21 th November

A topological space X is separable if it contains a countable subspace which is dense in X

Example $C^0([0,1])$ is separable because $\mathbb{R}[x]$ is dense and in turn $\mathbb{R}[x]$ is separable

Lemma E Banach space, E' the dual.
E' separable $\Rightarrow$ E separable

Pf Let $\{f_n\}$ be dense in E'. Let $\{x_n\}$ be a sequence in E built as follows:

1) $\|x_n\|_E = 1$

2) $f_n(x_n) \geq \frac{1}{2}\|f_n\|_{E'}$

We claim that $\overline{\operatorname{Span}\{x_n : n \in \mathbb{N}\}} =: L$ is $L = E$.

Suppose $L \subsetneq E$. Then by Hahn Banach

$\exists$ $f \in E'$ $f \not\equiv 0$ s.t. $f|_L \equiv 0$

Since $\{f_n\}_{n\in\mathbb{N}}$ is dense in E' $\exists$ $\{f_{n_k}\}$ a subsequence s.t. $\lim_{k\to+\infty} f_{n_k} = f$ in E' $\|f_{n_k} - f\|_{E'} \xrightarrow{k\to+\infty} 0$

$$\underbrace{\|f_{n_k} - f\|_{E'}}_{\substack{\downarrow k\to+\infty \\ 0}} \geq f_{n_k}(x_{n_k}) - f(x_{n_k}) = f_{n_k}(x_{n_k}) \geq \frac{\|f_{n_k}\|_{E'}}{2} \xrightarrow{k\to+\infty} \frac{\|f\|_{E'}}{2} > 0$$

Contradiction $\Rightarrow$ $L = E$

$$\|f_{n_k} - f\|_{E'} = \sup\{f_{n_k}(x) - f(x) : \forall x \in E \text{ s.t. } \|x\|_E = 1\}$$

Example $L^\infty((-1,1))$ is not separable.

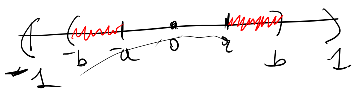
Suppose by contradiction it is separable. Then let $\{f_n\}_{n\in\mathbb{N}}$ form a dense subspace.

$\forall\, a \in (0,1)$ let $I_a = (-a,a)$

and let 1_{I_a} = characteristic function of I_a.

Let $0 < a < b \leq 1$

$1_{I_b} - 1_{I_a} = 1_{I_b \setminus I_a}$



$\| 1_{I_b \setminus I_a} \|_{L^\infty} = 1 = \| 1_{I_b} - 1_{I_a} \|_{L^\infty} = 1$

$\Rightarrow \{ D_{L^\infty}(1_{I_a}, \frac{1}{2}) \}_{a \in (0,1)}$ these are mutually disjoint disks.

Since $\{f_n\}_{n\in\mathbb{N}}$ is dense we could define a function

$(0,1) \ni a \longrightarrow n_a \in \mathbb{N}$ st $f_{n_a} \in D_{L^\infty}(1_{I_a}, \frac{1}{2})$

$(0,1) \longrightarrow \mathbb{N}$

$a \longrightarrow n_a$ we get an injective map.

this is impossible.

$\ell^\infty(\mathbb{N})$ is not separable

$\ell^1(\mathbb{N})$ is separable $\qquad \ell^\infty(\mathbb{N}) = (\ell^1(\mathbb{N}))'$

$L^1(\mathbb{R})$ separable $\qquad s = \sum_{j \in J} a_j \chi_{E_j}$ $\quad E_j$ intervals

$C^0_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$

$C^\infty_c(\mathbb{R})$ for L^∞ norm is separable.

If E is reflexive and separable then E' is reflexive and separable.

E reflexive $\Leftrightarrow$ E' reflexive

E reflexive and separable $\Rightarrow$ E'' separable $\Rightarrow$ E' separable.

<u>Theorem</u> E Banach space, E' separable. Then $(D_E(0,1), \sigma(E,E'))$ is metrizable.

The viceversa is true also.

<u>Proof</u> If E' is separable $\Leftrightarrow$ $\exists$ a sequence $\{f_n\}$ in $D_{E'}(0,1)$ dense in the disk.

