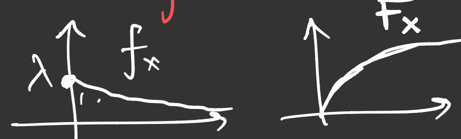
Lezione 14:

DENSITA' DI PROB. → f FUNZ. DI DISTRIB. → F

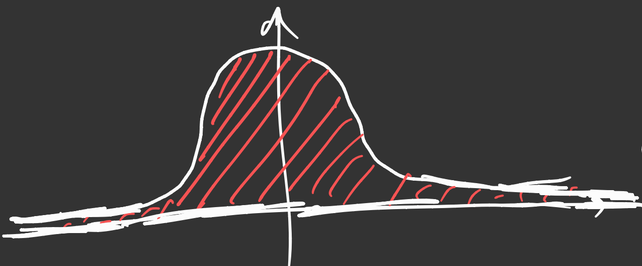
$$F = \int f, \quad f = \frac{dF}{dx}$$

	DISTRIBUZ.	X	$\mathbb{P}(X=k)$	Dimostraz.	f_X	F_X	$\mathbb{E}[X]$	$Var[X]$	σ_X
DISCRETE	BERNOULLI (n,p) o BINOMIALE	...	$\binom{n}{k} p^k (1-p)^{n-k}$	Thm Newton $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$					
	POISSON	il numero di							
CONTINUE	UNIFORME (a,b)				...	...			
	ESPONENZ. (λ)			$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	...	$\begin{cases} 1-e^{-\lambda t} & t>0 \\ 0 & t \le 0 \end{cases}$			
	NORMALE o GAUSSIANA				...	...			

aggiungete i grafici di f_X e di F_X



III.5. DISTRIBUZIONE NORMALE:

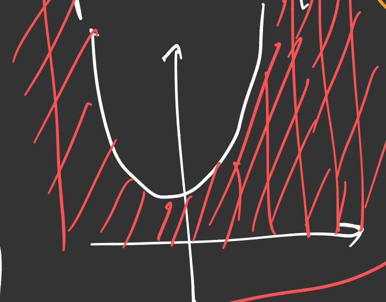


a quale funzione può corrispondere?

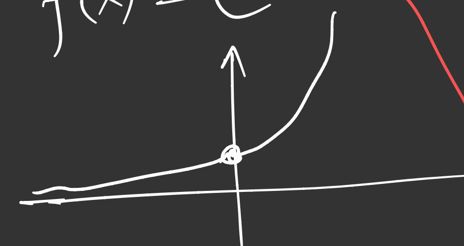
- asintoti
- minimi massimi
- scomporla
- derivata / integrale
- simmetria:
- funzioni elementari

$f(x)$ pari. $f(x) = f(-x)$

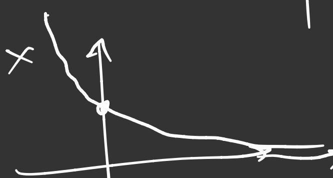
per esempio x^2



$f(x) = e^x$

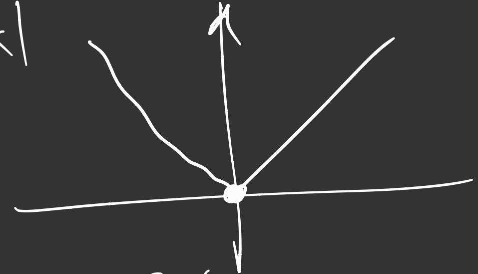


$f(x) = e^{-x}$

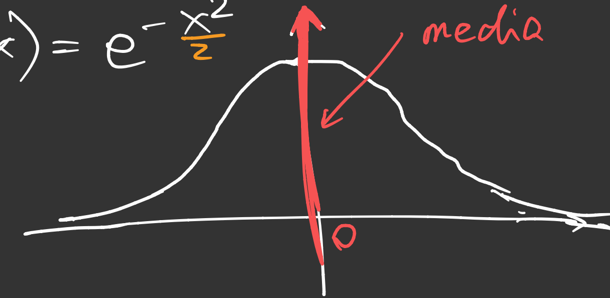


$e^{-x^2} \leftarrow \cancel{e^{-|x|}}$

$f(x) = |x|$



$f(x) = e^{-\frac{x^2}{2}}$



" $f^{\text{Normale}}(x) := e^{-\frac{x^2}{2}}$ " (quasi)

Dobbiamo controllare che sia una buona densità di probabilità, cioè che sia per esempio normalizzata.

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

vogliamo controllare questo.

