1,..N}$, where $|\psi_n\rangle$ are normalized states, not necessarily orthogonal to each other, and p_n are positive numbers summing to 1. The idea is that the state of a system is described by $|\psi_n\rangle$; we do not know which one, we know only the probability distribution. A statistical mixture can be associated the density matrix

$$\rho = \sum_{n=1}^{N} p_n |\psi_n\rangle\langle\psi_n|$$

2. Two statistical mixtures $\{(|\psi_n\rangle, p_n)\}_{n = 1,..N}$ and $\{(|\varphi_m\rangle, q_m)\}_{m = 1,..M}$ are said to be **equivalent** if they give rise to the same density matrix:

$$\sum_{n=1}^{N} p_n |\psi_n\rangle\langle\psi_n| = \sum_{m=1}^{M} q_m |\phi_m\rangle\langle\phi_m|$$

We focus now to the cases when the states $|\psi_n\rangle$ are orthogonal to each other and, similarly, the states $|\varphi_m\rangle$ are orthogonal to each other.

3. **Equivalent statistical mixtures can be created** (ideally, instantly) **at a distance**, by exploiting entanglement. Suppose that Alice and Bob, who are arbitrarily far away from each other, share an entangled pure (for simplicity) state $\rho_{AB} = |\Psi\rangle\langle\Psi|$.

Alice decides to measure one of two possible observables O^A_1 and O^A_2, whose associate operators are:

$$\hat{O}^{\mathrm{A}}_k = \sum_n o^{\mathrm{A}}_k P^{\mathrm{A}}_k, \qquad P^{\mathrm{A}}_k = |o^{\mathrm{A}}_k\rangle\langle o^{\mathrm{A}}_k|, \; k = 1, 2$$

By the Born rule and collapse postulate, Bob will end up with having the state:

$$\begin{aligned} \rho^{\mathrm{B}}_k &= \sum_n \mathrm{Tr}[P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}} \rho_{\mathrm{AB}}] \frac{P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}} \rho_{\mathrm{AB}} P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}}}{\mathrm{Tr}[P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}} \rho_{\mathrm{AB}}]}, \; k = 1, 2 \\ &= \sum_n P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}} \rho_{\mathrm{AB}} P^{\mathrm{A}}_{k,n} \otimes I^{\mathrm{B}}, \end{aligned}$$

Born rule
Collapse

It is easy to show that

$$\rho^{\mathrm{B}}_1 = \rho^{\mathrm{B}}_2 = \rho^{\mathrm{B}}$$

This means that the two statistical mixtures

$$\{(|\psi_n\rangle, p_n\} \qquad |\psi_n\rangle = \frac{P^{\mathrm{A}}_{1,n} \otimes I^{\mathrm{B}} |\Psi\rangle}{\| P^{\mathrm{A}}_{1,n} \otimes I^{\mathrm{B}} |\Psi\rangle \|}, \; p_n = \| P^{\mathrm{A}}_{1,n} \otimes I^{\mathrm{B}} |\Psi\rangle \|^2$$

$$\{(|\phi_n\rangle, q_n\} \qquad |\phi_n\rangle = \frac{P^{\mathrm{A}}_{2,n} \otimes I^{\mathrm{B}} |\Psi\rangle}{\| P^{\mathrm{A}}_{2,n} \otimes I^{\mathrm{B}} |\Psi\rangle \|}, \; q_n = \| P^{\mathrm{A}}_{2,n} \otimes I^{\mathrm{B}} |\Psi\rangle \|^2$$

are equivalent

Being equivalent, the two mixtures cannot be distinguished. This is the essence of the **no-signaling theorem**, which will be discussed later in connection with nonlocality.

This is an experimental fact (up to experimental error), therefore any modification of QM must takes this into account.

Given this, let us come back to our nonlinear dynamics. Assume that after Alice's measurement, Bob waits a little bit for the nonlinear dynamics to build up, before performing a measurement. The state $|\psi_n\rangle$ changes into $|\psi^F_n\rangle$, and likewise $|\varphi_n\rangle$ changes into $|\varphi^F_n\rangle$. We assume the final states normalized (otherwise troubles would emerge in making sense of the wave function). Not that the probabilities p_n and q_m do not change because they are related to our ignorance about the true state of the system

Then the two initially equivalent statistical mixtures need no remain equivalent any longer. For example, consider the two equivalent statistical mixtures

$$\{(|\psi_1\rangle, 1/2), (|\psi_2\rangle, 1/2)\}, \quad \{(|\psi_+\rangle, 1/2), (|\psi_-\rangle, 1/2)\}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other and $|\psi_\pm\rangle = [|\psi_1\rangle \pm |\psi_2\rangle]/\sqrt{2}$. Suppose the nonlinear dynamics is such that

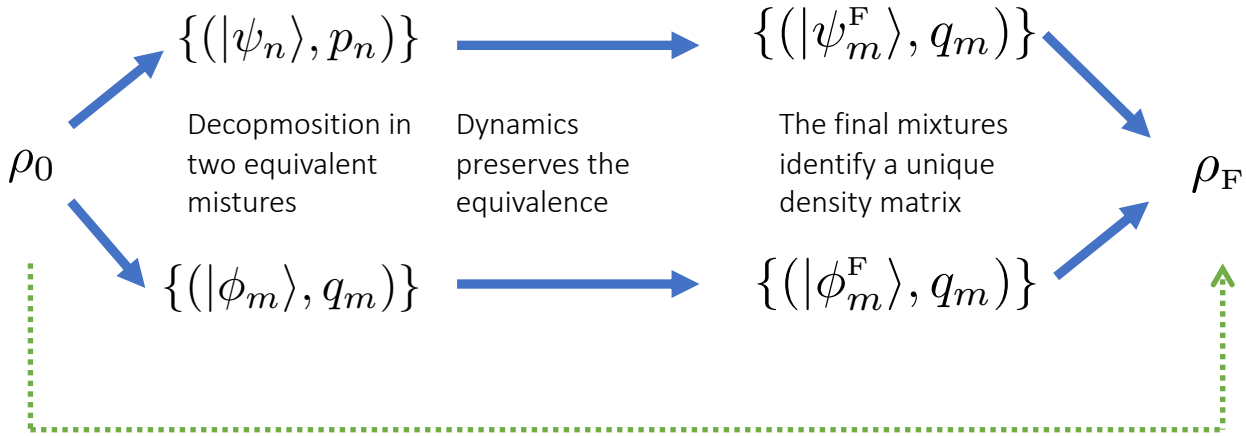
$$\begin{aligned} |\psi_1\rangle &\rightarrow |\psi_1^{\mathrm{F}}\rangle \\ |\psi_2\rangle &\rightarrow |\psi_2^{\mathrm{F}}\rangle \neq |\psi_1^{\mathrm{F}}\rangle \\ |\psi_+\rangle &\rightarrow |\psi_1^{\mathrm{F}}\rangle \\ |\psi_-\rangle &\rightarrow |\psi_1^{\mathrm{F}}\rangle \end{aligned}$$

Clearly, **the initially equivalent mixtures loose their equivalence** after some time. If they are not equivalent anymore, **they can be distinguished** by some measurement. Hence a **faster-than-light protocol** is possible.

The faster-than-light effect here discussed, which is related to the collapse of the wave function and quantum nonlocality, has nothing to do with working with nonrelativistic QM. It would remain in relativistic QFT.

If we demand **that faster-than-light is not possible, then the nonlinear dynamics must be such that the equivalence among statistical mixtures must be preserved over time**. This poses a severe constraint.

If the statistical equivalence is preserved over time, then one can define an evolution for the density matrix:



The nonlinear dynamics for the state vector uniquely identifies a dynamics for the density matrix. A key property is that **the dynamics for the density matrix is, by construction, linear**.