Let us define $\forall\, x \in E$

$$[x] = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x)|$$

$$[x] = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x)| \le \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\|f_n\|_{E'}}_{\le 1} \|x\|_E \le$$

$$\le \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_E = \|x\|_E$$

$$[x] \le \|x\|_E$$

$(D_E(0,1), d)$ $\qquad d(x,y) = [x-y]$

we prove that the topology induced by d is the $\sigma(E,E')$

Let $x_0 \in E$

V be an open neigh for $\sigma(E,E')$ of x_0

$g_1, \dots, g_n \in E'$ $\quad \varepsilon > 0$

$$V = \{ x \in D_E(0,1) : |g_j(x - x_0)| < \varepsilon, \ j = 1, \dots, n \}$$

We want to show that if $r > 0$

$$U_r = \{ x \in D_E(0,1) : [x - x_0] < r \}$$

$\exists\, r$ st $U_r \subseteq V$ $\Rightarrow$ the topology of d is as fine as that of $\sigma(E,E')$

$\exists$ $f_{j_1}, \dots, f_{j_n}$ s.t.

$$\|f_{j_i} - g_i\|_{E'} < \frac{\varepsilon}{4} \qquad \forall\ i = 1, \dots n$$

$$|f_{j_i}(x) - f_{j_i}(x_0)| = 2^{j_i} \frac{1}{2^{j_i}} |f_{j_i}(x - x_0)| \le 2^{j_i} \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} |f_\alpha(x - x_0)|$$

$$= 2^{j_i}\, [x - x_0]$$

$$\le \boxed{2^{j_i}\, r < \frac{\varepsilon}{4}}$$

$$|g_i(x - x_0)| = |(g_i(x-x_0) - f_{j_i}(x-x_0)) + f_{j_i}(x-x_0)|$$

$$\le |g_i(x-x_0) - f_{j_i}(x-x_0)| + \underbrace{|f_{j_i}(x-x_0)|}_{< \frac{\varepsilon}{4}}$$

$$\le \underbrace{\|g_i - f_{j_i}\|_{E'}}_{< \frac{\varepsilon}{4}} \underbrace{\|x - x_0\|_E}_{< 2} + \frac{\varepsilon}{4}$$

$$\le \frac{3}{4}\varepsilon < \varepsilon$$

I have shown $\exists\, r > 0$ st

$$\{ x \in D_E(0,1) : [x - x_0] < r \} \subseteq V = \{ x \in D_E(0,1) : |g_i(x - x_0)| < \varepsilon, \ i = 1, \dots, n \}$$

$(D_E(0,1), d)$ has stronger topology of $(D_E(0,1), \sigma(E,E'))$

It is easy to show that $(D_E(0,1), \sigma(E,E'))$ is stronger than $(D_E(0,1), d)$

Let us fix $x_0 \in D_E(0,1)$ $\{ x \in D_E(0,1) : [x - x_0] < r \} = U_r$

We need to find an open neigh V for $\sigma(E,E')$ of x_0 st $V \subseteq U_r$ $\qquad V = \{x : |g_i(x) - g_i(x_0)| < \varepsilon, \ i = 1, \dots n\}$

U_r is the set of points s.t.

$$[x - x_0] = \sum_{i=1}^{\infty} \frac{1}{2^i} |f_i(x - x_0)| < r \Rightarrow$$

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} |f_i(x - x_0)| \le \sum_{i=n+1}^{\infty} \frac{1}{2^i} \underbrace{\|f_i\|_{E'}}_{\le 1} \underbrace{\|x - x_0\|_E}_{\le 2}$$

$$\le \sum_{i=n+1}^{\infty} \frac{1}{2^i}\, 2 = \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} \sum_{i=0}^{\infty} \frac{1}{2^i}$$

$$= 2^{-(n-1)}$$

I take n st $2^{-(n-1)} < \frac{r}{4}$

and then take

$f_1, \dots, f_n$

$$V = \{ x \in D_E(0,1) : |f_j(x - x_0)| < \frac{r}{4} \text{ for } j = 1, \dots n \}$$

Now I show that $x \in V \Rightarrow [x - x_0] < r$

$$[x - x_0] = \sum_{j=1}^{n} \frac{1}{2^j} \underbrace{|f_j(x - x_0)|}_{< \frac{r}{4}} + \underbrace{\sum_{j=n+1}^{\infty} \frac{1}{2^j} |f_j(x - x_0)|}_{< \frac{r}{4}}$$

$$\le \underbrace{\sum_{j=1}^{\infty} \frac{1}{2^j}}_{1} \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$$

