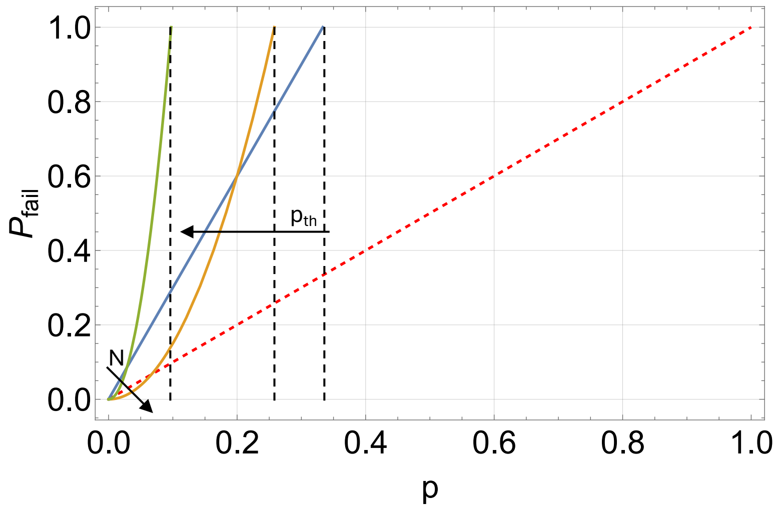


Fig. 7.6: Comparison of the failing probabilities $P_{\text{fail}}$ for different values $N$ of the qubits with a single layer of encoding. The arrow indicates the direction of increasing values of $N$, while the vertical black dashed lines indicate how the curves cannot define a threshold probability.

## 7.2 Stabiliser formalism

The stabiliser formalism exploits the fact that there exist operations, namely stabilisers, that can be used to detect errors without changing the state of the logical qubit. While the single stabiliser can only tell if there was an error, without establishing which error, a specific set of stabilisers can identify the specific errors that occurred, and thus providing the information to correct it.

### *7.2.1 Inverting quantum channels*

The following scheme summarises the QEC philosophy. Given a qubit, this is encoded in a logical qubit made of a set of physical qubits. The interaction with the surrounding environment (or other faulty components of the physical circuit) leads to errors in the state. The action of these errors can be described in terms of a CPTP map. The QEC code applies a recovery CPTP map, which — up to a certain probability — gives back the same initial state, as there were no errors, cf. Fig. 7.7.

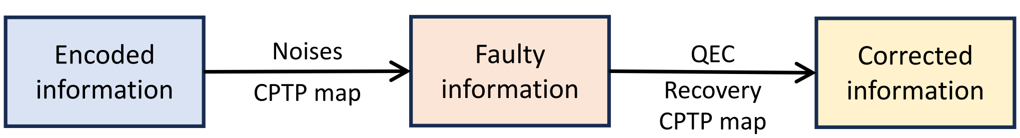


Fig. 7.7: Schematic representation of the QEC scheme, where a recovery CPTP map is applied to recover the information as there were no errors.

The question is: when is this possible?

Consider two CPTP maps $\mathcal{E}$ and $\mathcal{R}$, which describe respectively the occurrence of environmental errors and the recovery map. Their action is

$$\mathcal{B}(\mathbb{H}) \xrightarrow{\mathcal{E}} \mathcal{B}(\mathbb{H}') \xrightarrow{\mathcal{R}} \mathcal{B}(\mathbb{H}''), \tag{7.75}$$

where $\mathcal{B}(\mathbb{H})$ is the space of all the linear operators acting on $\mathbb{H}$. We want to have no effects of the errors, i.e. $\mathcal{R} \circ \mathcal{E} = \text{id}$, with $\mathbb{H} = \mathbb{H}' = \mathbb{H}''$.