Take $\rho = \lambda_1 \rho_1 + \lambda_2 \rho_2$ (convex combination of ρ_1 and ρ_2). Let us consider a decomposition, among the many possible ones, of the two density matrices ρ_1 and ρ_2 :

$$\rho_k \quad = \quad \sum_n p_n^k |\psi_n^k\rangle\langle\psi_n^k|, \qquad k = 1, 2$$

Then ρ admits the decomposition

$$\rho \quad = \quad \lambda_1 \sum_n p_n^1 |\psi_n^1\rangle\langle\psi_n^1| + \lambda_2 \sum_n p_n^2 |\psi_n^2\rangle\langle\psi_n^2|$$

Suppose that the states $|\psi^k{}_n\rangle$ evolve into $|\psi^{k,F}{}_n\rangle$. Then the density matrix ρ evolves into (remember that any decomposition is good to compute the time evolution of the density matrix, because equivalence is preserve):

$$\begin{aligned}\rho_{\rm F} &= \lambda_1 \sum_n p_n^1 |\psi_n^{1,{\rm F}}\rangle\langle\psi_n^{1,{\rm F}}| + \lambda_2 \sum_n p_n^2 |\psi_n^{2,{\rm F}}\rangle\langle\psi_n^{2,{\rm F}}| \\ &= \lambda_1 \rho_{1,{\rm F}} + \lambda_2 \rho_{2,{\rm F}}\end{aligned}$$

The conclusion is: **to avoid faster-than-light signaling, the (nonlinear) dynamics for the state vector must be such that, at the density matrix level, the evolution is linear**.

The basic reason for the result is that ”ignorance propagates linearly”, i.e. if a system is initially either in state x with probability p and in state y with probability q, and if x evolves into X and y into Y, then the system will end up in state X with probability p and in state Y with probability q.

It is rather obvious that a deterministic nonlinear dynamics for the state vector cannot become linear at the density matrix level: these dynamics are excluded at the fundamental level. Extra degrees of freedom, in the form of stochastic terms are needed in order to wash out nonlinearities of the state vector’s dynamics at the density matrix level. This poses a strong constraint of the possible forms of the dynamics.

Physics books are full of nonlinear deterministic Schrödinger’s equations, such as the Gross-Pitaevskij equation, or the Schrödinger Newton equation. These are phenomenological equations, which are very useful and valid within their domain of validity. They cannot be considered or promoted to fundamental equations, for the reasons explained here above.

Nonlinear stochastic Schrödinger equation

Which kind of dynamics can we write, considering the previous constraint? We assume that the dynamics is **Markovian**, which in general is a good working hypothesis. There has been quite an extensive work also with non-Markovian models, though.

At the density matrix level, the dynamics is of the QDS type and by construction is completely positive, since it is defined at the wave function level. Therefore it is of the **Lindblad type:**

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2} \left\{ L_i^\dagger L_i, \rho \right\} \right)$$

Then, mathematically speaking, the **goal** is to find a **dynamics for the wave function which, at the density matrix level, reproduces the Lindblad equation.**

Since the Lindblad equation consists of a unitary part, plus a non-unitary term, it is rather natural to look for an equation for the wave function which also consists of two parts: a unitary part (the standard Schrödinger dynamics) and a non unitary (nonlinear and stochastic) part.

As a working model, we assume that the non unitary part occurs as a **jump process**, i.e. at random times. One can consider a continuous process, and there is an extended literature at this regard, based on stochastic differential equations for the wave function. We will not touch on this

A very natural model for a jump process is a **Poisson process**

A stochastic process $(N(t))_{t\geq 0}$ is said to be a *counting process* if $N(t)$ counts the total number of 'events' that have occurred up to time t. Hence, it must satisfy:

(i) $N(t) \geq 0$ for all $t \geq 0$.
(ii) $N(t)$ is integer-valued.
(iii) If $s < t$, then $N(s) \leq N(t)$.
(iv) For $s < t$, the increment $N((s,t]) \stackrel{\text{def}}{=} N(t) - N(s)$ equals the number of events that have occurred in the interval $(s,t]$.

A counting process is said to have *independent increments* if the numbers of events that occur in disjoint time intervals are independent, that is, the family $(N(I_k))_{1\leq k\leq n}$ consists of independent random variables whenever $I_1, ..., I_n$ forms a collection of pairwise disjoint intervals. In particular, $N(s)$ is independent of $N(s+t) - N(s)$ for all $s, t \geq 0$.

A counting process is said to have *stationary increments* if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval $(s, s+t]$, i.e. $N((s, s+t])$ has the same distribution as $N((0,t])$ for all $s, t \geq 0$.

One of the most important types of counting processes is the Poisson process, which can be defined in various ways.

Definition 1.1. [The Axiomatic Way]. A counting process $(N(t))_{t\geq 0}$ is said to be a *Poisson process with rate (or intensity)* λ, $\lambda > 0$, if:

(PP1) $N(0) = 0$.
(PP2) The process has independent increments.
(PP3) The number of events in any time interval of length t is Poisson distributed with mean λt. That is, $N((s,t]) \stackrel{d}{=} Poi(\lambda t)$ for all $s, t \geq 0$:

$$\mathbb{P}(N((s,t]) = n) \;=\; e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0.$$

If $\lambda = 1$, then $(N(t))_{t\geq 0}$ is also called *standard Poisson process*.

Definition 1.2. [By Infinitesimal Description]. A counting process $(N(t))_{t\geq 0}$ is said to be a Poisson process with rate λ, $\lambda > 0$, if:

(PP1) $N(0) = 0$.
(PP4) The process has stationary and independent increments.
(PP5) $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$.
(PP6) $\mathbb{P}(N(h) \geq 2) = o(h)$.

Only a reference value

Markovianity + time tranlaltion invariance

Two events are not too close to eaxh other

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $o(h)$ (for $h \to 0$), if

$$\lim_{h\to 0} \frac{f(h)}{h} \;=\; 0.$$

Consider then what happens, at the density matrix level in an infinitesimal amount of time:

$$\rho(t+dt) = (1-\lambda dt)\left(\rho(t) - \frac{i}{\hbar}[H,\rho(t)]\right) + \lambda dt T[\rho(t)]$$

No event → Schrödinger evolution

Event → collapse term

i.e.

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H,\rho(t)] - \lambda\left(\rho(t) - T[\rho(t)]\right)$$

which is a special type of Lindblad equation, with:

$$T[\rho(t)] = \sum_i \gamma_i L_i \rho(t) L_i^\dagger, \qquad \sum_i \gamma_i L_i^\dagger L_i = 1$$

Going back to the wave function, the nonlinear process cannot be deterministic, otherwise the nonlinearity does not cancel at the density matrix level. So it has to be stochastic. Suppose that an initial state $|\psi\rangle$ is transformed into:

$$|\psi\rangle \;\rightarrow\; |F_k(\psi)\rangle, \text{ with probability } p_k$$

with $|F_k(\psi)\rangle$ normalized state vector. Then it must be:

$$\sum_k p_k |F_k(\psi)\rangle\langle F_k(\psi)| = \sum_i L_i |\psi\rangle\langle\psi| L_i^\dagger$$

we have absorbed the coefficients γ_l into the Lindblad operators L_i. The right hand and hand side of the equation represent two equivalent unravelings and by theorem 2.6 page 103 on Nielsen-Chuang

$$\sqrt{p_k}|F_k(\psi)\rangle = \underbrace{\sum_i u_{ki} L_i|\psi\rangle}_{\text{Theorem 2.6}} = \underbrace{\tilde{L}_k|\psi\rangle}_{\text{New Lindblad term - unitary freedom of the Lindblad equation}}$$

Then, by dropping the tilde:

$$|F_k(\psi)\rangle = \frac{L_k|\psi\rangle}{\sqrt{p_k}}$$

and since the left hand side is normalized, we arrive at the final result:

$$|\psi\rangle \;\rightarrow\; \frac{L_k|\psi\rangle}{\|L_k|\psi\rangle\|}, \quad \text{with probability } p_k = \|L_k|\psi\rangle\|^2$$

together with the constraint $\sum_k L_k^\dagger L_k = 1$

which also ensures that the probabilities p_k sum to 1.

This is the **general form of a collapse model, based on a discrete Poisson process**. The collapse model then reads:

1. A wave function is associated to a physical system, as in standard QM.
2. The wave function evolves according to the Schrödinger equation, except that at random times it is subject to a random collapse as described above.

For different systems, collapses occur independently (minimal choice). As for any theory, collapse models declare what there is, and how it changes in time. Everything lese follows from here. Note that a new parameter has been introduced, the collapse rate λ. The operators L_i identify the model; there can be different choices.

