29 November

$(\ell^1(\mathbb{N}))' = \ell^\infty(\mathbb{N})$

$c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N})$

$c_0(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{R} : \lim_{n\to+\infty} f(n) = 0 \}$

$(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$

$\ell^1(\mathbb{N}) \subseteq (c_0(\mathbb{N}))'$

$f \in \ell^1 \qquad g \in c_0(\mathbb{N})$

$g \to \langle f, g \rangle_{\ell^1 \times c_0} = \sum_{n=1}^{\infty} f(n)\, g(n)$ is an element of $(c_0(\mathbb{N}))'$

$J : \ell^1(\mathbb{N}) \hookrightarrow (\ell^\infty(\mathbb{N}))' \subset (c_0(\mathbb{N}))'$

Assume by contradiction that

$\phi \in (c_0)' \setminus \ell^1$

$u : \mathbb{N} \longrightarrow \mathbb{R}$

$u(n) = \phi(e_n) \qquad e_n \in c_0$

$$e_n(m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

We will show $u \in \ell^1$. Once we have this

then $\phi = \langle u, \cdot \rangle_{\ell^1 \times c_0}$

because $c_0 = \overline{\text{Span}\{\text{finite linear combinations of } e_n\text{'s}\}}$

$\phi\left(\sum_{j=1}^{k} \alpha_j e_{n_j}\right) = \sum_{j=1}^{k} \alpha_j \underbrace{\phi(e_{n_j})}_{u(n_j)} = \langle u, \sum_{j=1}^{k} \alpha_j e_{n_j} \rangle_{\ell^1 \times c_0}$

$\phi = \langle u, \cdot \rangle$ as functionals in c_0

Since $u \notin \ell^1(\mathbb{N})$

$\Rightarrow \sum_{j=1}^{\infty} |u(j)| = +\infty$

$\Rightarrow \forall\ M > 0\ \exists\ N$ s.t.

$M \leq \sum_{j=1}^{N} |u(j)| = \sum_{j=1}^{N} \text{sign}(u(j))\, \overset{\phi(e_j)}{u(j)} =$

$= \phi\left(\sum_{j=1}^{N} \text{sign}(u(j))\, e_j\right)$

$= \langle \sum_{j=1}^{N} \text{sign}(u(j))\, e_j, \phi \rangle_{c_0 \times c_0'}$

$\leq \underbrace{\left| \sum_{j=1}^{N} \text{sign}(u(j))\, e_j \right|_{\ell^\infty}}_{\leq 1} |\phi|_{c_0'}$

$\Rightarrow M \leq |\phi|_{c_0'} \quad \forall\ M > 0$ a contradiction

$\Rightarrow u(n) = \phi(e_n) \quad u \in \ell^1$.

Theorem (Young's convolution inequality)

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ $\quad p, q \in [1, \infty]$

Set

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy$$

Then

$$\|f * g\|_{L^r(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$$

when $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. $\qquad \mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$

$$L^p(\mathbb{R}) \times L^q(\mathbb{R}^d) \longrightarrow L^r(\mathbb{R}^d)$$
$$(f, g) \longrightarrow f * g$$

Pf $\quad r = \infty$ is just Hölder inequality

$r < \infty \Rightarrow p < \infty$, $q < \infty$

$$\left| \int f g \, dx \right| \le |f|_p \, |g|_{p'}$$

$h \in L^{r'}(\mathbb{R}^d)$

$$\left| \int h(x)\, f * g(x)\, dx \right| \le |h|_{L^{r'}} |f|_{L^p} |g|_{L^q}$$

