28 Novembe

Next academic year Differential Equation will be taught by Stefano Scrobogna
Fluid Dynamics

E. Carneiro [ICTP] Topics in Adv. Analysis 1
Harmonic Analysis and Number theory

somebody from SISSA will teach. Topics 2.

We will prove that for $1 < p < \infty$ $L^p(X)$ is reflexive.

Teorem $2 \le p < \infty$ is $L^p(X)$ is reflexive.

Pf We will prove that $L^p(X)$ are uniformly convex

$\|f\|_{L^p} \le 1, \quad \|g\|_{L^p} \le 1$ $\quad \|f-g\|_{L^p} \ge \varepsilon \Rightarrow \exists\, \delta > 0$ st. $\left\|\frac{f+g}{2}\right\|_{L^p} < 1-\delta$

Clarkson inequality

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \le \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right)$$

$$\left\|\frac{f+g}{2}\right\|_p^p \le 1 - \left\|\frac{f-g}{2}\right\|_p^p$$

if

$$\le 1 - \frac{\varepsilon^p}{2^p}$$

$$\left\|\frac{f+g}{2}\right\|_p \le 1 - \underbrace{\left(1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{\frac{1}{p}}\right)}_{\delta > 0}$$

Milman Pettis $\Rightarrow$ reflexivity Clarkson is based on

1) $\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}\left(|a|^p + |b|^p\right)$ $\quad 2 \le p < \infty$

2) $\alpha^p + \beta^p \le \left(\alpha^2 + \beta^2\right)^{\frac{p}{2}}$ $\quad 2 \le p < \infty$

2) $\Rightarrow$ 1 $\quad \alpha = \left|\frac{a+b}{2}\right| \quad \beta = \left|\frac{a-b}{2}\right|$

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \left(\frac{(a+b)^2}{4} + \frac{(a-b)^2}{4}\right)^{\frac{p}{2}} =$$

$$= \left(\frac{a^2}{2} + \frac{b^2}{2}\right)^{\frac{p}{2}} = f\left(\frac{a^2}{2} + \frac{b^2}{2}\right)$$ $\quad f(t) = t^{\frac{p}{2}}$ convex $\quad \frac{p}{2} \ge 1$

$$\le \frac{1}{2} f(a^2) + \frac{1}{2} f(b^2) =$$

$$= \frac{1}{2} a^p + \frac{1}{2} b^p$$

② $\quad \alpha^p + \beta^p \le \left(\alpha^2 + \beta^2\right)^{\frac{p}{2}}$ ✓ $\quad q = \frac{p}{2}$

$a = \alpha^2 \quad b = \beta^2$

$$a^q + b^q \le (a+b)^q$$

$$\left(\frac{a}{a+b}\right)^q + \left(\frac{b}{a+b}\right)^q \le 1$$ ✓ $\quad q \ge 1$

$$\left(\frac{a}{a+b}\right)^q + \left(\frac{b}{a+b}\right)^q \le \frac{a}{a+b} + \frac{b}{a+b} = 1$$ ✓

Thm L^p $1<p<2$ is reflexive

Pf $L^p \xhookrightarrow{T} (L^{p'})'$ $\frac{1}{p'}+\frac{1}{p}=1$

$f \longrightarrow Tf$ $p'=\frac{p}{p-1}$

$\langle Tf, g\rangle_{(L^{p'})' \times L^{p'}} = \int f g \, dx$

$\left| \int f g \, dx \right| \leq |f|_p \; |g|_{p'}$

$T: L^p \hookrightarrow (L^{p'})'$ and it is an isometry

$|Tf|_{(L^{p'})'} = |f|_{L^p}$ $\forall f \in L^p$

$|Tf|_{(L^{p'})'} \leq |f|_{L^p}$ $\int f g \leq |f|_p$

$|Tf|_{(L^{p'})'} = \sup \{ \langle Tf, g\rangle_{(L^{p'})' \times L^{p'}} : \forall_g |g|_{p'} = 1 \}$

$\leq |f|_p$

$\langle Tf, g\rangle_{(L^{p'})' \times L^p} = \int f g \, d\mu =$ $g(x) = |f(x)|^{p-2} f(x)$ $\in L^{p'}$

$|g(x)|^{p'} = \left(|f(x)|^{p-1} \right)^{p'=\frac{p}{p-1}}$

$= |f(x)|^p \in L^1 \Rightarrow |g|^{p'} \in L^1 \Rightarrow g \in L^{p'}$

$\int |f(x)|^p \, d\mu = |f|_p^p$

$|g|_{p'} = \left(\int |g|^{p'} d\mu \right)^{\frac{1}{p'}} = \left(\int |f|^p d\mu \right)^{\frac{p-1}{p}} =$

$= |f|_p^{p-1}$

$|g|_{p'} |Tf|_{(L^{p'})'} \geq \langle Tf, g\rangle_{(L^{p'})' \times L^{p'}} = |f|_p^p$

$\cancel{|f|_p^{p-1}} \; |Tf|_{(L^{p'})'} \geq |f|_p^{\cancel{p}\,1}$

$|Tf|_{(L^{p'})'} \geq |f|_p \Longrightarrow$ they are equal

$\Rightarrow$ So T is an isometry $\Rightarrow$ $R(T)$ is a closed subspace of $(L^{p'})'$ $2<p'<\infty$

$L^{p'}$ reflexive $\Rightarrow (L^{p'})'$ is reflexive $\Rightarrow R(T)$ is reflexive