In realtà: $\boxed{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}}$

$\boxed{\text{DEF}}$ (DENSITÀ DI PROBABILITÀ GAUSSIANA o NORMALE)

$$f_X^{\text{Gauss o normale}}(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F_X^{\text{Gauss}}(t) := \int_{-\infty}^{t} f_X^{\text{Gauss}}(x)\,dx$$

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x \cdot e^{-\frac{x^2}{2}}\,dx}_{= \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Bigg|_{-\infty}^{+\infty}$$

$$= 0 - 0 = 0 \implies \mathbb{E}[X] = 0.$$

$\left\{\begin{array}{l} \text{media} \\ \text{speranza matem.} \\ \text{valor medio} \end{array}\right.$

$$\mathrm{Var}[X] := \int_{-\infty}^{\infty} f_X(x)(x-0)^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot x^2 = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \underbrace{\left(-x e^{-\frac{x^2}{2}}\right)}_{\frac{d}{dx} e^{-\frac{x^2}{2}}} dx$$

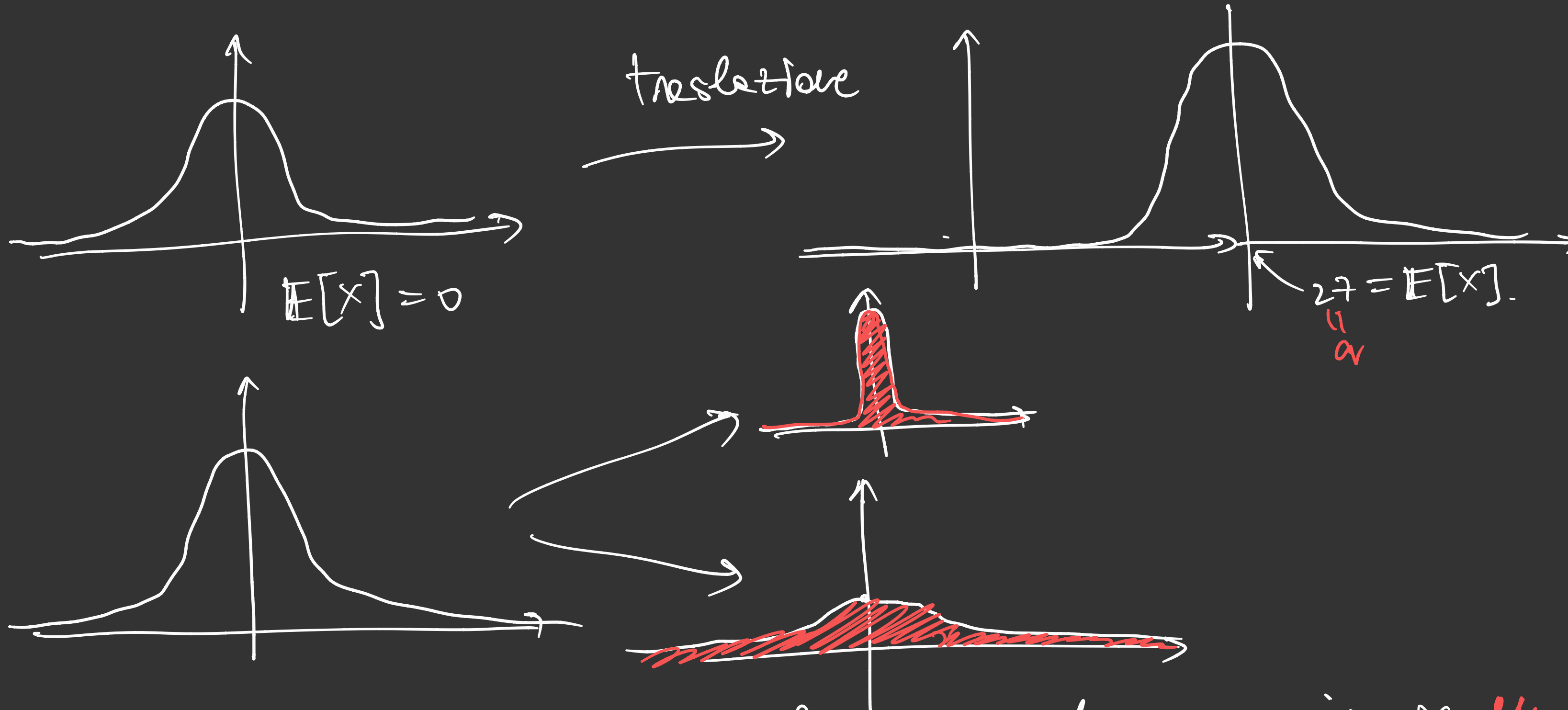
$$= -\frac{1}{\sqrt{2\pi}}\Big(\underbrace{x e^{-\frac{x^2}{2}}\Big|_{-\infty}^{+\infty}}_{=0-0=0} - \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx}_{=\sqrt{2\pi}}\Big)$$

<u>Memo</u>: $\int (fg)' = \int f'g + \int fg' \Rightarrow \int f\cdot g' = [fg]_\circ^\circ - \int f'g$

$$= (-)(-)\frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1 \Rightarrow Var[X] = 1 \Rightarrow \sigma_X = 1$$

f_X^{Gauss} ha media $= 0$ e $\begin{cases} \text{varianza} = 1. \\ \text{deviaz. standard} = 1. \end{cases}$

E se volessimo avere una media $\neq 0$?
e una deviazione standard $\neq 1$?



