28 Novembre

Teor (cambio variabile per integrale definito)

Siano $[a,b]$ e J due intervalli con

$$y(x) : [a,b] \to J \quad \text{e} \quad f(y) : J \to \mathbb{R}$$

con $y(x) \in C^1([a,b])$ e $f \in C^0(J)$

Allora vale

$$\int_a^b f(y(x))\, y'(x)\, dx = \int_{y(a)}^{y(b)} f(y)\, dy$$

Dim

1° teor fond.

$$\int_a^b f(y(x))\, y'(x)\, dx \overset{\downarrow}{=} \left(\int f(y(x))\, y'(x)\, dx\right)\Big]_a^b$$

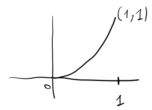
$$= \left(\int f(y)\, dy\right)(y(x))\Big]_a^b$$

$$= \int f(y)\, dy \Big]_{y(a)}^{y(b)} = \int_{y(a)}^{y(b)} f(y)\, dy$$

cambio variabile per integrale indefinito

$$\int_a^b f(y(x))\, y'(x)\, dx = \int_{y(a)}^{y(b)} f(y)\, dy$$

$$\int_0^1 \sqrt{1+4x^2}\, dx$$



$y = x^2$

$(x^2)' = 2x$

In generale se $f \in C^1([a,b])$ allora

$$\int_a^b \sqrt{1+(f'(x))^2}\, dx$$ è la lunghezza del grafico di f

$$\int_0^1 \sqrt{1+4x^2}\, dx =$$

$y = 2x$

$dy = 2\, dx$

$$= \frac{1}{2}\int_0^2 \sqrt{1+y^2}\, dy =$$

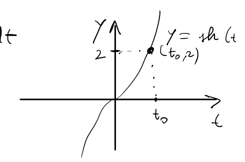
$y = sh\, t$

$dy = ch\, t\, dt$

$$= \frac{1}{2}\int_0^{t_0} \sqrt{1+sh^2 t}\; ch\, t\, dt$$

$1 + sh^2(t) = ch^2(t)$

$$= \frac{1}{2}\int_0^{t_0} \sqrt{ch^2(t)}\; ch(t)\, dt$$



$$= \frac{1}{2}\int_0^{t_0} ch(t)\; ch(t)\, dt$$

$$= \frac{1}{2}\int_0^{t_0} ch^2(t)\, dt =$$

sia qui t_0 l'unica soluzione di $sh(t) = 2$

Verifichiamo se $ch^2(t) = \frac{1+ch(2t)}{2}$

$$ch^2(t) = \left(\frac{e^t+e^{-t}}{2}\right)^2 = \frac{e^{2t}+e^{-2t}+2}{4} = \frac{\frac{e^{2t}+e^{-2t}+2}{2}}{2}$$

