Esercizi svolti su integrazione

1. Calcolare i seguenti integrali indefiniti "immediati":

$$\int \frac{\log^3 x}{x} dx \qquad \int \frac{dx}{x\log^3 x} \qquad \int x^2 e^{x^3}\,dx$$

$$\int \frac{\text{arctg}^4 x}{1+x^2} dx \qquad \int \frac{x}{\sqrt{(1-x^2)^3}} dx \qquad \int \frac{1+\cos x}{x+\sin x} dx$$

$$\int \frac{x^3}{1+x^8} dx \qquad \int \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx \qquad \int \frac{\sin 2x}{1+\sin^2 x} dx.$$

2. Calcolare i seguenti integrali con il metodo di decomposizione in somma:

$$\int (4x^4+3x^2+5x)dx \qquad \int \frac{x^3+x+1}{x^2+1} dx \qquad \int (1+2x^3)^2 dx$$

$$\int (1+\cos x)^2 dx \qquad \int \text{ctg}^2 x\,dx \qquad \int \cos^3 x\,dx$$

$$\int \frac{x+1}{x(1+x^2)} dx \qquad \int \frac{1}{1+e^{2x}} dx \qquad \int \frac{1}{x^2(2+x^2)} dx.$$

3. Calcolare i seguenti integrali con la tecnica di integrazione per parti:

$$\int x^3 \text{sh}x\,dx \qquad \int x^2 \text{ch}(3x)dx \qquad \int x^4\cos(2x)dx$$

$$\int e^{2x}\sin(3x)dx \qquad \int e^{-3x}\cos(2x)dx \qquad \int \arcsin x\,dx$$

$$\int x^3\log x\,dx \qquad \int \sqrt[3]{x}\log x\,dx \qquad \int \frac{\log x}{\sqrt[4]{x}} dx$$

$$\int \log^3 x\,dx \qquad \int x^3\sin(x^2)dx \qquad \int x^5 e^{-x^3} dx$$

$$\int \frac{x^3}{\sqrt{1-x^2}} dx \qquad \int x^3\log^2 x\,dx \qquad \int x\sin^2 x\,dx$$

$$\int \sqrt{1+x^2}dx \qquad \int \cos^4 x\,dx \qquad \int \log(\sqrt{x+1}+\sqrt{x-1})dx.$$

4. Calcolare i seguenti integrali con la tecnica di integrazione per sostituzione:

$$\int \text{ctg}x\,dx \qquad \int_0^1 x(3x^2+1)^7 dx \qquad \int_{-1}^0 \sqrt[3]{(2x+1)^2}\,dx$$

$$\int_1^2 \frac{1}{\sqrt{(3x-2)^3}} dx \qquad \int \frac{x^5}{\sqrt{x^3-1}} dx \qquad \int_0^1 \sqrt{e^x-1}dx$$

$$\int \frac{dx}{x(2+\log^2 x)} \qquad \int \frac{1}{\sqrt{(1-x^2)^3}} dx \qquad \int \frac{1}{\sqrt{(1+x^2)^3}} dx$$

$$\int \frac{1}{\sqrt{(x^2-1)^3}} dx \qquad \int \frac{1}{x^2\sqrt{1+x^2}} dx \qquad \int \frac{x^2}{\sqrt{x^2-1}} dx.$$

5. Calcolare i seguenti integrali di funzioni razionali:

$$\int \frac{x^3}{x^2+7x+12}dx \qquad \int \frac{x^3+x^2-x}{x^2+x-6}dx \qquad \int \frac{x^2+x+2}{x^2+4}dx$$
$$\int \frac{x^2-2x-1}{x^2-4x+4}dx \qquad \int \frac{x^2-10x+10}{x^3+2x^2+5x}dx \qquad \int \frac{3x^2-x}{(x+1)^2(x+2)}dx$$
$$\int \frac{dx}{x^4-1} \qquad \int \frac{x^3-2}{x^2(x^2+1)}dx \qquad \int \frac{dx}{x^4-x}.$$

6. Calcolare i seguenti integrali riconducendoli a integrali di funzioni razionali:

$$\int \frac{e^{3x}+2e^{2x}+3e^x}{e^x+1}dx \qquad \int \frac{e^x}{e^{2x}-5e^x+6}dx \qquad \int \frac{1-3x}{\sqrt{x}-2}dx$$
$$\int \frac{dx}{x(x^{\frac{1}{4}}-1)} \qquad \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \qquad \int \mathrm{ctg}^5 x dx$$
$$\int \frac{dx}{2\sin x+\cos x-1} \qquad \int \frac{\sin 2x}{6\sin x-\cos 2x+5}dx \qquad \int \frac{1+\cos x}{1+\sin x}dx$$
$$\int \frac{\mathrm{tg}^3 x+\mathrm{tg} x}{\mathrm{tg} x+4}dx \qquad \int \frac{\mathrm{tg} x}{\sin^2 x+1}dx \qquad \int x\log(1-2x-3x^2)dx.$$