The GRW model

Since the problem with Quantum Mechanics is why we do not see objects delocalized in space, the assumption is that collapses occur in space. Therefore the discrete index k becomes a continuous index **x**, the point in space where the collapse occurs

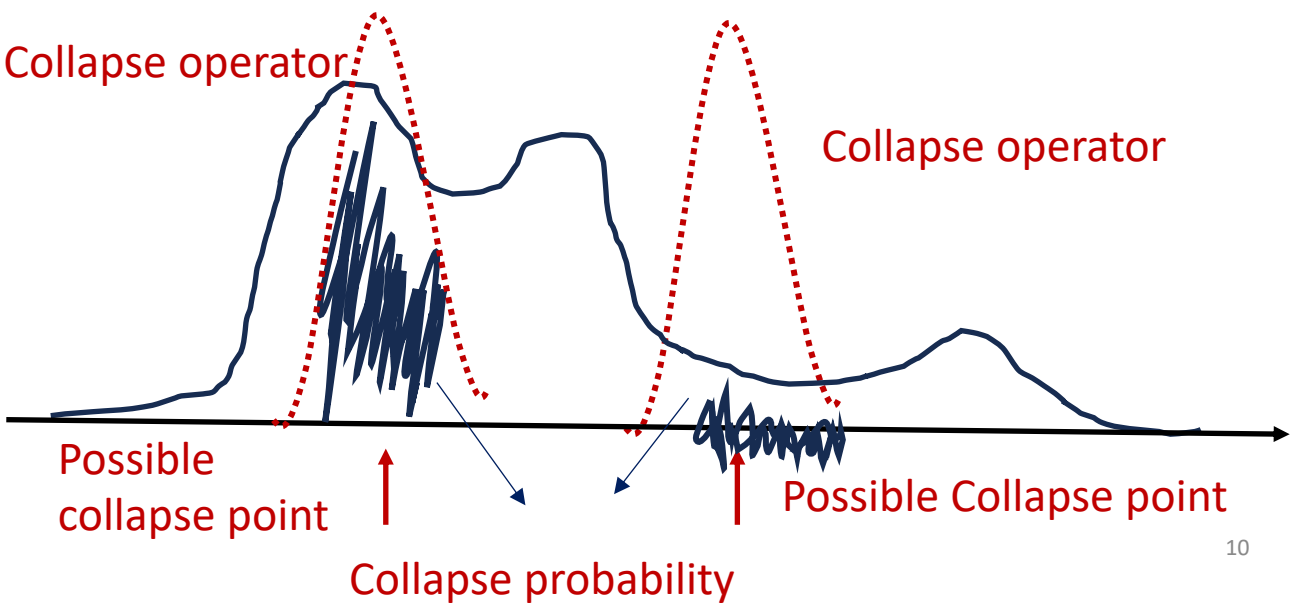
Given a system on N particles, the collapse operator associated to the i-th particle is taken equal to:

$$L^{(i)}(\mathbf{x}) = \frac{1}{(\pi r_{\mathrm{C}})^{3/4}} \exp\left[\frac{(\mathbf{q}_i - \mathbf{x})^2}{2r_{\mathrm{C}}^2}\right]$$

where $\mathbf{q}_i$ is the position operator of particle i. In the position representation, the action of the operator amounts to multiplying the wave function by a Gaussian.

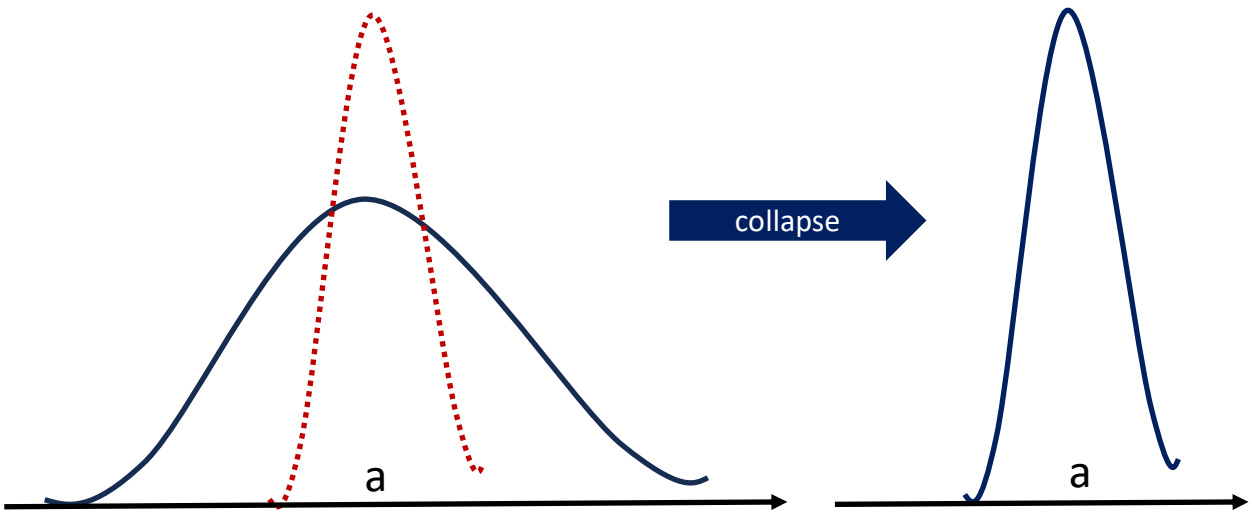
A second parameter appears, the size r_c of the Gaussian function; in total, we have two new parameters. The rest of this section is meant to explain how the new dynamics works.

Example 1. Collapse probability. Consider a particle in 1 dimension, whose state is delocalized in space



The collapse is more likely to occur where the the wave function is more appreciably different from 0. This is the ultimate reason for the recovering of the Born rule, which we haven't talked about, so far.

Example 2. Large Gaussian. Consider a particle in a Gaussian state, centered in x = a with width $r \gg r_c$.



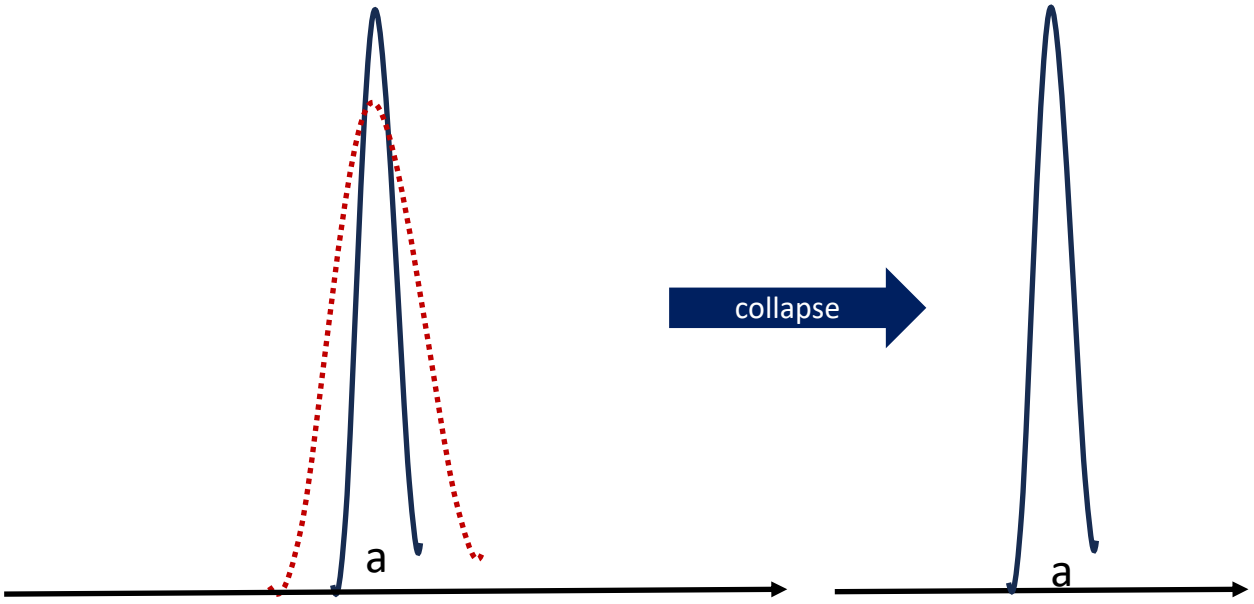
The most probable collapse will occur around x = a. let as assume for simplicity that it occurs exactly in a.

$$\psi(x) = \frac{1}{\sqrt[4]{\pi r}} e^{\frac{(x-a)^2}{2r^2}} \quad \rightarrow \quad \frac{1}{N} e^{\frac{(x-a)^2}{2r_c^2}} e^{\frac{(x-a)^2}{2r^2}}$$

$$\simeq \quad \frac{1}{\sqrt[4]{\pi r_c}} e^{\frac{(x-a)^2}{2r_c^2}}$$

The initially spread out wave function has been localized in space, with resolution equal to r_c.