$$= \frac{\frac{e^{2t}+e^{-2t}}{2}+1}{2} = \frac{ch(2t)+1}{2}$$

$$= \frac{1}{4}\int_0^{t_0} (1+ch(2t))\, dt$$

$$= \frac{1}{4}\int_0^{t_0} dt + \frac{1}{4}\int_0^{t_0} ch(2t)\, dt$$

$$= \frac{t_0}{4} + \frac{1}{4}\left.\frac{sh(2t)}{2}\right]_0^{t_0}$$

$$= \frac{t_0}{4} + \frac{1}{8}\, sh(2t_0) =$$

ci chiediamo se

$$sh(2t_0) = 2\, sh(t_0)\, ch(t_0)$$

$$sh(2t) = \frac{e^{2t}-e^{-2t}}{2} = \frac{1}{2}(e^t - e^{-t})(e^t + e^{-t})$$

$$= 2\, \frac{e^t - e^{-t}}{2}\; \frac{e^t+e^{-t}}{2}$$

$$= \frac{t_0}{4} + \frac{1}{4}\, sh(t_0)\, ch(t_0)$$

Qui

$sh(t_0) = 2$

$ch(t_0) = \sqrt{1+sh^2(t_0)} = \sqrt{1+2^2} = \sqrt{5}$

Inversione $\quad y = sh(t)$

$$y = \frac{e^t - e^{-t}}{2}$$

$$2y = e^t - e^{-t}$$

$$2y e^t = e^{2t} - 1$$

$$(e^t)^2 - 2y e^t - 1 = 0$$

$$(e^t)_\pm = \cancel{y} \pm \sqrt{\cancel{y}^2 + 1}$$

$$(e^t)_\pm = \begin{cases} \boxed{y + \sqrt{y^2+1} > 0} \\ y - \sqrt{y^2+1} < 0 \end{cases}$$

Verifichiamo che

$$y - \sqrt{y^2+1} < 0 \iff y < \sqrt{y^2+1}$$

~~Se~~ $y \leq 0$ è vero. Se $y > 0$

$$y < \sqrt{y^2+1} \iff \cancel{y^2} < \cancel{y^2} + 1 \iff 0 < 1$$

$$e^t = (e^t)_+ = y + \sqrt{y^2+1}$$

$$e^t = y + \sqrt{y^2+1}$$

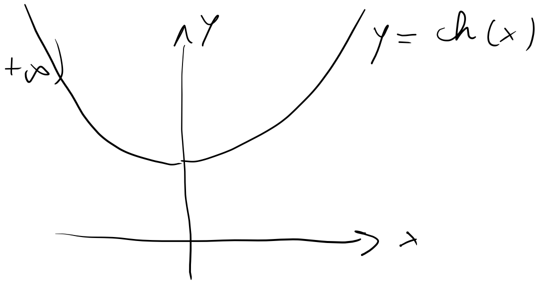
$$t = \lg\left(y + \sqrt{y^2+1}\right)$$

per $y = 2$

$$t_0 = \lg\left(2 + \sqrt{5}\right)$$

$ch : [0, +\infty) \to [1, +\infty)$

è biettiva



$y = ch(x) \qquad x \geq 0$

$y = \frac{e^x + e^{-x}}{2} \qquad 2y = e^x + e^{-x} \quad \cdot e^x$

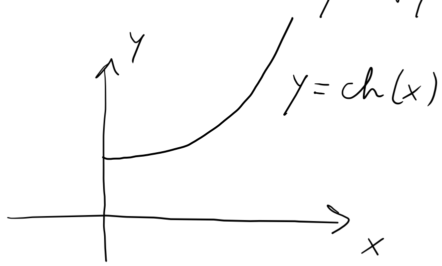
$2y e^x = e^{2x} + 1$

$(e^x)^2 - 2y e^x + 1 = 0$

$(e^x)_\pm = y \pm \sqrt{y^2 - 1}$

$y + \sqrt{y^2-1} > 1$

$y - \sqrt{y^2-1} < 1$ se $y > 1$



$x \geq 0$

$\Rightarrow e^x \geq e^0 = 1 \quad \Rightarrow \quad e^x = (e^x)_+ = y + \sqrt{y^2-1}$

$x = \lg(y + \sqrt{y^2-1})$

Verifichiamo che se $y > 1$ allora

$\boxed{y - \sqrt{y^2-1} < 1}$

$y - 1 < \sqrt{y^2-1}$

$\Updownarrow$

$(y-1)^2 < y^2 - 1$

$\cancel{y^2} - 2y + 1 < \cancel{y^2} - 1$

$0 < 2y - 2 = 2(y-1) \Leftrightarrow y > 1$ vero

$\boxed{y > 1}$

$$\int_1^2 \lg^2(x)\, dx = \qquad\qquad y = \lg x$$

$$dy = \frac{1}{x}\, dx$$

$$= \int_0^{\lg 2} y^2\, e^y\, dy \qquad\qquad x = e^y$$

$$dx = x\, dy = e^y\, dy$$

$$= \int_0^{\lg 2} y^2 \left(e^y\right)' dy = \int_0^{\lg 2} x^2 \left(e^x\right)' dx$$