Now, we show that the following two statements are equivalent:

1) Given a CPTP map $\mathcal{E}$, it exists another CPTP map $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E} = \text{id}$.
2) For any Kraus representation of $\mathcal{E}$, which is defined through the set of Kraus operators $\hat{E}_i$, then one has that

$$\hat{E}_i^\dagger \hat{E}_j = \mu_{ij} \hat{\mathbb{1}}, \tag{7.76}$$

where $\mu_{ij} \in \mathbb{C}$ are the coefficients of the density matrix of the environment imposing the map $\mathcal{E}$. Namely,

$$\hat{\mu} = \sum_{ij} \mu_{ij} \left| e_i \right\rangle \left\langle e_j \right| . \tag{7.77}$$

First, we prove that 1) implies 2). Consider $|\psi\rangle \in \mathbb{H}$, to which we apply the map $\mathcal{E}$, or better the unitary operation that acts as $\mathcal{E}$ on $\mathbb{H}$ alone. This gives

$$|\psi\rangle \xrightarrow{\mathcal{E}} \sum_j \hat{E}_j \left|\psi\right\rangle \left|e_j\right\rangle , \tag{7.78}$$

where $|e_j\rangle$ is the state of the environment with which the system is entangling. Now, we apply the map $\mathcal{R}$, or better its corresponding unitary:

$$\xrightarrow{\mathcal{R}} \sum_{jk} \hat{R}_k \hat{E}_j \left|\psi\right\rangle \left|e_j\right\rangle \left|a_k\right\rangle , \tag{7.79}$$

where $|a_k\rangle$ is the state of an ancilla, whose interaction defines the map $\mathcal{R}$. Since we want that the map $\mathcal{R}$ works as a recovery map for the map $\mathcal{E}$, we have to impose that

$$\sum_{jk} \hat{R}_k \hat{E}_j \left|\psi\right\rangle \left|e_j\right\rangle \left|a_k\right\rangle = |\psi\rangle \otimes (\dots), \tag{7.80}$$

where $(\dots)$ is a suitable state of the environment and the ancilla. In this way, the entanglement between the system and the environment is transferred to the environment and the ancilla, with no correlation to the state of the system. This is possible only if

$$\hat{R}_k \hat{E}_j = \alpha_{kj} \hat{\mathbb{1}}. \tag{7.81}$$

In such a case, one has that $(\dots) = \sum_{jk} \alpha_{kj} \left|e_j\right\rangle \left|a_k\right\rangle$ and

$$\begin{aligned} \hat{E}_i^\dagger \hat{E}_j &= \sum_k \hat{E}_i^\dagger \hat{R}_k^\dagger \hat{R}_k \hat{E}_j, \\ &= \sum_k \alpha_{ki}^* \alpha_{kj} \hat{\mathbb{1}}, \end{aligned} \tag{7.82}$$

where we used that

$$\sum_k \hat{R}_k^\dagger \hat{R}_k = \hat{\mathbb{1}}. \tag{7.83}$$

Then, we can define

$$\mu_{ij} = \sum_k \alpha_{ki}^* \alpha_{kj}. \tag{7.84}$$

Notably, we have that $\mu_{ji}^* = \mu_{ij}$, indeed

$$\mu_{ji}^* \hat{\mathbb{1}} = (\mu_{ij}\hat{\mathbb{1}})^\dagger = (\hat{E}_i^\dagger \hat{E}_j)^\dagger = \hat{E}_j^\dagger \hat{E}_i = \mu_{ij}\hat{\mathbb{1}}. \tag{7.85}$$

Moreover, we have that

$$\sum_i \mu_{ii}\hat{\mathbb{1}} = \sum_i \hat{E}_i^\dagger \hat{E}_i = \hat{\mathbb{1}}, \tag{7.86}$$

and that

$$\mu_{ii}\hat{\mathbb{1}} = \hat{E}_i^\dagger \hat{E}_i, \tag{7.87}$$

is a positive operator, which implies that $\mu_{ii} > 0$. Thus, $\{\mu_{ij}\}_{ij}$ have all the properties to be the coefficients of a density matrix, and this proves that 1) implies 2).