Example 3. Narrow gaussian. Consider a particle still in a Gaussian state, centered in x = a, with width $r \ll r_c$



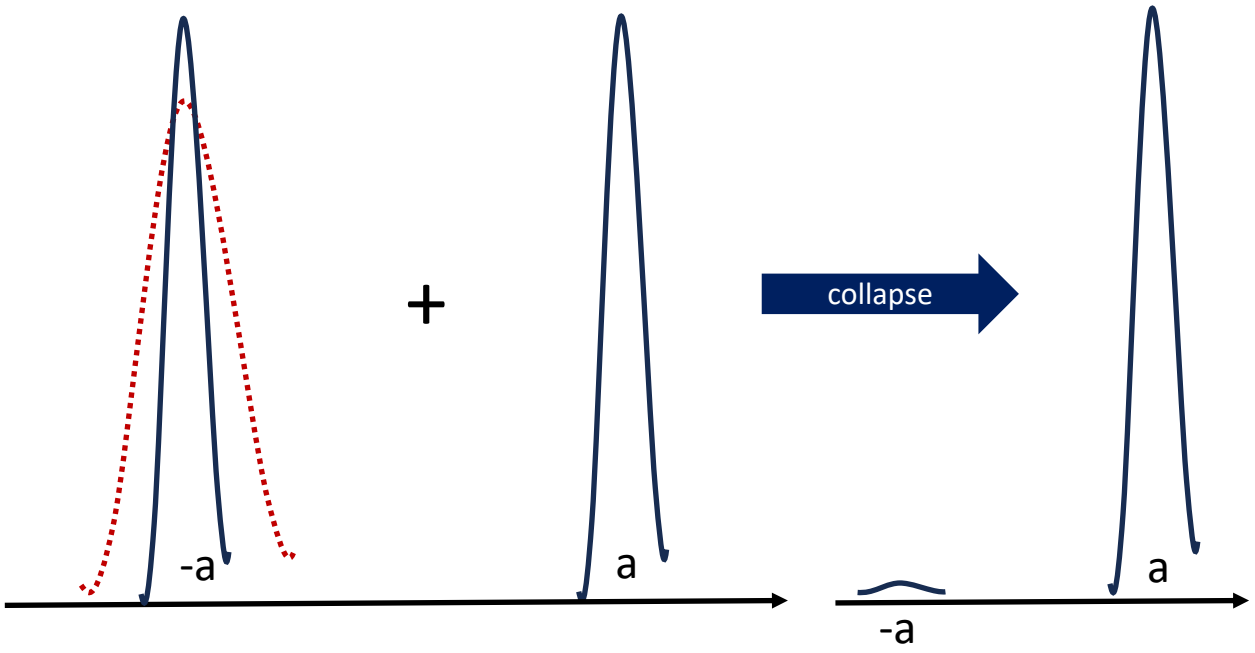
Now we have

$$\psi(x) = \frac{1}{\sqrt[4]{\pi r}} e^{-\frac{(x-a)^2}{2r^2}} \quad \rightarrow \quad \frac{1}{N} e^{-\frac{(x-a)^2}{2r_c^2}} e^{-\frac{(x-a)^2}{2r^2}}$$

$$\simeq \quad \frac{1}{\sqrt[4]{\pi r_c}} e^{-\frac{(x-a)^2}{2r}}$$

The state hasn't changed much.

Example 4. Now consider a particle whose state is the superposition of two Gaussian states, one centered in x = a and the other in x = -a, with $2a \gg r_c$, each having spread $r \ll r_c$.



The collapse has roughly ½ probability to occur on the left at x = –a, and ½ probability to occur on the right at x = a. Suppose it occurs no the right. Then:

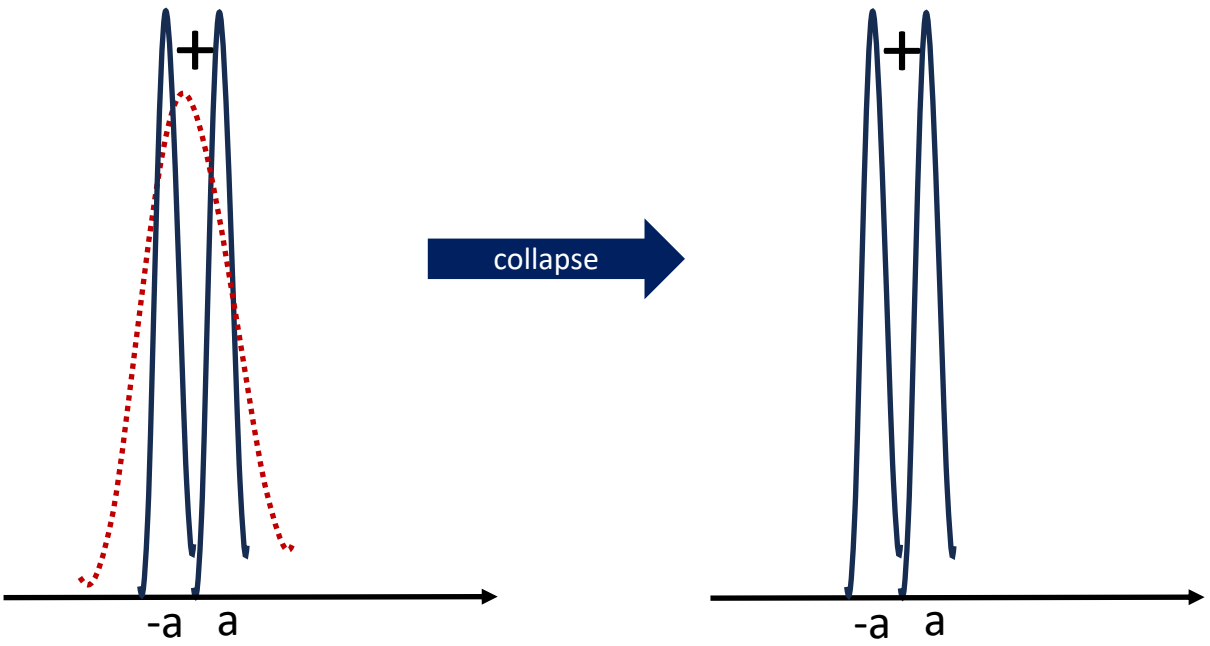
$$\psi(x) = \frac{1}{N}\left[e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right] \quad \rightarrow \quad \frac{1}{N'}e^{-\frac{(x-a)^2}{2r_c^2}}\left[e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right]$$

$$\simeq \quad \frac{1}{N'}\left[e^{-\frac{2a^2}{2r_c^2}}e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right]$$

$\simeq 0$

Where only the approximation $r \ll r_c$ has been used. The Gaussian on the right has been exponentially suppressed, and is essentially pert of the tail of the Gaussian on the right.

Example 5. Now consider a particle whose state is the superposition of two Gaussian states, one centered in x = a and the other in x = -a, with $2a \ll r_c$, each having spread $r \ll r_c$.



As before, the collapse is more likely to occur around x = -a or x = a. Suppose it occurs at x =a. The calculation is the same as before, since only the approximation $r \ll r_c$ has been used :

$$\psi(x) = \frac{1}{N}\left[e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right] \quad \rightarrow \quad \frac{1}{N'}e^{-\frac{(x-a)^2}{2r_c^2}}\left[e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right]$$

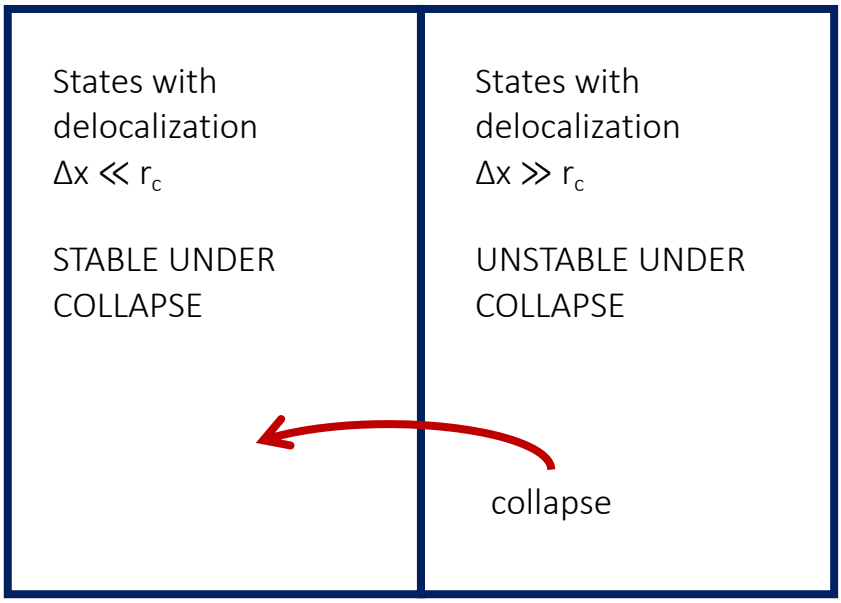
$$\simeq \quad \frac{1}{N'}\left[e^{-\frac{2a^2}{2r_c^2}} e^{-\frac{(x+a)^2}{2r^2}} + e^{-\frac{(x-a)^2}{2r^2}}\right]$$

$\simeq 1$

Now however the Gaussian on the left is not exponentially suppressed: both terms of the superposition survive.