$$= x^2 e^x \Big]_0^{\lg 2} - 2 \int_0^{\lg 2} x \left(e^x\right)' dx$$

$$= \lg^2 2 \quad 2 - 2 \left(x e^x \Big]_0^{\lg 2} - \int_0^{\lg 2} e^x\, dx \right)$$

$$= 2 \lg^2 2 - 2 \left(2 \lg 2 - e^x \Big]_0^{\lg 2} \right) =$$

$$= 2 \lg^2 2 - 4 \lg 2 + 2\, e^x \Big]_0^{\lg 2}$$

$$= 2 \lg^2 2 - 4 \lg 2 + \overbrace{4 - 2}^{2}$$

$$\int_1^2 \lg^2(x)\, dx = 2 + 2 \lg^2 2 - 4 \lg 2$$

$$= 2 \left(1 + \lg^2 2 - 2 \lg 2 \right) \overset{?}{>} 0$$

$$1 + \lg^2 2 - 2 \lg 2 > 0$$

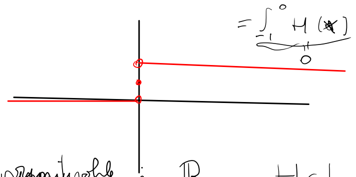
$$(1 - \lg 2)^2 > 0 \iff 1 - \lg 2 > 0$$

so che $\quad 1 = \lg e > \lg 2 > 0$

Esempio di funzione non primitivabile

$$H(x)=\begin{cases}1 & x>0\\ \frac{1}{2} & x=0\\ 0 & x<0\end{cases}$$

$$\int_{-1}^{7} H(x)\,dx = \int_{-1}^{0} H(x)\,dx + \int_{0}^{7} H(x)\,dx = \underbrace{\int_{-1}^{0} H(x)\,dx}_{0} + \underbrace{\int_{0}^{7} H(x)\,dx}_{7}$$



Non è primitivabile in $\mathbb{R}$. $H \in L_{loc}(\mathbb{R})$

Supponiamo per assurdo che $G(x)$ sia una primitiva di H in $\mathbb{R}$ e consideriamo

$$F(x)=\int_0^x H(t)\,dt$$

Per definizione $G'(x)=H(x)$ $\forall x\in\mathbb{R}$

Dal teorema fond del calcolo

$$F'(x)=H(x) \qquad \forall x\neq 0$$

Inoltre $F\in C^0(\mathbb{R})$.

Sia $K(x)=F(x)-G(x)$

$K\in C^0(\mathbb{R})$ come differenza di funzioni continue

$K'(x)$ esiste $\forall x\neq 0$ $\left(F'_d(0)=H(0^+)=1,\ F'_s(0)=H(0^-)=-1 \right)$ $\Rightarrow F'(0)$ non esiste.

$$K'(x)=F'(x)-G'(x) = H(x)-H(x)=0$$

$K\in C^0(\mathbb{R})$ e $K'(x)=0$ $\forall x\neq 0$.

Per Lagrange in $[0,+\infty)$ $K(x)\equiv c=K(0)$

Basta usare Lagrange. Per $x>0$

$$\frac{K(x)-K(0)}{x-0}=\underbrace{K'(c)}_{=0} \text{ per un opportuno}$$

$0<c<x$. Siccome $K'(c)=0 \Rightarrow \frac{K(x)-K(0)}{x}=0 \iff K(x)=K(0)$

Ma lo stesso è vero anche in $(-\infty,0]$, $K(x)=K(0)$

Basta usare Lagrange. Per $x<0$

$\exists$ $x<c<0$ t.c. $\frac{K(x)-K(0)}{x}=K'(c)=0 \Rightarrow K(x)=K(0)$

Ma allora $K(x)=\underbrace{K(0)}_{=c}$ $\forall x\in\mathbb{R}$.

$$c=K(x)=F(x)-G(x)$$
$$F(x)=G(x)+c$$

Ma allora $\exists F'(0)=G'(0)$ (esiste $F'(0)$ con...)

Ma sappiamo che $F'(0)$ non esiste perché $F'_d(0)=1$ $F'_s(0)=0$