7. Sia $g(x)=(1+x^2)e^{-|x+1|}$. Si calcoli la primitiva G di g in $\mathbb{R}$ tale che $\lim_{x\to+\infty} G(x)=3$.
8. Calcolare tutte le primitive di $f(x)=\frac{xe^{|x+1|}}{e^x}$ in $\mathbb{R}$. Calcolare poi la primitiva F tale che $F(0)=0$.
9. Sia $f(x)=\begin{cases} \sqrt{|x|} & x<1 \\ \frac{1}{4+x^2} & x\geq 1. \end{cases}$ Determinare la primitiva generalizzata di f che si annulla per $x=0$.
10. Sia
$$f(x)=\begin{cases} x^3\sin(\pi x^2) & x\leq 1 \\ x^2-8x+16 & x>1. \end{cases}$$
Determinare la primitiva generalizzata di f che si annulla in $x_0=0$.
11. a) Si calcoli $I_n=\int_1^2 \frac{nx}{(x^2+\frac{1}{n})^n}dx$, $\forall n\in\mathbb{N}$, $n>0$. b) Si calcoli $\lim_{n\to+\infty} I_n$.
12. Si calcoli la primitiva che si annulla in $x_0=0$ della seguente funzione definita su $(-\infty,3)$: $f(x)=\frac{x+2}{(|x|+3)(x-3)}$.
13. Determinare la primitiva che si annulla in $x_0=0$ della funzione $f(x)=\frac{8}{(|x|+2)^2}$ definita in $\mathbb{R}$.
14. Data la funzione
$$f(x)=\begin{cases} x|6x-2|+1 & x\leq 0 \\ (x+1)e^{x/2} & x>0, \end{cases}$$
a) determinare la primitiva F di f tale che $F(-1)=0$; b) calcolare $\int_{-2}^1 f(t)dt$; c) calcolare $\int_{-1/3}^2 f(3t)dt$.

Svolgimento esercizi

1.

$$\int \frac{\log^3 x}{x}dx = \frac{\log^4 x}{4} + c \qquad \int \frac{dx}{x\log^3 x} = -\frac{1}{2\log^2 x} + c \qquad \int x^2 e^{x^3} dx = \frac{e^{x^3}}{3} + c$$

$$\int \frac{\text{arctg}^4 x}{1+x^2}dx = \frac{\text{arctg}^5 x}{5} + c \quad \int \frac{x}{\sqrt{(1-x^2)^3}}dx = \frac{1}{\sqrt{1-x^2}} + c \quad \int \frac{1+\cos x}{x+\sin x}dx = \log|x+\sin x| + c$$

$$\int \frac{x^3}{1+x^8}dx = \frac{\text{arctg} x^4}{4} + c \quad \int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}dx = \frac{\arcsin^3 x}{3} + c \quad \int \frac{\sin 2x}{1+\sin^2 x}dx = \log(1+\sin^2 x) + c.$$

2.

$$\int (4x^4 + 3x^2 + 5x)dx = \frac{4}{5}x^5 + x^3 + \frac{5}{2}x^2 + c \qquad \int \frac{x^3+x+1}{x^2+1}dx = \frac{x^2}{2} + \text{arctg} x + c$$

$$\int (1+2x^3)^2 dx = \frac{4}{7}x^7 + x^4 + x + c \qquad \int (1+\cos x)^2 dx = \frac{3}{2}x + 2\sin x + \frac{\sin x\cos x}{2} + c$$

$$\int \text{ctg}^2 x\, dx = -x - \text{ctg} x + c \qquad \int \cos^3 x\, dx = \sin x - \frac{\sin^3 x}{3} + c$$

$$\int \frac{x+1}{x(1+x^2)}dx = \text{arctg} x + \log|x| - \frac{\log(1+x^2)}{2} + c \quad \int \frac{1}{1+e^{2x}}dx = x - \frac{1}{2}\log\left|1+e^{2x}\right| + c$$

$$\int \frac{1}{x^2(2+x^2)}dx = -\frac{1}{2x} - \frac{\sqrt{2}}{4}\text{arctg}\frac{x}{\sqrt{2}} + c.$$

3.