$$|h|_{L^{r'}} = |f|_{L^p} = |g|_{L^q} = 1$$

$$I(f, g, h) = \int h(x) \int f(y) g(x-y)\, dy\, dx$$

$$|I(f, g, h)| \le I(|f|, |g|, |h|)$$

$f \ge 0$, $g \ge 0$, $h \ge 0$

$I(f, g, h) = \int$ $\qquad y \to -y \qquad \le 1$

$$\int_{\mathbb{R}^d} f(x-y) g(y)\, dy \qquad z = x - y \qquad -y = x - z$$

$$= \int_{\mathbb{R}^d} f(z) g(x-z) \underbrace{|\det \text{Jacobian}|}_{1} dz$$

$f * g = g * f$

Our inequality is equivalent to

$I(f, g, h) \le 1 \qquad \forall f \ge 0, g \ge 0, h \ge 0$

$|f|_{L^p} = 1$, $|g|_{L^q} = 1$, $|h|_{L^{r'}} = 1$

$$I(f, g, h) = \int f(y) g(x-y) h(x)\, dx\, dy$$

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} = 1 + 1 - \frac{1}{r'}$$

$$2 = \frac{1}{r'} + \frac{1}{p} + \frac{1}{q}$$

$$\left(2 - \frac{1}{p} - \frac{1}{q}\right) r' = 1 \qquad \left(2 - \frac{1}{p} - \frac{1}{r'}\right) q = 1$$

$$\left(2 - \frac{1}{r'} - \frac{1}{q}\right) p = 1$$

$$\left(1 - \frac{1}{p}\right) r' + \left(1 - \frac{1}{q}\right) r' = 1$$

$$\left(1 - \frac{1}{p}\right) q + \left(1 - \frac{1}{r'}\right) q = 1$$

$$\left(1 - \frac{1}{r'}\right) p + \left(1 - \frac{1}{q}\right) p = 1$$

$$I(f, g, h) = \int f(y) g(x-y) h(x)\, dx\, dy =$$

$$= \int \left(f^p(y) g^q(x-y)\right)^{\frac{1}{r}} \left(f^p(y) h^{r'}(x)\right)^{\frac{1}{q'}} \left(g^q(x-y) h^{r'}(x)\right)^{\frac{1}{p'}} dx\, dy$$

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} = 2 - \frac{1}{p'} - \frac{1}{q'}$$

$$\frac{1}{r} + \frac{1}{p'} + \frac{1}{q'} = 1$$

$$\le \left(\int dy\, f^p(y) \underbrace{\int g^q(x-y)\, dx}_{1} \right)^{\frac{1}{r}} \underbrace{\left(\int f^p(y) h^{r'}(x)\, dx\, dy \right)^{\frac{1}{q'}}}_{\left(\int f^p(y) dy \int h^{r'}(x) dx\right)^{\frac{1}{q'}} = 1} \left(\int g^q(x-y) h^{r'}(x)\, dx\, dy \right)^{\frac{1}{p'}} = 1$$

Prop: $f \in C^k_c(\mathbb{R}^d)$ $g \in L^1_{loc}(\mathbb{R}^d)$

Then $f * g \in C^k(\mathbb{R}^d)$

$\nabla^j (f * g) = \nabla^j f * g \quad \forall \ j \leq k.$

Pf $F(x,y) = f(x-y)\, g(y)$

$F(x,\cdot) = f(x-\cdot)\, g(\cdot) \in L^1(\mathbb{R}^d) \quad \forall x$

$f(x-y)\, g(y)$ has support in $-\text{supp} f + x$

$K = \text{supp} f$ $\quad f(x-y) \neq 0$

$x - y \in K$

$-y \in K - x$

$y \in -K + x$

there will be $\hat{K}$

$$\int \underbrace{f(x-y)\, g(y)}_{F(x,y)} \, dy$$

$x_n \to x$ compact

For $\hat{K} \supseteq \bigcup_{n=1}^{\infty} (-K + x_n)$

$F(x,y) = \chi_{\hat{K}}(y)\, F(x,y)$

$F(x_n, y) = \chi_{\hat{K}}(y)\, F(x_n, y)$

$|F(x_n,y)| \leq \chi_{\hat{K}}(y)\, |f(x_n - y)\, g(y)| \leq$

$\leq \chi_{\hat{K}}(y)\, \|f\|_{L^\infty} |g(y)| \in L^1(\mathbb{R}_y)$

$$\lim_{n \to +\infty} \int F(x_n, y)\, dy = \lim_{n\to+\infty} \int f(x_n - y)\, g(y)\, dy$$

$$= \int \lim_{n \to +\infty} f(x_n - y)\, g(y)\, dy = \int f(x-y) g(y)\, dy$$

$f \in C^0_c$ $\quad g \in L^1_{loc} \Rightarrow f * g \in C^0$

$k = 1$ $\quad f \in C^1_c(\mathbb{R}^d)$

$f(x+h-y) - f(x-y) - h \cdot \nabla f(x-y) = h \cdot I(x-y, h)$

$$I(x-y,h) = \int_0^1 [\nabla f(x+sh-y) - \nabla f(x-y)]\, ds$$

$$f(x+h-y) - f(x-y) = \int_0^1 \frac{d}{ds}[f(x+sh-y)]\, ds$$

$\nabla f \in C^0_c(\mathbb{R}^d, \mathbb{R}^d) \Rightarrow \nabla f$ is uniformly cont in $\mathbb{R}^d$

