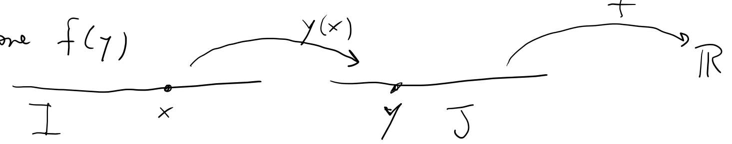
27 Novembre

Teor (Cambio di variabile integrale indefinito)

Siano I e J due intervalli, $y: I \to J$

$f: J \to \mathbb{R}$, $y \in C^1(I)$ e $f \in C^0(J)$

Denotiamo con $\int f(y)\,dy$ le primitive della funzione $f(y)$



Allora vale la formula

$$\int f(y(x))\, y'(x)\,dx = \left(\int f(y)\,dy\right)(y(x))$$

Osservazione 1) Di solito nei testi, con un abuso di notazione, si trova scritto la formula

$$\int f(y(x))\; y'(x)\,dx = \int f(y)\,dy$$

2) $$\int f(y(x))\; \frac{dy(x)}{\cancel{dx}}\,\cancel{dx} = \int f(y)\;dy$$

Dim. Vogliamo dimostrare l'uguaglianza

$$\int f(y(x))\,y'(x)\,dx = \left(\int f(y)\,dy\right)(y(x))$$

Basta dimostrare che i due termini hanno la medesima derivata. Per la definizione di primitiva si ha

$$\frac{d}{dx}\int f(y(x))\,y'(x)\;dx = \underline{f(y(x))\;y'(x)}\,.$$

$$\frac{d}{dx}\left(\int f(y)\,dy\right)(y(x)) = ?$$

Applico la regola della catena

$$\frac{d}{dx}F(y(x)) = \frac{d}{dy}F(y(x))\;\frac{dy(x)}{dx}$$

Nel nostro caso

$$\frac{d}{dx}\left(\int f(y)\,dy\right)(y(x)) = \frac{d}{dy}\left(\int f(y)\,dy\right)(y(x))\;\;\frac{dy(x)}{dx}$$

$$= f(y(x))\;\;\frac{dy(x)}{dx}$$

$$\int f(y(x)) \frac{dy}{dx} \, dx = \int f(y) \, dy$$

$$\int \tan x \, dx = \int \frac{\operatorname{sen} x}{\cos x} \, dx$$

$$y = \cos x$$

$$\frac{dy}{dx} = -\operatorname{sen} x$$

$$dy = -\sin x \, dx$$

$$= -\int \frac{dy}{y}$$

$$= -\lg|y| + c = -\lg|\cos x| + c$$

$$\int \frac{x+1}{x+7}\,dx =$$

$u = x + 7$

$du = dx$

$x = u - 7$

$$= \int \frac{u-6}{u}\,du$$

$$= \int \left(1 - \frac{6}{u}\right) du = \int du - 6 \int \frac{1}{u}\,du =$$

$$= u - 6 \lg|u| + C$$

$$= x+7 - 6 \lg|x+7| + C$$

$$= x+7 - \lg(x+7)^6 + C$$

$$\int \sqrt{ax+b}\, dx$$

$$y = ax+b$$

$$dy = a\, dx$$

$$= \frac{1}{a} \int \sqrt{y}\, dy =$$

$$= \frac{1}{a} \frac{y^{3/2}}{3/2} + c = \frac{2}{3a} (ax+b)^{3/2} + c$$

$n = 0, 1, 2, \ldots$

$m = 1, 2, 3, \ldots$

$$\int \cos^{2n+1}(x) \sin^m(x)\, dx$$

$$\cos(x) = (\operatorname{sen}(x))' \qquad \cos^{2n}(x) = (\cos^2(x))^n$$

$$\int \cos^{2n}(x) \operatorname{sen}^m(x) \cos x\, dx =$$

$y = \operatorname{sen} x$

$dy = \cos x\, dx$

$$= \int (1 - \sin^2(x))^n \sin^m(x) \cos x\, dx$$

$$= \int (1 - y^2)^n y^m\, dy$$

$$\int \cos^7(x) \sin^2(x)\, dx$$

$y = \sin x$

$dy = \cos x\, dx$

$$= \int (1 - y^2)^3 y^2\, dy =$$

$$= \int (1 - 3y^2 + 3y^4 - y^6) y^2\, dy$$

$$= \int (y^2 - 3y^4 + 3y^6 - y^8)\, dy$$

$$= \int y^2 dy - 3\int y^4 dy + 3\int y^6 dy - \int y^8 dy$$

$$= \frac{y^3}{3} - \frac{3}{5} y^5 + \frac{3}{7} y^7 - \frac{y^9}{9} + c$$

dove $y = \operatorname{sen} x$.