The conclusion is that not all superposition states collapse; only those with spread larger than r_c do collapse. The other ones are not appreciably affected (of course, they also change a bit, but not much). This is the physical meaning of the parameter r_c.

We can summarize the situation in the following way



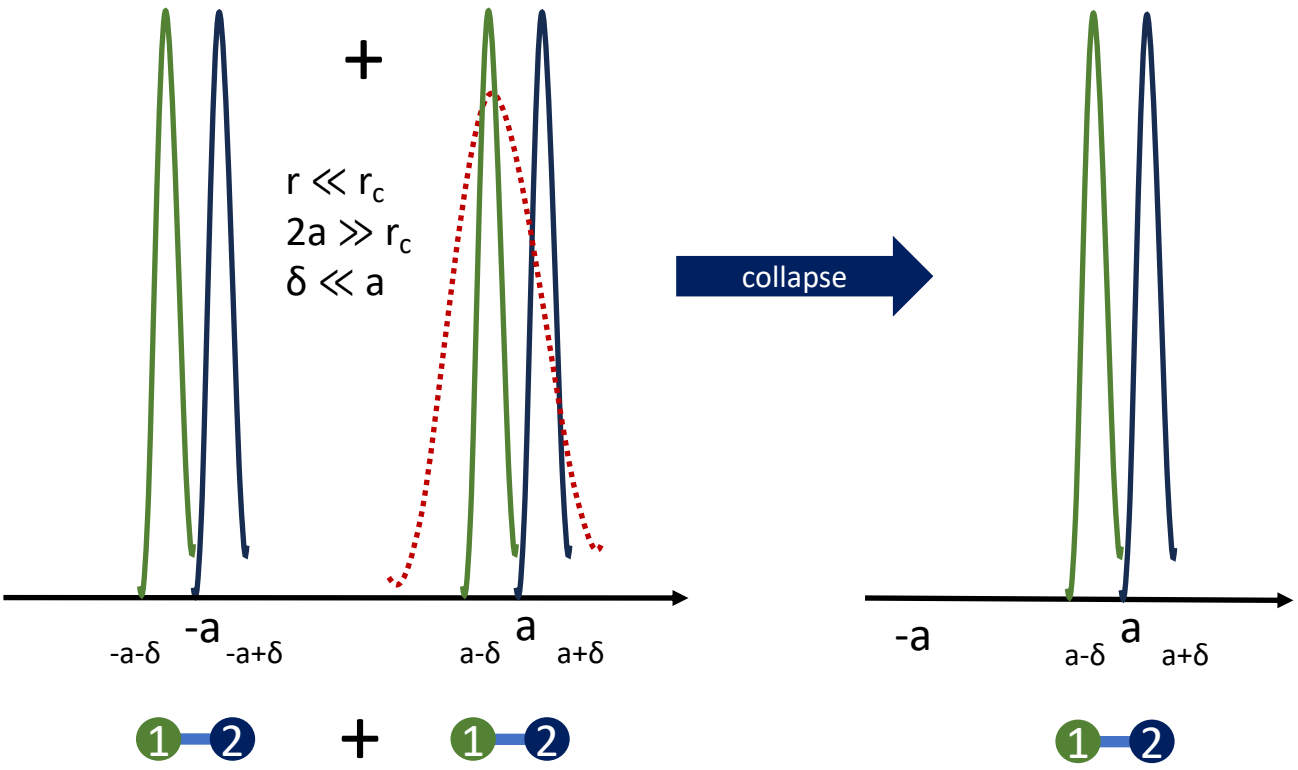
Hilbert space of the system

These considerations suggest the numerical value one should attribute to r_c: a mesoscopic distance, which kills macroscopic superpositions ($\Delta x \gg r_c$) but preserves microscopic ones ($\Delta x \ll r_c$), which have been tested experimentally with success. The number suggested in the literature is:

$$r_c \simeq 10^{-5} \text{ cm}$$

Amplification mechanism

This is a crucial property of collapse models. So far we considered only what happens for one particle. Suppose we have **two particles** (the generalization to an arbitrary number of particles tis straightforward), which form a bounded system, like a diatomic molecule, an example of a **rigid object**. The composite system is in a superposition state as in the picture



The initial state of the two-particle system is:

$$\psi(x,y) = \frac{1}{N}\left[e^{-\frac{(x+a+\delta)^2}{2r^2}} e^{-\frac{(y+a-\delta)^2}{2r^2}} + e^{-\frac{(x-a+\delta)^2}{2r^2}} e^{-\frac{(y-a-\delta)^2}{2r^2}}\right]$$

The collapse processes for the two particles are independent. Suppose that particle 1 (with coordinate x) suffers a collapse around x = a – δ. From what seen before, this occurs with probability about ½.

The wave function changes into:

$$\psi(x,y) \to \simeq \frac{1}{N'}\left[e^{-\frac{2(a-\delta)^2}{2r_c^2}} e^{-\frac{(x+a+\delta)^2}{2r^2}} e^{-\frac{(y+a-\delta)^2}{2r^2}} + e^{-\frac{(x-a+\delta)^2}{2r^2}} e^{-\frac{(y-a-\delta)^2}{2r^2}}\right]$$

$\simeq 0$

The first term of the superposition is exponentially suppressed. This means that the **collapse for one particle has killed the superposition of the entire system**.

This is the **amplification mechanism**: when we have a composite system, like a **macroscopic object**, in a macroscopic superposition ($\Delta x \gg r_c$) "here" plus "there" (what is called a Schrödinger's cat state), then **the collapse of one of its constituents cause the collapses of the entire wave function.** Given that the collapse process for each constituent are taken to be independent, this means that:

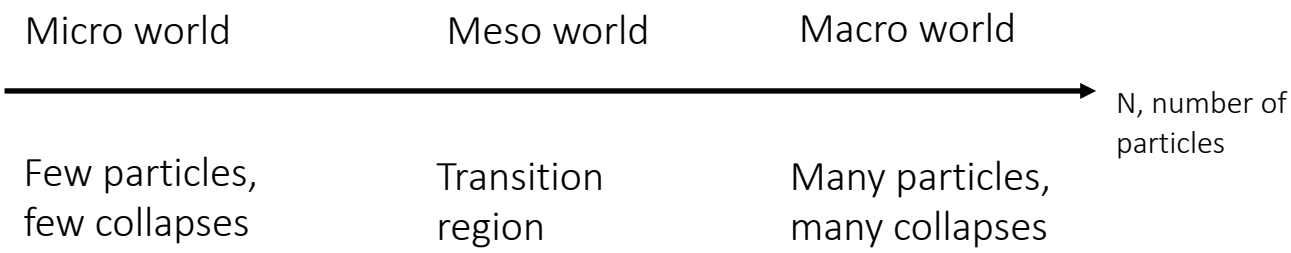
- Collapse rate for one particle: λ
- Effective collapse rate for and N-particle object: $N\lambda$

