23 Novembre

Espansione di Hermite $\qquad R(x) = \frac{P(x)}{Q(x)} \qquad \deg Q > \deg P$

$\frac{1}{x^3-3x^2+2x} = \qquad \deg 1 = 0 \qquad \deg \text{denom} = 3$

$= \frac{1}{x(x-1)(x-2)}$

$P(x)$ polinomio di grado n

$P(x) = a_n x^n + \dots + a_0$

e supponiamo che le radici siano

x_1 di molteplicità $m_1 (\geq 1)$

x_2 $\dots\dots$ $m_2 (\geq 1)$

$\vdots$

x_k $\dots\dots$ $m_k (\geq 1)$

allora $m_1 + \dots + m_k = n$ e inoltre

$P(x) = a_n (x-x_1)^{m_1} \cdots (x-x_k)^{m_k}$

Esistono costanti A, B, C t.c.

$R(x) = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \qquad \forall x$ diverso da $0, 1, 2$

Moltiplichiamo per x

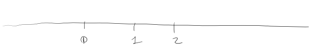
$R(x)\,x = \frac{1}{(x-1)(x-2)} = A + x\left[\frac{B}{x-1} + \frac{C}{x-2}\right]$ Vale per ogni $x \neq 0, 1, 2$

L'ultima uguaglianza vale anche in $x = 0$

$R(x)\,x\big|_{x=0} = \frac{1}{(x-1)(x-2)}\Big|_{x=0} = \frac{1}{(-1)(-2)} = \boxed{\frac{1}{2} = A}$

$R(x) = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \qquad \cdot (x-1)$

$R(x)(x-1) = \frac{1}{x(x-2)} = B + (x-1)\left[\frac{A}{x} + \frac{C}{x-2}\right]$

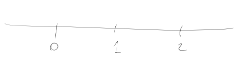


L'ultima uguaglianza vale anche per $x = 1$

$R(x)(x-1)\big|_{x=1} = \frac{1}{x(x-2)}\Big|_{x=1} = \frac{1}{1(-1)} = -1 = B$

$R(x) = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \qquad \cdot (x-2)$

$R(x)(x-2) = \frac{1}{x(x-1)} = C + (x-2)\left[\frac{A}{x} + \frac{B}{x-1}\right]$



Uguaglianza vale anche in $x = 2$

$R(x)(x-2)\big|_{x=2} = \frac{1}{2} = C$

$A = \frac{1}{2} = C \qquad B = -1$

$\frac{1}{x(x-1)(x-2)} = \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}$

$$\frac{1}{x(x-1)(x-2)} = \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}$$

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \quad \cdot x$$

$$\frac{1}{(x-1)(x-2)} = A + \frac{x}{x-1} B + \frac{x}{x-2} C \qquad x \to +\infty$$

$$\downarrow x \to +\infty \qquad \downarrow 1 \qquad \downarrow 1$$

$$0 = A + B + C$$

$$\int \frac{1}{x^3-3x^2+2x}\,dx = \int \left[\frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x-2}\,dx$$

$$= \frac{1}{2}\lg|x| - \lg|x-1| + \frac{1}{2}\lg|x-2| + C$$

$$\lim_{x \to 0} \frac{\sin(2x) - \tan(2x + x^3)}{x\,(1 - \cos^2 x)} =$$

$$\text{den} = x\,(1 - \cos^2 x) = x\,\underbrace{(1 + \cos x)}_{\to 2}\,\underbrace{\frac{(1 - \cos x)}{x^2}}_{\to \frac{1}{2}}\,x^2$$

$$= x^3\,(1 + o(1))$$

$$\lim_{x \to 0} \frac{\sin(2x) - \tan(2x + x^3)}{x^3} = f(x)$$

$$\text{num} = \sin(2x) - \frac{\sin(2x + x^3)}{\cos(2x + x^3)} = \sin(2x) - \frac{\sin(2x)\cos(x^3)}{\cos(2x + x^3)} - \frac{\sin(x^3)\cos(2x)}{\cos(2x + x^3)}$$

$$f(x) = \underbrace{\frac{\sin(2x)}{2x}}_{\to 1}\,2\,\frac{1 - \frac{\cos(x^3)}{\cos(2x + \cancel{x}^3)}}{x^2} - \underbrace{\frac{\sin(x^3)}{x^3}}_{\to 1}\,\underbrace{\frac{\cos(2x)}{\cos(2x + \cancel{x}^3)}}_{\to 1}$$

$$\text{mio limite} = 2\,\underbrace{\lim_{x \to 0} \frac{1 - \frac{\cos(x^3)}{\cos(2x + x^3)}}{x^2}}_{-2} - 1 = -5$$

$$\frac{1 - \frac{\cos(x^3)}{\cos(2x + x^3)}}{\cancel{x}^2} = \underbrace{\frac{1}{\cos(2x + x^3)}}_{\to 1}\,\frac{\cos(2x + x^3) - \cos(x^3)}{x^2}$$

$$\frac{\cos(2x + x^3) - \cos(x^3)}{x^2} = \frac{\cos(2x + x^3) - 1}{x^2} + \underbrace{\frac{1 - \cos(x^3)}{x^2\,x^4}}_{\to \frac{1}{2}}\,\underbrace{x^4}_{\to 0}$$

$$\underbrace{\frac{\cos(2x + x^3) - 1}{(2x + x^3)^2}}_{\to -\frac{1}{2}}\,\underbrace{\frac{(2x + x^3)^2}{x^2}}_{\to 4} \longrightarrow -2$$