$f(x)$: traslare significa mandare x in $x - \mu$

modificare la pendenza " moltiplicare o dividere per un valore

$$x \longmapsto \frac{x}{\sigma}$$

Adesso diciamo che:

1) lo shift μ diventerà proprio la nuova media

2) il parametro σ " " " " deviaz. standard.

Dobbiamo controllare! La proposta corrente è

$$\boxed{f^{Gauss}_{X,\mu,\sigma}(x) := f^{Gauss}_X\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}}$$

cioè sospettiamo che:

1) $\int_{-\infty}^{\infty} x f^{Gauss}_{X,\mu,\sigma}(x)\, dx = \mu.$

2) $\int_{-\infty}^{\infty} (x-E[X])^2 f^{Gauss}_{X,\mu,\sigma}\, dx = \sigma^2.$

$dx = dy \cdot \sigma$

$dy = \frac{dx}{\sigma}$

$$1)\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sigma y+\mu)\,e^{-\frac{y^2}{2}}\,dy \qquad \boxed{y=\frac{x-\mu}{\sigma}}$$

$$= \frac{\sigma}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} y\, e^{-y^2/2}}_{=0} + \frac{\mu}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-y^2/2}\,dy}_{=\sqrt{2\pi}} = \mu\cdot\frac{\cancel{\sqrt{2\pi}}}{\cancel{\sqrt{2\pi}}} = \mu$$

$dx = dy\cdot\sigma$

$$\boxed{y=\frac{x-\mu}{\sigma}}$$

proprio come prima

$$2)\ \mathrm{Var}[X] = \int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{\sigma^2}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} y^2\cdot e^{-y^2/2}\,dy}_{\text{già calcolato prima!}}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\cdot\sqrt{2\pi} = \sigma^2$$

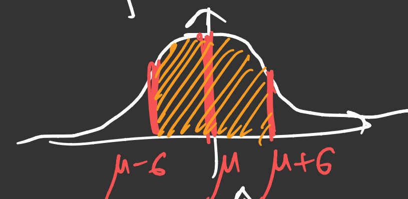
Allora la deviaz. standard $\sigma_X^{\text{Gauss}} := \sqrt{\mathrm{Var}[X]} = \sqrt{\sigma^2} = \sigma$

DEF X variabile aleatoria continua è NORMALE o GAUSSIANA di media μ e deviaz. standard σ

$$\Longleftrightarrow \quad \boxed{f_X = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}.$$

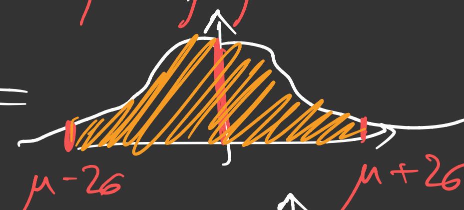
Ritroviamo gli intervalli di confidenza che avevamo già visto!

$F_X(\mu+\sigma) - F_X(\mu-\sigma) =$



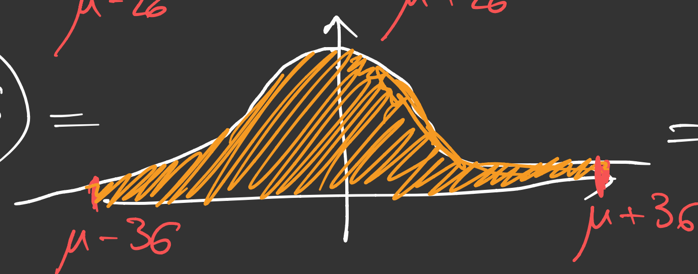
$= 68,27\%$

$F_X(\mu+2\sigma) - F_X(\mu-2\sigma) =$



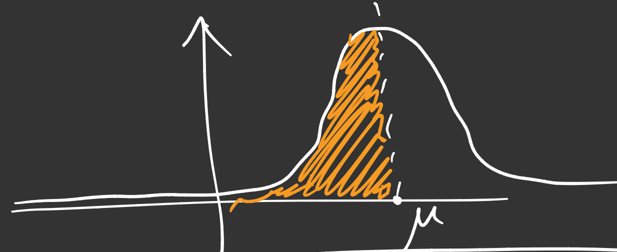
$= 95,45\%$

$F_X(\mu+3\sigma) - F_X(\mu-3\sigma) =$



$= 99,73\%$

$F_X(\mu) =$



$= 50\% = \frac{1}{2}$

Allora $\mu = F_X^{-1}\left(\frac{1}{2}\right)$! Quindi μ è anche la **MEDIANA**.

Idea del teorema del limite centrale:

$(X_i)_i$ = collezione infinita di variabili aleatorie
$X_1, X_2, X_3, \ldots$
limitate, indipendenti, di media $= \mu$
e di deviazione standard $= \sigma$.

$$\Rightarrow \frac{X_1 + X_2 + \ldots + X_n}{n} \xrightarrow[n \to \infty]{} X = \text{Gaussiana}(\mu, \sigma)$$