This allows to choose λ such that, for microscopic systems the collapses are rare and almost negligible for all practical purposes, while for macroscopic objects the amplification mechanism makes sure that their wave function is very rapidly localized. The original value suggested was

$$\lambda \simeq 10^{-16}\ s^{-1}$$

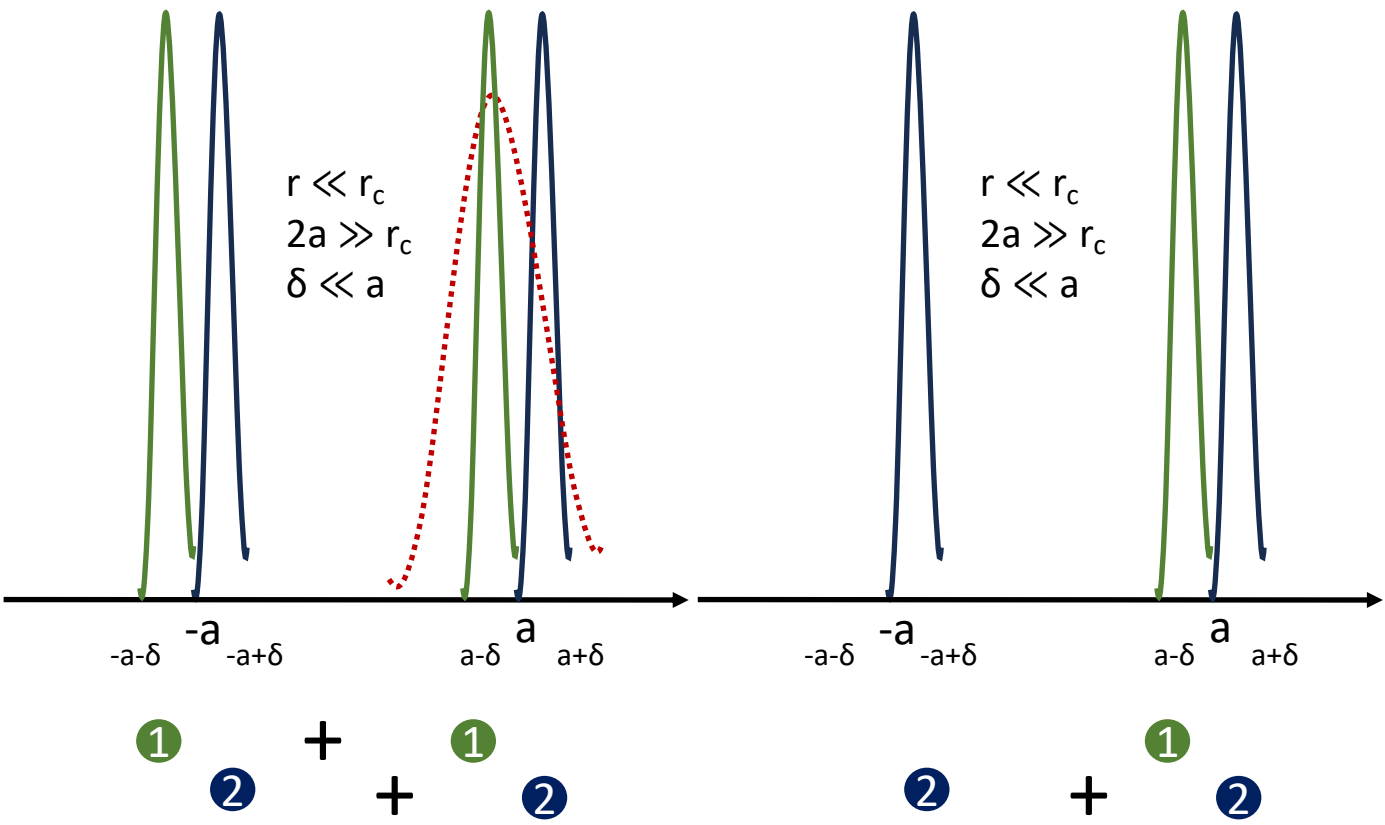
For example, for a macroscopic object with $N \simeq 20^{24}$ (Avogadro's number) particles, we have $\Lambda_{MACRO} = N\lambda \simeq 10^{24} \times 10^{-16}\ s^{-1} = 10^{8}\ s^{-1}$. This means that once every 10^{-8} s (almost immediately) there occurs a collapse somewhere in the object, which kills the macroscopic superposition, if present: macroscopic objects are always well localized in space.

A way to represent this is:



Yet, not always a system with a large number of particles is so heavily affected by the collapse process. Consider again the two particle system, now mimicking an **ideal gas** rather than a rigid object: the two particles are independent from each other, which is represented by a factorized wave function:

$$\psi(x,y) = \frac{1}{N}\left[e^{-\frac{(x+a+\delta)^2}{2r^2}} + e^{-\frac{(x-a+\delta)^2}{2r^2}}\right] \otimes \left[e^{-\frac{(y+a-\delta)^2}{2r^2}} + e^{-\frac{(y-a-\delta)^2}{2r^2}}\right]$$



Suppose again that particle 1 (with coordinate x) suffers a collapse around x = a – δ; once again, this occurs with probability about ½. The wave function changes to:

$$\psi(x,y) \to \simeq \frac{1}{N}\left[e^{-\frac{2(a-\delta)^2}{2r_c^2}} e^{-\frac{(x+a+\delta)^2}{2r^2}} + e^{-\frac{(x-a+\delta)^2}{2r^2}}\right] \otimes \left[e^{-\frac{(y+a-\delta)^2}{2r^2}} + e^{-\frac{(y-a-\delta)^2}{2r^2}}\right]$$

$\simeq 0$

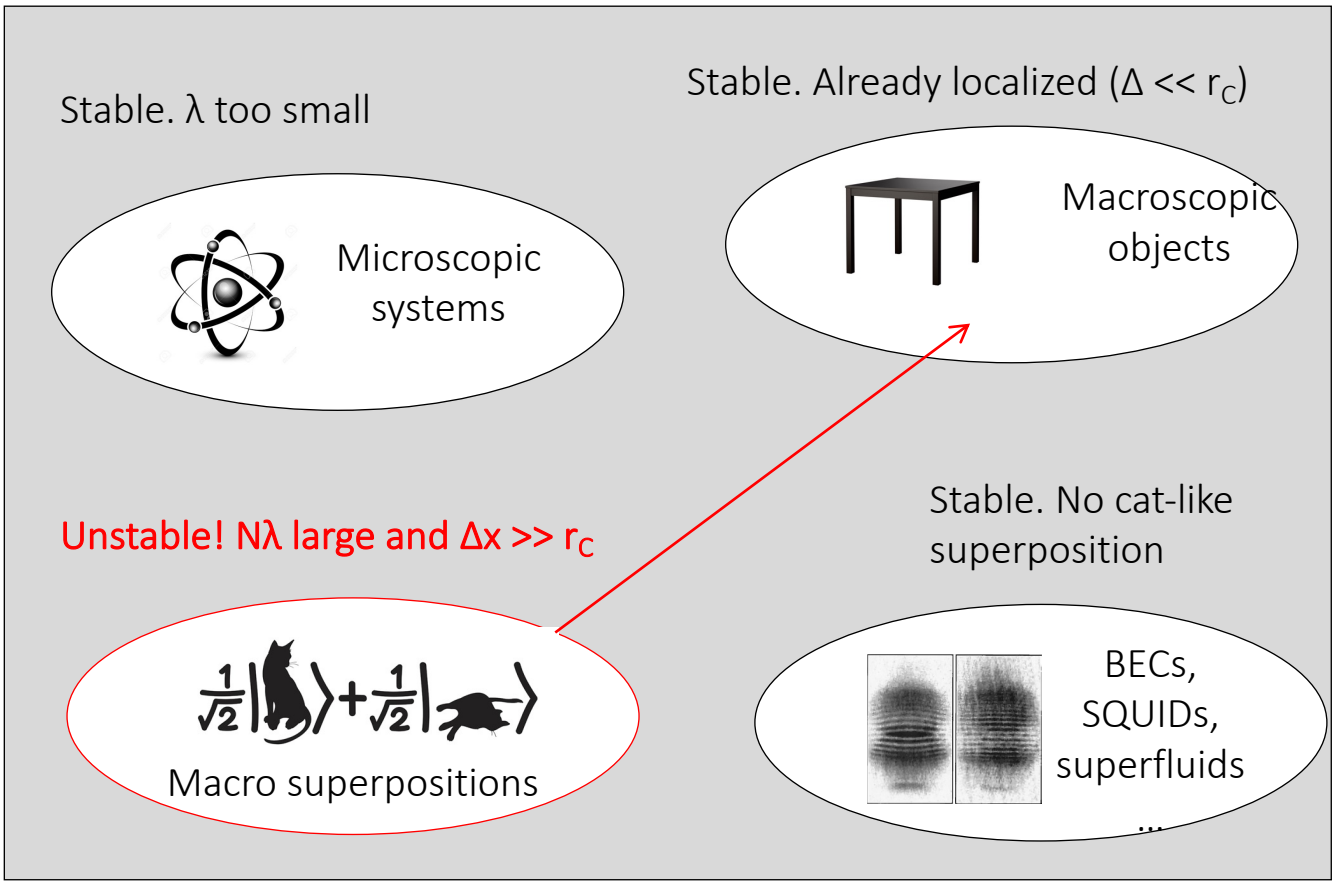
The collapse has localized the state of particle 1, but not that of particle 2: there is **no amplification mechanism**.