$$\lim_{x\to 0} \frac{\sin(2x)-\tan(2x+x^3)}{x^3} =$$

$$= \lim_{x\to 0}\left[\underbrace{\frac{\sin(2x)-\tan(2x)}{x^3}}_{\to -4} + \underbrace{\frac{\tan(2x)-\tan(2x+x^3)}{x^3}}_{\to -1}\right]$$

$$\frac{\sin(2x) - \frac{\sin(2x)}{\cos(2x)}}{x^3} = \frac{\sin(2x)}{2x}\, 2\, \frac{1-\frac{1}{\cos(2x)}}{x^2}$$

$$= \underbrace{\frac{\sin(2x)}{2x}}_{\to 1}\, 2\, \underbrace{\frac{1}{\cos(2x)}}_{\to 1}\, \underbrace{\frac{\cos(2x)-1}{4x^2}}_{\to -\frac{1}{2}}\, 4 \longrightarrow -4$$

$$\frac{\tan(2x)-\tan(2x+x^3)}{x^3} =$$

$$\tan(2x+x^3) = \frac{\sin(2x+x^3)}{\cos(2x+x^3)} = \frac{\sin(2x)\cos(x^3)+\cos(2x)\sin(x^3)}{\cos(2x)\cos(x^3)-\sin(2x)\sin(x^3)}$$

$$\Rightarrow \cdot \frac{\frac{1}{\cos(2x)\cos(x^3)}}{\frac{1}{\cos(2x)\cos(x^3)}}$$

$$= \frac{\tan(2x)+\tan(x^3)}{1-\tan(2x)\tan(x^3)}$$

$$\frac{\tan(2x) - \frac{\tan(2x)+\tan(x^3)}{1-\tan(2x)\tan(x^3)}}{x^3} =$$

$$= \underbrace{\frac{\tan(2x)}{2x}}_{\to 1}\, 2\, \frac{1-\frac{1}{1-\tan(2x)\tan(x^3)}}{x^2} - \underbrace{\frac{\tan(x^3)}{x^3}}_{\to 1}\, \underbrace{\frac{1}{1-\tan(2x)\tan(x^3)}}_{\to 1}$$

$$= (2+o(1))\, \underbrace{\frac{1}{1-\tan(2x)\tan(x^3)}}_{\to 1}\, \underbrace{\frac{\cancel{1}-\tan(2x)\tan(x^3)\cancel{-1}}{x^2}}_{\to 0} - 1 + o(1)$$

$$\longrightarrow -1$$

$$\frac{\sin(2x) - \tan(2x+x^3)}{x^3}$$

$$\sin(y) = y - \frac{y^3}{6} + o(y^3)$$

$$\sin(2x) = 2x - \frac{8}{6}x^3 + o(x^3)$$

$$\tan'(x) = 1 + \tan^2(x) \qquad \tan'(0) = 1$$

$$\tan''(x) = 2\tan(x)\left(1 + \tan^2(x)\right)$$

$$\tan'''(0) = 2\tan'(0) = 2$$

$$\tan(y) = y + \frac{2}{6}y^3 + o(y^3) = y + \frac{1}{3}y^3 + o(y^3)$$

$$\tan(2x+x^3) = 2x + x^3 + \frac{1}{3}\left(2x+x^3\right)^3 + o(x^3)$$

$$= 2x + x^3\left[1 + \frac{8}{3}\right] + o(x^3)$$

$$\sin(2x) - \tan(2x+x^3) = \cancel{2x} - \frac{4}{3}x^3 - \cancel{2x} - \left(1 + \frac{8}{3}\right)x^3 + o(x^3)$$

$$= -5x^3 + o(x^3)$$

$$\lim_{x \to 0} \frac{\sin(2x) - \tan(2x + x^3)}{x^3} =$$

$$= \lim_{x \to 0} \frac{2\cos(2x) - (1 + \tan^2(2x+x^3))(2+3x^2)}{3x^2}$$

$$= \frac{2}{3} \lim_{x \to 0} \left[\frac{\cos(2x) - 1}{x^2} - \frac{\tan^2(2x+x^3)}{x^2} \right] - \underbrace{\lim_{x \to 0} \frac{(1+\tan^2(2x+x^3))\, 3\cancel{x^2}}{3\cancel{x^2}}}_{\to 1}$$

$$= \frac{2}{3} \lim_{x \to 0} \underbrace{\frac{\cos(2x) - 1}{4x^2}}_{\to -\frac{1}{2}} 4 - \frac{2}{3} \lim_{x \to 0} \underbrace{\frac{\tan^2(2x+x^3)}{4x^2}}_{\to 1} 4 - 1$$

$$= -\frac{4}{3} - \frac{8}{3} - 1 = -5$$

$$\lim_{y \to 0} \frac{\tan y}{y} = 1$$

$$\lim_{x \to 0} \frac{\tan^2(2x+x^3)}{x^2} = \lim_{x \to 0} \underbrace{\frac{\tan^2(2x+x^3)}{(2x+x^3)^2}}_{\to 1} \underbrace{\frac{(2x+x^3)^2}{x^2}}_{\to 4}$$

$$\lim_{x \to 0} \frac{(2x+x^3)^2}{x^2} = \lim_{x \to 0} \frac{\cancel{x^2}(2+x^2)^2}{\cancel{x^2}} = 4$$

$$= \lim_{x \to 0} \frac{4x^2 + \cancel{4x^4 + x^6}}{x^2} = \lim_{x \to 0} 4 = 4$$