$$\int x^3 \text{sh} x\, dx = (x^3+6x)\text{ch} x - 3(x^2+2)\text{sh} x + c \quad \int x^2 \text{ch}(3x)dx = \frac{1}{27}\text{sh}(3x)(9x^2+2) - \frac{2}{9}x\,\text{ch}(3x) + c$$

$$\int x^4\cos(2x)dx = \frac{1}{2}\sin(2x)(x^4 - 3x^2 + \frac{3}{2}) + \cos(2x)(x^3 - \frac{3}{2}x) + c$$

$$\int e^{2x}\sin(3x)dx = \frac{2}{13}e^{2x}[\sin(3x) - \frac{3}{2}\cos(3x)] \quad \int e^{-3x}\cos(2x)dx = -\frac{3}{13}e^{-3x}[\cos(2x) - \frac{2}{3}\sin(3x)]$$

$$\int \arcsin x\, dx = x\arcsin x + \sqrt{1-x^2} + c \qquad \int x^3\log x\, dx = \frac{x^4\log x}{4} - \frac{x^4}{16} + c$$

$$\int \sqrt[3]{x}\log x\, dx = \frac{3}{4}x^{4/3}\log x - \frac{9}{16}x^{4/3} + c \qquad \int \frac{\log x}{\sqrt[4]{x}}dx = \frac{4}{3}x^{3/4}\log x - \frac{16}{9}x^{3/4} + c$$

$$\int \log^3 x\, dx = x(\log^3 x - 3\log^2 x + 6\log x - 6) \quad \int x^3\sin(x^2)dx = \frac{-x^2\cos(x^2) + \sin(x^2)}{2} + c$$

$$\int x^5 e^{-x^3}dx = \frac{e^{x^3}(x^3-1)}{3} + c \qquad \int \frac{x^3}{\sqrt{1-x^2}}dx = -x^2\sqrt{1-x^2} - \frac{2}{3}(1-x^2)^{3/2} + c$$

$$\int x^3\log^2 x\, dx = \frac{x^4}{4}(\log^2 x - \frac{\log x}{2} + \frac{1}{8}) + c \qquad \int x\sin^2 x\, dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x) + c$$

$$\int \sqrt{1+x^2}dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\text{settsh} x + c \qquad \int \cos^4 x\, dx = \frac{\sin x\cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x) + c$$

$$\int \log(\sqrt{x+1} + \sqrt{x-1})dx = x\log(\sqrt{x+1} + \sqrt{x-1}) - \frac{\sqrt{x^2-1}}{2} + c.$$

4.

$$\int \mathrm{ctg}x\,dx = \log|\sin x| + c$$

$$\int_{-1}^{0} \sqrt[3]{(2x+1)^2}\,dx = \frac{1}{2}\int_{-1}^{1} t^{2/3}dt = \frac{3}{5}$$

$$\int \frac{x^5}{\sqrt{x^3-1}}dx = \frac{2}{9}(x^3-1)^{3/2} + \frac{2}{3}(x^3-1)^{1/2}$$

$$\int \frac{dx}{x(2+\log^2 x)} = \frac{1}{\sqrt{2}}\mathrm{arctg}\left(\frac{\log x}{\sqrt{2}}\right) + c$$

$$\int \frac{1}{\sqrt{(1+x^2)^3}}dx = \frac{x}{\sqrt{1+x^2}} + c$$

$$\int \frac{1}{x^2\sqrt{1+x^2}}dx = -\frac{\sqrt{1+x^2}}{x} + c$$

$$\int_0^1 x(3x^2+1)^7dx = \frac{1}{6}\int_1^4 t^7dt = \frac{4^8-1}{48}$$

$$\int_1^2 \frac{1}{\sqrt{(3x-2)^3}}dx = \frac{1}{3}\int_1^4 t^{-3/2}dt = \frac{1}{3}$$

$$\int_0^1 \sqrt{e^x-1}dx = 2\sqrt{e-1} - 2\mathrm{arctg}\sqrt{e-1}$$

$$\int \frac{1}{\sqrt{(1-x^2)^3}}dx = \frac{x}{\sqrt{1-x^2}} + c$$

$$\int \frac{1}{\sqrt{(x^2-1)^3}}dx = -\frac{x}{\sqrt{x^2-1}} + c$$

$$\int \frac{x^2}{\sqrt{x^2-1}}dx = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\mathrm{settch}\,x + c.$$

5.

$$\int \frac{x^3}{x^2+7x+12}dx = \frac{x^2}{2} - 7x + 64\log|x+4| - 27\log|x+3| + c$$

$$\int \frac{x^3+x^2-x}{x^2+x-6}dx = \frac{x^2}{2} + 2\log|x-2| + 3\log|x+3| + c$$

$$\int \frac{x^2+x+2}{x^2+4}dx = x + \frac{1}{2}\log(x^2+4) - \mathrm{arctg}\frac{x}{2} + c$$

$$\int \frac{x^2-2x-1}{x^2-4x+4}dx = x + 2\log|x-2| + \frac{1}{x-2} + c$$

$$\int \frac{x^2-10x+10}{x^3+2x^2+5x}dx = 2\log|x| - \frac{1}{2}\log(x^2+2x+5) - \frac{13}{2}\mathrm{arctg}\frac{x+1}{2} + c$$

$$\int \frac{3x^2-x}{(x+1)^2(x+2)}dx = 14\log|x+2| - 11\log|x+1| - \frac{4}{x+1} + c$$

$$\int \frac{dx}{x^4-1} = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\mathrm{arctg}x + c$$

$$\int \frac{x^3-2}{x^2(x^2+1)}dx = \frac{1}{2}\log(x^2+1) + \frac{2}{x} + 2\mathrm{arctg}x + c$$

$$\int \frac{dx}{x^4-x} = \frac{1}{3}\log|x-1| - \log|x| + \frac{1}{3}\log(x^2+x+1) + c.$$