This shows that the amplification mechanism occurs only when the **multi-particle wave function is in a specific state, an entangled state of a significant number of particles composing the system**. In all other cases, the collapse remains ineffective.

This is important, because there is an increasing number of experimental results with cold atoms, superconductivity, superfluidity, where a larger and larger number of particles are set in a collective quantum state. All these cases turn out to be sufficiently stable against the collapse, because none of them is a Schrödinger's cat state of the form "here" + "there".

Ironically, entanglement, which is considered the characteristic feature of QM, is what is needed to recover classicality.

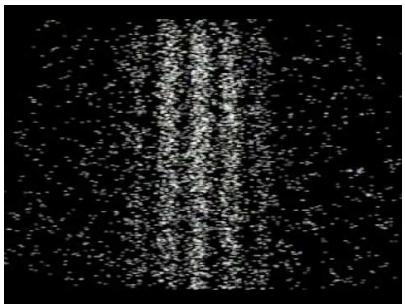
The overall picture, which emerges from collapse models, is the following



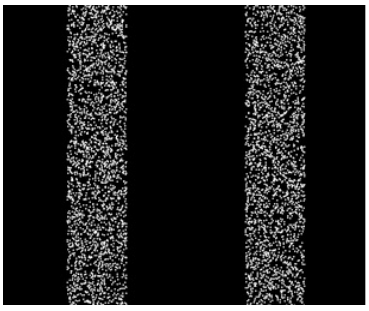
Testing collapse models

By changing the dynamics, **collapse models make predictions, which differ from standard quantum mechanical predictions**.

The direct way to test these models is via interferometric experiments: one takes a system as massive as possible, in order to trigger the amplification mechanism, and creates a superposition state, with Δx as large as possible, keeping at as long as possible.



Prediction of
quantum mechanics
(no environmental noise)



Prediction of
collapse models
(no environmental noise)

If one detects quantum coherence, than QM is right and Collapse models are wrong; if on the other hand there is a loos of coherence even if environmental noises are kept low, than one has to wonder what is happening in the experiment.

Such experiment are difficult to perform: it is difficult to create large mass macroscopic superpositions, for a variety of reasons.

The alternative way is to test them non-interferometrically. Such kind of experiments are based on the following property of collapse models.

Consider again example 3 of one particle in a narrow Gaussian state. Before we said that the collapse is practically ineffective. There we assumed that the collapse occurs exactly at x=a, which is the point with the highest probability density. But also nearby point have a similar, though slightly smaller, probability density to be the collapse point.

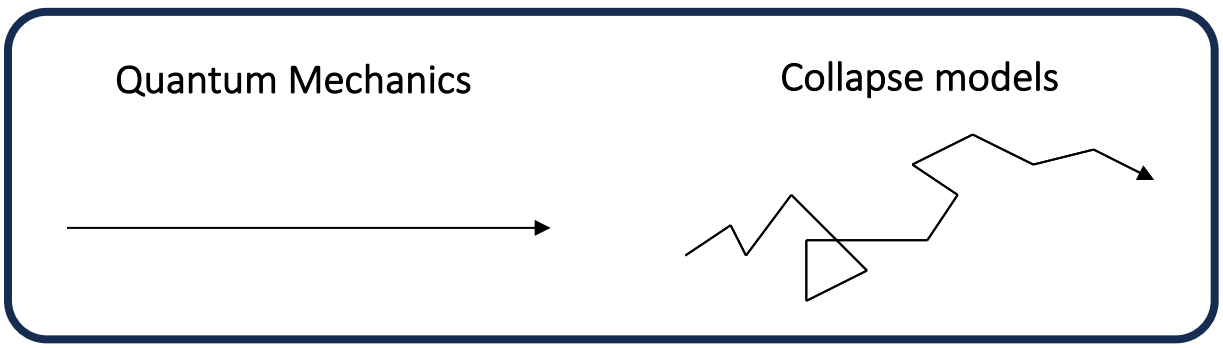
Suppose then that the collapse occurs near a, as a+δ. Then, under the usual approximation $r \ll r_C$, we have:

$$\psi(x) = \frac{1}{\sqrt[4]{\pi r}} e^{-\frac{(x-a-\delta)^2}{2r^2}} \quad \rightarrow \quad \frac{1}{N} e^{-\frac{(x-a)^2}{2r_c^2}} e^{-\frac{(x-a)^2}{2r^2}}$$

$$\simeq \quad \frac{1}{\sqrt[4]{\pi r_c}} e^{-\frac{(x-a-\delta')^2}{2r}}, \qquad \delta' = \frac{r^2}{r^2 + r_c^2}\delta$$

The shape an size of the wave function hasn't changed significantly, but now the center has slightly moved, in this case to the right: **its mean position has shifted**.

Since the probability that the collapse occurs exactly in x = a is zero, then every collapse causes a (slight) shift of the particle's position → random motion



The important point is that, while the collapse of the wave function is appreciable only when $\Delta x \gg r_C$, the random motion occurs also when the wave function is well localized in space. Therefore **there is no need to create quantum superpositions in order to observe this effect**. This experimental advantage can be used to test these models, via high-precision position measurements, of microscopic or macroscopic systems.

Note that this effect is an unavoidable feature of any model which collapses the wave function in space. It is another way of saying that the collapse, being nonlinear, requires random terms in order to generate a linear evolution at the density matrix level, in order to avoid superluminal signaling.

Density matrix

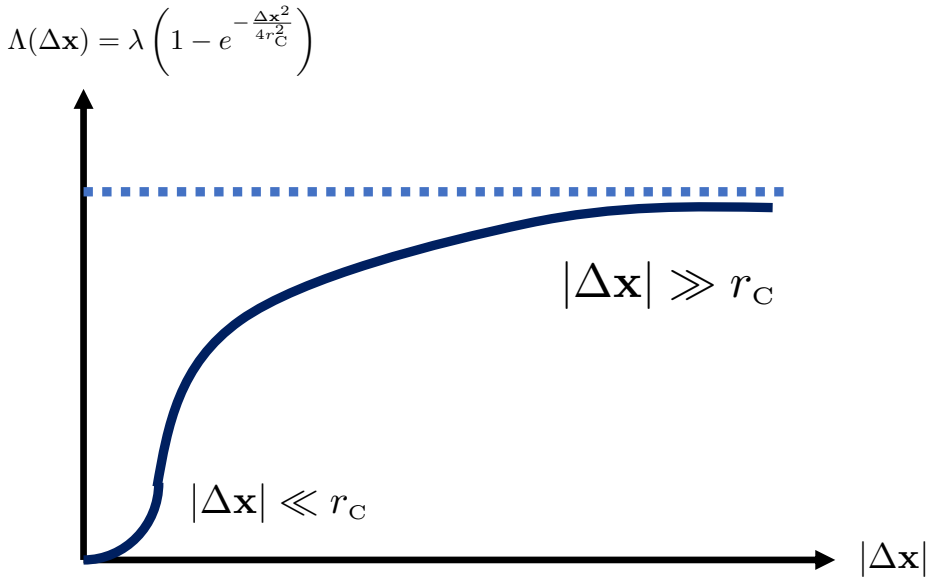
From the previous analysis, the one particle Lindblad equation for the GRW model is:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H,\rho(t)] - \lambda\left(\rho(t) - T[\rho(t)]\right)$$
$$T[\rho(t)] = \int d^3x L(\mathbf{x})\rho(t)L(\mathbf{x})$$

In the position representation is reads

$$\frac{d}{dt}\rho(\mathbf{x},\mathbf{y},t) = -\frac{i}{\hbar}[H,\rho(\mathbf{x},\mathbf{y},t)] - \lambda\left(1 - e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4r_C^2}}\right)\rho(\mathbf{x},\mathbf{y},t)$$

Note that the Lindblad terms determine the decay of the off-diagonal elements of the density matrix (in position) es an effect of the collapse. Their strength is:



- Quadratic increase for small Δx: good for experiments
- Saturation for $|\Delta x| \gg r_c$: it does not help experimentally to create too delocalized states, much larger than r_c.