$\Rightarrow \quad |I(z,h)| \leq o(1)$ $\quad o(1)$ depending only on h

$o(1) \xrightarrow{h \to 0} 0$

$|f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)| \leq |h| \ o(1)$

For x fixed and $|h| \leq 1$. As before $\exists$ $\hat{K}$ compact

$$\left| \frac{f(x+h-y) - f(x-y) - h \cdot \nabla f(x-y)}{|h|} \right| \leq o(1)\, \chi_{\hat{K}}(y)$$

$f*g(x+h) - f*g(x) - h \cdot \nabla f * g(x) =$

$$\lim_{h\to 0} \left| \int_{\mathbb{R}^d} \frac{[f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)]}{|h|}\, g(y)\, dy \right|$$

$$\leq \lim_{h \to 0} \int_{\hat{K}} \left| \frac{f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)}{|h|} \right| |g(y)|\, dy$$

$$\leq \int \underbrace{\lim_{h\to 0} o(1)}_{0} \quad |g(y)|\, dy = 0$$

Theorem $\rho \in L^1(\mathbb{R}^d)$ $\int \rho(x)\,dx = 1$

$\rho_\varepsilon(x) \stackrel{?}{=} \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$.

Then $\forall f \in L^p(\mathbb{R}^d)$ $1 \le p < \infty$

we have $\lim_{\varepsilon \to 0^+} \rho_\varepsilon * f = f$ in $L^p(\mathbb{R}^d)$

In L^∞ is false but if we replace

$L^\infty(\mathbb{R}^d)$ with $f \in C_0^0(\mathbb{R}^d)$

$$\lim_{\varepsilon \to 0^+} \rho_\varepsilon * f = f$$

Pf Let $f \in C_c^0(\mathbb{R}^d)$ $z = \frac{y}{\varepsilon}$ $y = \varepsilon z$

$$\rho_\varepsilon * f(x) - f(x) = \int \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) f(x-y)\,dy - f(x)$$

$$= \int \rho(z) f(x - \varepsilon z)\,dz - \int \rho(z)\,dz\, f(x)$$

$$= \int \rho(y) \left[f(x - \varepsilon y) - f(x)\right] dy$$

$$|\rho_\varepsilon * f - f|_{L^p} = \left| \int \rho(y) \left[f(x - \varepsilon y) - f(x)\right] dy \right|_{L^p}$$

$$\le \int dy |\rho(y)| \; |f(\cdot - \varepsilon y) - f|_{L^p} =$$

$\Delta(y) = |f(\cdot - y) - f(\cdot)|_{L^p}$

$$= \int dy\, |\rho(y)| \; \Delta(\varepsilon y)$$

$\lim_{y \to 0} \Delta(y) = 0$

$$\lim_{y \to 0} \int_K |f(x-y) - f(x)|^p \,dx = 0$$

$\le |\rho(y)| \cdot 2 |f|_{L^p}$

$$|\rho_\varepsilon * f - f|_{L^p} \le \int |\rho(y)| \; \Delta(\varepsilon y)\,dy$$

$$|\Delta(y)| = |f(\cdot - y) - f(\cdot)|_{L^p} \le 2\,|f|_{L^p}$$

$$\Rightarrow \lim_{\varepsilon \to 0} \int |\rho(y)|\, \Delta(\varepsilon y)\,dy = \int |\rho(y)| \underbrace{\lim_{\varepsilon \to 0} \Delta(\varepsilon y)}_{0} dy$$

So we proved $\rho_\varepsilon * f \to f$ in L^p $\forall f \in C_c^0(\mathbb{R}^d)$