$T: L^p \longrightarrow R(T)$ is an isomorphism

$\Rightarrow L^p$ is reflexive

Theorem $\quad 1 < p < \infty \qquad \phi \in (L^p(X))'$

$\Rightarrow \ \exists \ u \in L^{p'}(X)$ s.t. $\quad \phi = Tu$

$\phi(f) = \int f u \quad \forall \ f \in L^p(X) \qquad$ (u is unique)

Pf $\qquad T : L^p \longrightarrow (L^{p'})' \qquad k \in L^{p'}$, $\; Jk \downarrow J$

If $\quad R(T) \subsetneq (L^{p'})' \quad \Rightarrow \quad h \in (L^{p'})'' \quad h \neq 0$, $\; h = Jk$

$\langle Tf, Jk \rangle_{(L^{p'})' \times (L^{p'})''} = 0 \quad \forall \ f \in L^p$

$= \langle Tf, k \rangle_{(L^{p'})' \times L^{p'}} = \int f k = 0 \qquad \forall f \in L^p$

$f = |k|^{p'-2} k \qquad p = \frac{p'}{p'-1}$

$|f|^p = \left(|k|^{p'-1}\right)^p = |k|^{p'}$

$\int |f|^p = \int |k|^{p'} \qquad k \in L^{p'} \Rightarrow f \in L^p$

$0 = \int f k = \int |k|^{p'} \Rightarrow \quad k = 0 \quad$ contradiction because $k \neq 0$

$k = u$

$\Rightarrow \qquad R(T) = (L^{p'})'$

$\Rightarrow$ statement any $\phi \in (L^{p'})'$ is of form

$\phi = Tu \quad$ with $\; u \in L^p$

<u>Theorem</u> If $\phi \in (L^1(X))'$ $\quad X$ σ-finite

Then $\exists\, u \in L^\infty(X)$ s.t.

$$\phi(f) = \int u f$$

<u>Pf</u> σ finite $\quad X = \bigcup_{1 \le n < N} X_n \qquad N \in \mathbb{N} \cup \{\infty\}$

and each X_n has finite measure $\mu(X_n) < \infty$

We can assume $\{X_n\}$ increasing.

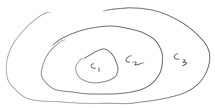
There is a $w \in L^2(X)$ st $\forall n\ \exists\, C_n > 0$

st. $w(x) > C_n > 0 \quad \forall x \in X.$

If $\mu(X) = \infty \qquad \{c_n\} \qquad c_n > 0$

$$c_1^2 \mu(X_1) + \sum_{2 \le n < \infty} c_n^2 \mu(X_n \setminus X_{n-1}) < \infty$$

$$w(x) = \begin{cases} c_1 & \text{if } x \in X_1 \\ c_{n+1} & \text{if } x \in X_{n+1} \setminus X_n \end{cases}$$



$\phi : L^1(X) \to \mathbb{R}$

$f \in L^2 \longrightarrow \mathbb{R} \qquad f \to \langle \phi, f \overset{L^1}{w} \rangle_{(L^1)' \times L^1} = \int f g$

$\Rightarrow \exists\, g \in L^2$ st $\langle \phi, f w \rangle_{(L^1)' \times L^1} = \int f g$

$u = \frac{g}{w}$

$$\left| \int f g \right| = \left| \langle \phi, f w \rangle_{(L^1)' \times L^1} \right| \le |\phi|_{(L^1)'}\, |f w|_{L^1}$$

Want to show $u \in L^\infty$. We will show $|u|_{L^\infty} = |\phi|_{(L^1)'}$

First we show $|u|_{L^\infty} \le |\phi|_{(L^1)'}$.

Let $C > |\phi|_{(L^1)'}$

$$A_\pm = \{ x : \pm u(x) > C \}$$

We show $\mu(A_\pm) = 0 \qquad \mu(A_+) = 0$

$$C \int_{A_+ \cap X_n} w \le \int_{A_+ \cap X_n} w u = \int_{A_+ \cap X_n} g =$$

$$= \int 1_{A_+ \cap X_n}\, g = \langle \phi, 1_{A_+ \cap X_n} w \rangle_{(L^1)' \times L^1}$$

$$\le |\phi|_{(L^1)'} \int_{A_+ \cap X_n} w$$

$\Rightarrow \mu(A_+ \cap X_n) = 0 \qquad \forall n.$

$\Rightarrow \mu(A_+) = 0$

$|u|_{L^\infty} \le |\phi|_{(L^1)'}$

Next we claim

$\langle \phi, f \rangle_{(L^1)' \times L^1} = \int f u \qquad \forall f \in L^1$

$f \in L^1 \cap L^2 \qquad 1_{X_n} f \qquad \langle \phi, F w \rangle = \int F g \quad \forall F \in L$

$$\langle \phi, 1_{X_n} f \rangle = \langle \phi, 1_{X_n} \frac{f}{w}\, w \rangle = \int 1_{X_n} \frac{f}{w} \overset{u}{g} =$$

$$= \int 1_{X_n} f u$$

$1_{X_n} f \longrightarrow f$ in L^1 by dominated convergence

$\langle \phi, f \rangle = \int f u \qquad \forall f \in L^1 \cap L^2$

$\therefore$ But $L^1 \cap L^2$ is dense in L^1

$f_n \longrightarrow f \in L^1 \quad \forall f$

$\langle \phi, f_n \rangle = \int f_n u$

$\downarrow$

$\langle \phi, f \rangle = \int f u \le |f|_{L^1}\, |u|_{L^\infty}$

$\Rightarrow |\phi|_{(L^1)'} \le |u|_{L^\infty} \Rightarrow$ they are equals.