The density matrix can also be written in the form

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] - \frac{\lambda}{2}[L(\mathbf{x}), [L(\mathbf{x}), \rho(t)]]$$

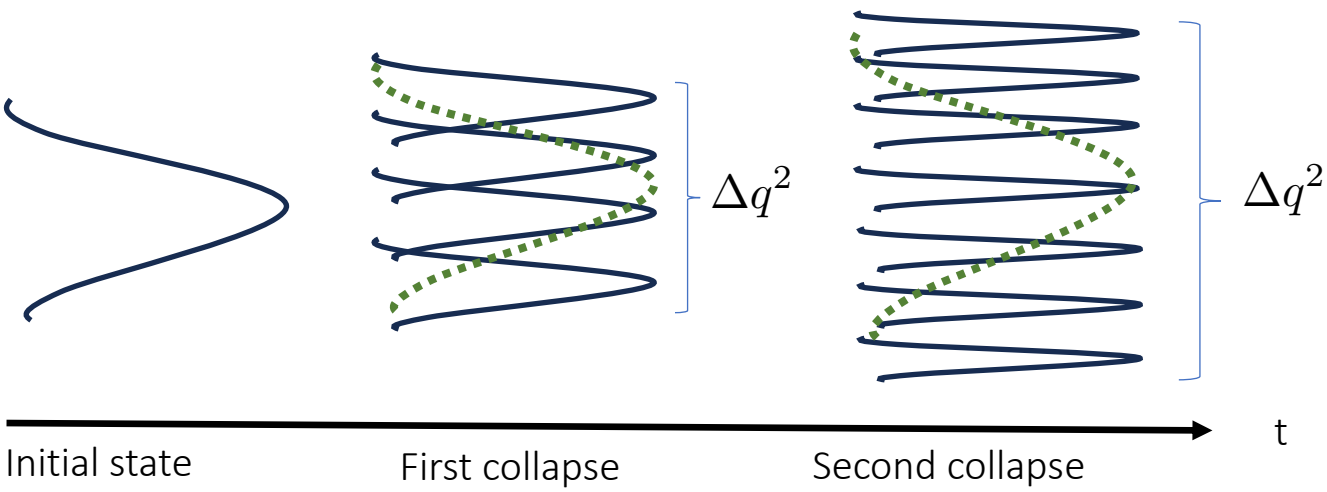
Remembering that L(**x**) is function of the position operator **q**, and considering the case of a **free particle**, one can quite easily derive the following results for the average position and momentum;

$$\begin{aligned} \langle \mathbf{q} \rangle_t &= \langle \mathbf{q} \rangle_t^{\mathrm{SCH}} \\ \langle \mathbf{p} \rangle_t &= \langle \mathbf{p} \rangle_t^{\mathrm{SCH}} \end{aligned}$$

where "SCH" refers to the pure Schrödinger evolution, without the collapse. On the average, the (free) particle moves like in standard quantum mechanics. For the variances we have:

$$\begin{aligned} \Delta q_t^2 = \langle (\mathbf{q} - \langle \mathbf{q} \rangle)^2 \rangle_t &= \Delta q_t^{2(\mathrm{SCH})} + \frac{\lambda \hbar^2}{6 r_{\mathrm{C}}^2 m^2} t^3 \\ \Delta p_t^2 = \langle (\mathbf{p} - \langle \mathbf{p} \rangle)^2 \rangle_t &= \Delta p_t^{2(\mathrm{SCH})} + \frac{\lambda \hbar^2}{2 r_{\mathrm{C}}^2} t \end{aligned}$$

Both the variance in position as well as in momentum increase over time; this is an effect of the underlying random process. The physical picture is the following:



Green = pure Schrödinger spread

In particular, for the kinetic energy we have:

$$\langle E\rangle_t = \langle E\rangle_t^{\mathrm{SCH}} + \frac{\lambda\hbar^2}{4r_{\mathrm{C}}^2 m}t$$

There is a linear increase of the mean energy, sourced by the collapse process. It is a form of violation of energy conservation.

All these results are useful to quantify the predictions of collapse models to be tested experimentally.

For a N-particle system, being the collapse events related to each particle independent from each other, the Lindblad equation reads:

$$\frac{d}{dt}\rho(t) \quad = \quad -\frac{i}{\hbar}[H,\rho(t)] - \lambda\sum_{n=1}^{N}(\rho(t) - T_n[\rho(t)])$$

where for simplicity we assumed that the collapse rate λ is the same for all particles, and $T_n[.]$ is the collapse operator for the n-th particle.

By writing the explicit expression for $T_n[.]$, moving to the center-of-mass (Q) and internal ($\{r_i\}$) coordinates, and taking the partial trace over the internal coordinates, one finds that

$$\mathrm{Tr}^{(\{r_i\})}(T_n[\rho]) = T_Q[\mathrm{Tr}^{(\{r_i\})}(\rho)] = T_Q[\rho_Q], \qquad \rho_Q \equiv \mathrm{Tr}^{(\{r_i\})}(\rho)$$

where ρ_Q is the (reduced) density matrix for the center of mass.

To discuss this point, let us introduce the center of mass and relative motion position operators Q and r_j $(j = 1, 2, \ldots, N-1)$, related to the operators q_i by

$$q_i = Q + \sum_{j=1}^{N-1} c_{ij} r_j \ . \tag{6.41}$$

Eq. (6.39), when the Hamiltonian H can be split into the sum of the center of mass and internal motion parts H_Q and H_r acting in the respective state spaces, reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{\mathrm{i}}{\hbar}[H_Q, \rho(t)] - \frac{\mathrm{i}}{\hbar}[H_r, \rho(t)] - \sum_i \lambda_i(\rho(t) - T_i[\rho(t)]) \ , \tag{6.42}$$

where the operator $T_i[\rho]$ can now be written as

$$T_i[\rho] = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-(\alpha/2)[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x]^2} \rho \mathrm{e}^{-(\alpha/2)[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x]^2} \ . \tag{6.43}$$

The dynamical evolution of the *center of mass* of the system is described by the statistical operator

$$\rho_Q = \mathrm{Tr}^{(r)}[\rho] \ , \tag{6.44}$$

obtained by taking the partial trace on the internal degrees of freedom of the statistical operator ρ for the complete N-particle system. Taking the r trace of the operation $T_i[\rho]$ one gets

$$\int \mathrm{d}r_1 \ldots \mathrm{d}r_{N-1} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-(\alpha/2)[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x]^2}$$

$$\cdot \langle r_1 \ldots r_{N-1} | \rho | r_1 \ldots r_{N-1} \rangle \mathrm{e}^{-(\alpha/2)[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x]^2} \ , \tag{6.45}$$

and integrating over x

so that, by shifting the integration variable x by the amount $\sum_j c_{ij} r_j$, one finds

$$\mathrm{Tr}^{(r)}(T_i[\rho]) = T_Q[\mathrm{Tr}^{(r)}(\rho)] \ , \tag{6.46}$$

where

$$T_Q[\,\cdot\,] = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-(\alpha/2)(Q-x)^2} [\,\cdot\,] \mathrm{e}^{-(\alpha/2)(Q-x)^2} \ . \tag{6.47}$$

If one takes the r trace of Eq. (6.42) one then gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_Q(t) = -\frac{\mathrm{i}}{\hbar}[H_Q, \rho_Q] - \sum_i \lambda_i(\rho_Q - T[\rho_Q]) \ . \tag{6.48}$$

Assuming also that the total Hamiltonian splits into a term H_Q for the center of mass, plus a term $H_{\{ri\}}$ for the internal degrees of freedom, we arrive at the following equation for the center of mass coordinate

$$\frac{d}{dt}\rho_Q(t) = -\frac{i}{\hbar}[H_Q, \rho_Q(t)] - N\lambda(\rho_Q(t) - T_Q[\rho_Q(t)])$$

which is the equation for one particle, with the collapse rate amplified N times. It is the mathematical manifestation of the amplification mechanism.

The equation for the internal degrees of freedom depends very much on the internal structure. For a **crystalline structure**, typical of rigid objects, by working with the phonon formalism, one can show that phonon states are excited over time, since the systems warms up subject to the collapse.

Open questions:

- The origin of the collapse: is it an intrinsic feature of Nature, or is it caused by some agent? Which agent? Penrose suggests it might be gravity.
- What is the role of the wave function? It is not a field in 3D space, therefore it does not allow for a naïve interpretation as representing the stuff in 3D space.

FURTHER MATERIAL

- QUMPL and measurement process
- CLS equation
- DP model
- Experimental bounds