We have proved that E' separable $\Rightarrow$ $(D_E(0,1), \sigma(E,E'))$ is metrizable

Theorem

$(D_{E'}(0,1), \sigma(E',E))$ is metrizable $\iff$ E is separable

$\Leftarrow$

$\Rightarrow$

Pf $\Leftarrow$ we start with $\{x_n\}_{n\in\mathbb{N}}$ a sequence dense in $D_E(0,1)$. We use this to define a metric in $D_{E'}(0,1)$

We first define a norm in E'

$$[f] = \sum_{j=1}^{\infty} \frac{1}{2^j} |f(x_j)|$$

This is a norm and it is easy to see that

$$[f] \leq \|f\|_{E'}$$

In $D_{E'}(0,1)$ $\quad d(f,g) = [f-g]$

$f_0 \in D_{E'}(0,1)$. We prove that the metric induces a topology in $D_{E'}(0,1)$ stronger than the $\sigma(E',E)$ top.

We pick a neigh. V of f_0 in $\sigma(E',E)$

$$V = \{ f \in D_{E'}(0,1) : |f(y_i) - f_0(y_i)| < \varepsilon,\ i = 1,\dots,m\}$$
$$y_1, \dots, y_m \in D_E(0,1)$$

$$U_r = \{ f \in D_{E'}(0,1) : [f - f_0] < r \}$$

we need to find $r>0$ st $U_r \subset V$.

For any $\|y_i - x_{n_i}\|_E < \frac{\varepsilon}{4}$

$$|f(y_i) - f_0(y_i)| = |f(y_i) - f(x_{n_i}) + f_0(x_{n_i}) - f_0(y_i) + f(x_{n_i}) - f_0(x_{n_i})|$$

$$\leq |f(y_i) - f(x_{n_i})| + |f_0(y_i - x_{n_i})| + |f(x_{n_i}) - f_0(x_{n_i})|$$

$$\leq \underbrace{\|f\|_{E'}}_{<1} \underbrace{\|y_i - x_{n_i}\|_E}_{<\frac{\varepsilon}{4}} + \underbrace{\|f_0\|_{E'}}_{<1} \underbrace{\|y_i - x_{n_i}\|_E}_{\frac{\varepsilon}{4}} + 2^{n_i} \frac{1}{2^{n_i}} |f(x_{n_i}) - f_0(x_{n_i})|$$

$$< \frac{\varepsilon}{2} + 2^{n_i} \underbrace{\sum_{j=1}^{\infty} |f(x_j) - f_0(x_j)| \frac{1}{2^j}}_{[f - f_0]}$$

$$< \frac{\varepsilon}{2} + 2^{n_i} r$$

$n_1, n_2, \dots, n_m$ are fixed

$2^{n_i} r < \frac{\varepsilon}{4} \quad \forall\, i = 1,\dots,m$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \frac{3}{4}\varepsilon \implies (f \in U_r \implies f \in V)$$

Proceeding like in the previous proof

for any neigh. U_r of f_0 $\exists$ V open neigh. of f_0 for the $\sigma(E',E)$ topology with $V \subset U_r$

Now I prove that if $(D_{E'}(0,1), \sigma(E',E))$ is metrizable then E is separable.

Let d be the metric in and

$$U_n = \{ f : d(f,0) < \tfrac{1}{n} \} \quad \forall\, n \in \mathbb{N}$$

$\exists$ $V_n \subset U_n$ of the form

$$V_n = \{ f \in D_{E'}(0,1) : |f(x)| < \varepsilon_n \text{ for all } x \in \Phi_n \}$$

where where $\varepsilon_n > 0$ and Φ_n is a finite subset in $D_E(0,1)$

$S = \bigcup_{n=1}^{\infty} \Phi_n$. We claim that

$V = \overline{\mathrm{Span}\{S\}}$ (V is separable)

$V = E$

If not $V \subsetneq E$ on by Hahn Banach

$\exists$ $f \in E'$ $f \neq 0$ st $f|_V = 0$ $\quad f(x) = 0\ \forall x \in S$

$f(x) = 0 \quad \forall\, x \in \Phi_n$

$\implies f \in V_n \implies f \in U_n \quad \forall n$

$\implies d(f,0) < \frac{1}{n} \quad \forall n$

$\implies f = 0$.