We now prove that 2) implies 1). We start from

$$\hat{E}_i^\dagger \hat{E}_j = \mu_{ij}\hat{\mathbb{1}}, \tag{7.88}$$

and we diagonalise $\mu_{ij}$. This implies a change the Kraus operators according to $\hat{E}_i \to \hat{F}_i$ such that

$$\hat{F}_i^\dagger \hat{F}_j = \delta_{ij} p_i \hat{\mathbb{1}}, \tag{7.89}$$

where $p_i > 0$ by definition. We then introduce the isometries $\hat{V}_i$ that are related to $\hat{F}_i$ via

$$\hat{F}_i = \sqrt{p_i}\hat{V}_i. \tag{7.90}$$

Then, as represented in Fig. 7.8, the map $\mathcal{E}$ is mapping $\mathbb{H}$ to different subspaces $\mathbb{H}_i'$ of $\mathbb{H}'$. Each of these

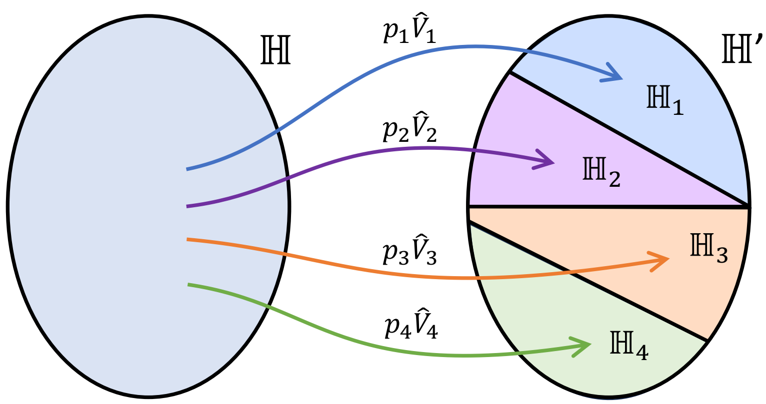


Fig. 7.8: Graphical representation of the mapping between $\mathbb{H}$ and $\mathbb{H}'$.

mapping is performed with a probability $p_i$ by the operator $\hat{V}_i \propto \hat{F}_i$. Notably, since we diagonalised $\mu_{ij}$, the different operators $\hat{V}_i$ are orthogonal, and thus are also the subspaces $\mathbb{H}_i'$. Now, given this set of isometries, we can construct the recovery Kraus operators as

$$\hat{R}_i = \hat{V}_i^\dagger, \tag{7.91}$$

which will act only on the corresponding subspace $\mathbb{H}_i'$, leaving the rest of Hilbert space $\mathbb{H}'$ untouched: indeed, for $i \neq j$ we have $\hat{V}_j^\dagger F_j \propto \hat{V}_j^\dagger V_j = 0$. Finally, we compose the maps $\mathcal{E}$ and $\mathcal{R}$:

$$\begin{aligned}\hat{\rho} \xrightarrow{\mathcal{E}} \mathcal{E}(\hat{\rho}) &= \sum_i p_i \hat{V}_i \hat{\rho} \hat{V}_i^\dagger, \\ \xrightarrow{\mathcal{R}} \mathcal{R}(\mathcal{E}(\hat{\rho})) &= \sum_j \hat{V}_j^\dagger \left( \sum_i p_i \hat{V}_i \hat{\rho} \hat{V}_i^\dagger \right) \hat{V}_j, \\ &= \sum_{ij} p_i \hat{V}_j^\dagger \hat{V}_i \hat{\rho} \hat{V}_i^\dagger \hat{V}_j, \\ &= \sum_i p_i \hat{\rho} = \hat{\rho},\end{aligned} \tag{7.92}$$

where we used that $\hat{V}_j^\dagger \hat{V}_i = \delta_{ij} \hat{\mathbb{1}}$.

### 7.2.2 Correctable errors

The generic scheme for QEC is the following. We are given $k$ qubits in an unknown state $|\psi\rangle \in \mathbb{H}$. We encode $|\psi\rangle$ in a larger number $n > k$ of qubits. These $n$ qubits are subject to errors, which are described in terms of a Kraus map $\mathcal{E}$ with the Kraus operators being $\hat{E}_i$ or equivalently $\hat{V}_i$, see Eq. (7.90). The recovery protocol employs some extra $n'$ ancillary qubits to apply the QEC, which inverts the error Kraus map under certain conditions.