$$\lim_{x\to 0} \frac{\log(1+x^2) - \sin^2(x)}{(1-\cos(x) + x^4)^2} =$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\frac{1-\cos x}{x^2} = \frac{1}{2} + o(1)$$

$$1-\cos x = \frac{x^2}{2} + o(x^2)$$

$$= \lim_{x\to 0} \frac{\log(1+x^2) - \sin^2(x)}{\left(\frac{x^2}{2} \cancel{+ o(x^2)}\right)^2}$$

$$= \lim_{x\to 0} \frac{\log(1+x^2) - \sin^2(x)}{\frac{x^4}{4}}$$

$$\log(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\log(1+x^2) = x^2 - \frac{x^4}{2} + o(x^4)$$

$$\sin(x) = x - \frac{x^3}{6} + o(x^3)$$

$$\sin^2(x) = x^2 - \frac{2}{6}x^4 + o(x^4) = x^2 - \frac{1}{3}x^4 + o(x^4)$$

$$4\ \frac{\log(1+x^2) - \sin^2(x)}{x^4} = \frac{\cancel{x^2} - \frac{x^4}{2} - \cancel{x^2} + \frac{1}{3}x^4 + o(x^4)}{x^4}\ 4$$

$$= 4\ \frac{-\frac{1}{6}x^4 + \cancel{o(x^4)}}{x^4} \xrightarrow{x\to 0} -\frac{2}{3}$$

$$\int \cos^{2n+1}(x) \quad \sin^{m}(x)\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$$\int \cos^{m}(x) \quad \sin^{2n+1}(x)\, dx$$

$u = \cos x$

$du = -\sin x\, dx$

$$\int \cos^2 x \quad \sin x\, dx$$

$$\int \cos^2(x) \sin^3(x)\, dx$$

$$\int \cos^2 x \quad \sin^2 x\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$$= \int \cos x \quad \sin^2 x \quad \cos x\, dx$$

$\cos x = \sqrt{1-\sin^2(x)}$

$$= \int \sqrt{1-u^2} \quad u^2 \, du$$

$$\int \cos^2 x \quad \sin^2 x\, dx =$$

$\cos 2x = 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$$= \int \frac{\cos(2x)+1}{2} \quad \frac{1-\cos(2x)}{2}\, dx$$

$\cos^2 x = \dfrac{\cos(2x)+1}{2}$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$$= \frac{1}{4}\int \left(1-\cos^2(2x)\right) dx = \frac{x}{4} - \frac{1}{4}\int \frac{\cos(4x)+1}{2}\, dx$$

$$= \frac{x}{8} - \frac{1}{8}\int \cos(4x)\, dx$$

$$= \frac{x}{8} - \frac{1}{8}\,\frac{\sin(4x)}{4} + C$$

$$\int \frac{1}{ax^2+bx+c} \qquad b^2-4ac<0$$

Vorremmo $\int \frac{1}{y^2+1}\,dy = \arctan(y)+c$

$$\frac{1}{a}\int \frac{1}{x^2+\frac{b}{a}x+\frac{c}{a}}\,dx =$$

$$= \frac{1}{a}\int \frac{1}{\left(x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}\right)-\frac{b^2}{4a^2}+\frac{c}{a}}\,dx$$

$$= \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}\,dx \qquad -\Delta = |\Delta|$$

$$= \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{|\Delta|}{4a^2}}\,dx$$

$$\frac{\sqrt{|\Delta|}}{2a}\,y = \left(x+\frac{b}{2a}\right)$$

$a>0$

$$\frac{\sqrt{|\Delta|}}{2a}\,dy = dx$$

$$= \frac{1}{a}\,\frac{\sqrt{|\Delta|}}{2a}\int \frac{1}{\frac{|\Delta|}{4a^2}y^2+\frac{|\Delta|}{4a^2}}\,dy$$

$$= \frac{1}{a}\,\frac{\frac{\sqrt{|\Delta|}}{2a}}{\frac{|\Delta|}{4a^2}}\int \frac{1}{y^2+1}\,dy$$

dove $y = \frac{2a}{\sqrt{|\Delta|}}\left(x+\frac{b}{2a}\right)$

$$= \frac{1}{\cancel{a}}\,\frac{1}{\frac{\sqrt{|\Delta|}}{2\cancel{a}}}\arctan(y)+c$$

$$= \frac{2}{\sqrt{\Delta}}$$