After the encoding, the relevant state will be $|\psi'\rangle \in \mathbb{H}_\mathrm{C}$, which is a subspace of $\mathbb{H}'$ and it is called code space. In particular, the entire Hilbert space $\mathbb{H}'$ is the union of the code space $\mathbb{H}_\mathrm{C}$ with the $\otimes_i \mathbb{H}'_i$. Here, the basis of each $\mathbb{H}'_i$ is obtained by applying the corresponding $\hat{V}_i$ to the basis of $\mathbb{H}_\mathrm{C}$. Under this perspective, one can say that also the basis of $\mathbb{H}_\mathrm{C}$ is generated in the same way, where the corresponding error operator is $\hat{V}_\mathrm{C} = \hat{\mathbb{1}}$. Now, since the subspace $\mathbb{H}'_i$ so constructed is orthogonal to $\mathbb{H}_\mathrm{C}$, then the error can be recovered. What is needed is a error syndrome measurement that identifies the subspace $\mathbb{H}'_i$ in which the state $|\psi'\rangle$ has been mapped. Given such a measurement, one can apply the corresponding Kraus recovery operator. This is graphically represented in Fig. 7.9

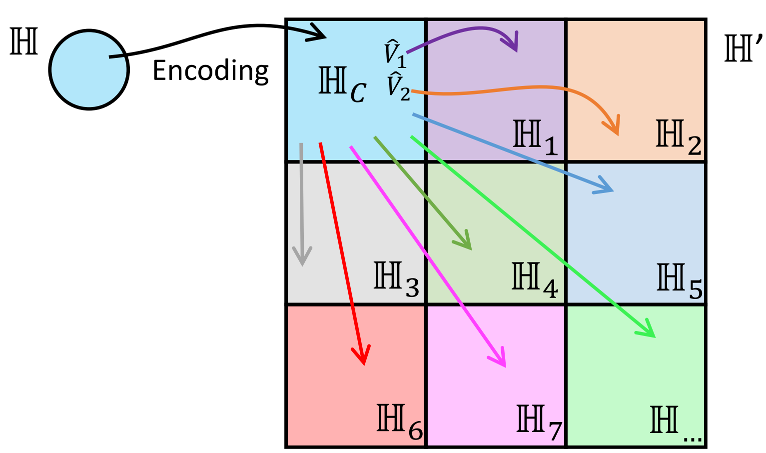


Fig. 7.9: Graphical representation of the division of the Hilbert space $\mathbb{H}'$ in the code space $\mathbb{H}$ and error spaces $\mathbb{H}_i$, which are linked by the operators $\hat{V}_i$.

One can invert the error map if the corresponding operators $\hat{E}_i$ satisfy

$$\hat{P}\hat{E}_i^\dagger\hat{E}_j\hat{P} = \mu_{ij}\hat{P}, \tag{7.93}$$

where $\hat{P}$ is a projector on the code space $\mathbb{H}_C$.

Notably, $\hat{P}$ acts as an identity operator if restricted on $\mathbb{H}_C$. This also implies that there exists a special set of Kraus operators, which allow to rewrite the error map, and thus also the recovery map, as a mixture of isometries. These are induced by a set of unitary operators $\hat{U}_i$ on the code space:

$$\hat{V}_i = \hat{U}_i\hat{P}, \tag{7.94}$$

such that

$$\hat{P}\hat{U}_i^\dagger\hat{U}_j\hat{P} = \delta_{ij}\hat{P}. \tag{7.95}$$

Then, one can map the state $\hat{V}_i|\psi'\rangle$ back to $\mathbb{H}_C$ by selecting the corresponding recovery Kraus operator $\hat{R}_i = \hat{V}_i^\dagger$.

An important note is the following. Let be $\mathcal{R}$ the recovery map for $\mathcal{E}$. Then, one has

$$\begin{aligned}\mathcal{E}(\hat{\rho}) &= \sum_i \hat{E}_i\hat{\rho}\hat{E}_i^\dagger,\\ \mathcal{R}(\hat{\rho}) &= \sum_k \hat{R}_k\hat{\rho}\hat{R}_k^\dagger,\end{aligned} \tag{7.96}$$

such that $\hat{R}_k\hat{E}_i = \alpha_{ki}\hat{\mathbb{1}}$. We define the map $\mathcal{D}$ as

$$\mathcal{D}(\hat{\rho}) = \sum_i \hat{D}_i\hat{\rho}\hat{D}_i^\dagger, \tag{7.97}$$

with $\hat{D}_i$ appertaining to the span of $\{\hat{E}_i\}_i$, i.e. $\hat{D}_i = \sum_j c_{ij}\hat{E}_j$. Then, the map $\mathcal{D}$ can be recovered with the same recovery map $\mathcal{R}$.

Consider the case of $n$ qubits. We want to construct the recovery map for errors due to the application of the Pauli operators. These are $\{\hat{\mathbb{1}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}^{\otimes n}$, and they form the basis of $\mathcal{B}(\mathbb{H}^{\otimes n})$. The operator $\hat{\mathbb{1}}$ is the identity, so is not associated to any error. The operator $\hat{\sigma}_y = i\hat{\sigma}_x\hat{\sigma}_z$. So, one needs to construct the recovery map $\mathcal{R}$ that corrects only errors due to $\hat{\sigma}_x$ and $\hat{\sigma}_z$. Now, for every Pauli operator different from the identity, we have two important properties: $\mathrm{Tr}[\hat{\sigma}_i] = 0$ and $\hat{\sigma}_i^2 = \hat{\mathbb{1}}$. They imply that their eigenvalues are $\pm 1$. Thus, one can divide the full Hilbert space $\mathbb{H}' = \mathbb{H}$, with $\dim(\mathbb{H}) = 2^n$, in two subspaces (of the same dimension), which are associated to the corresponding eigeinvalues, see Fig. 7.10. Namely, given $\hat{\sigma}_i$ we have $\mathbb{H}_{\sigma_i=1}$ and $\mathbb{H}_{\sigma_i=-1}$, with $\dim(\mathbb{H}_i) = 2^{n-1}$, whose union gives $\mathbb{H}'$. Suppose the code space $\mathbb{H}_C$ is defined in terms of the operators $\hat{E}_1$ and $\hat{E}_2$ as it follows: $\forall\,|\psi\rangle \in \mathbb{H}_C$, one has

$$\hat{E}_1|\psi\rangle = +1|\psi\rangle, \quad \text{and} \quad \hat{E}_2|\psi\rangle = +1|\psi\rangle. \tag{7.98}$$

Any other combination identifies an error subspace $\mathbb{H}_i'$.

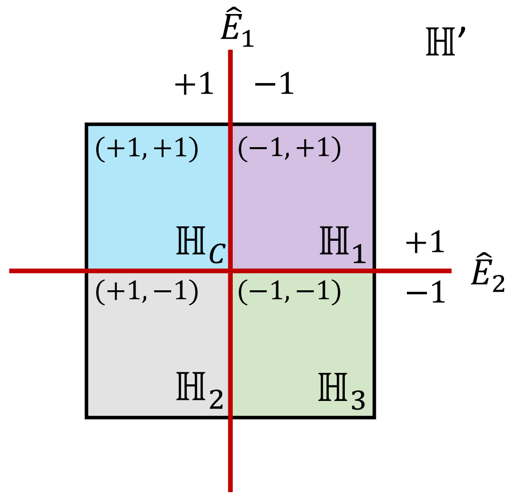


Fig. 7.10: Division of the Hilbert space $\mathbb{H}'$ with respect to the subspaces defined by the eigeinvalues of $\hat{E}_1$ and $\hat{E}_2$.

More in general, given the operators in $\mathcal{B}(\mathbb{H})$, they are part of one of the following families:

- They specify the subspaces we are going to divide $\mathbb{H}'$.
- They are responsible for errors. So they describe how a state moves from $\mathbb{H}_\mathrm{C}$ to $\mathbb{H}_i'$.
- They are responsible for logical operations within the individual subspaces $\mathbb{H}_\mathrm{C}$ and $\mathbb{H}_i$.