7.

$$G(x) = \begin{cases} -e^{-(x+1)}\left(x^2+2x+3\right)+3 & x \ge -1 \\ e^{x+1}\left(x^2-2x+3\right)-5 & x < -1. \end{cases}$$

8.

$$F(x) = \begin{cases} \frac{e}{2}x^2 + k & x \ge -1 \\ -\frac{e^{-2x-1}}{2}(x+\frac{1}{2}) + k + \frac{e}{4} & x < -1 \end{cases}$$

con $k \in \mathbb{R}$. La primitiva F tale che $F(0) = 0$ si ottiene per $k = 0$.

9.

$$F(x) = \begin{cases} -\frac{2}{3}(-x)^{3/2} & x < 0 \\ \frac{2}{3}x^{3/2} & 0 \le x < 1 \\ \frac{1}{2}\mathrm{arctg}\frac{x}{2} + \frac{2}{3} - \frac{1}{2}\mathrm{arctg}\frac{1}{2} & x \ge 1. \end{cases}$$

10.

$$F(x)=\begin{cases}\frac{1}{2\pi^2}\left[-\pi x^2\cos(\pi x^2)+\sin(\pi x^2)\right] & x\le 1\\ \frac{1}{3}x^3-4x^2+16x+\frac{1}{2\pi}-\frac{37}{3} & x>1.\end{cases}$$

11. a)

$$I_n=\begin{cases}\frac{n}{2-n}\left[\left(4+\frac{1}{n}\right)^{1-n}-\left(1+\frac{1}{n}\right)^{1-n}\right] & n\ne 1\\ \frac{1}{2}\log\frac{5}{2} & n=1.\end{cases}$$

b) $\lim_{n\to+\infty} I_n=\frac{1}{2e}$.

12.

$$F(x)=\begin{cases}\frac{1}{6}\log(x+3)+\frac{5}{6}\log(3-x)-\log 3 & 0\le x<3\\ \frac{5}{x-3}-\log(3-x)+\frac{5}{3}+\log 3 & x<0.\end{cases}$$

13.

$$F(x)=\begin{cases}\frac{4x}{x+2} & x\ge 0\\ \frac{4x}{2-x} & x<0.\end{cases}$$

14. a)

$$F(x)=\begin{cases}-2x^3+x^2+x-2 & x\le 0\\ 2(x-1)e^{x/2} & x>0.\end{cases}$$

b) $\int_{-2}^{1} f(t)dt=-16$. c) $\int_{-1/3}^{2} f(3t)dt=\frac{10}{3}e^3$.

6.

$$\int \frac{e^{3x}+2e^{2x}+3e^x}{e^x+1}dx=\frac{1}{2}e^{2x}+e^x+2\log(e^x+1)+c$$

$$\int \frac{e^x}{e^{2x}-5e^x+6}dx=\log\left|\frac{e^x-3}{e^x-2}\right|+c$$

$$\int \frac{1-3x}{\sqrt{x}-2}dx=-22\sqrt{x}-2x\sqrt{x}-44\log(\sqrt{x}-2)-6x+c$$

$$\int \frac{dx}{x(x^{\frac{1}{4}}-1)}=4\log\left|\frac{x^{1/4}-1}{x^{1/4}}\right|+c$$

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(1+\sqrt[6]{x})+c$$

$$\int \mathrm{ctg}^5 x dx=-\frac{1}{4\sin^4 x}+\frac{1}{\sin^2 x}+\log|\sin x|+c$$

$$\int \frac{dx}{2\sin x+\cos x-1}=\frac{1}{2}\log\left|\mathrm{tg}\frac{x}{2}\right|-\frac{1}{2}\log\left|\mathrm{tg}\frac{x}{2}-2\right|+c$$

$$\int \frac{\sin 2x}{6\sin x-\cos 2x+5}dx=2\log|\sin x+2|-\log|\sin x+1|+c$$

$$\int \frac{1+\cos x}{1+\sin x}dx=-\frac{2}{1+\mathrm{tg}\frac{x}{2}}+2\log\left|1+\mathrm{tg}\frac{x}{2}\right|-\log\left|1+\mathrm{tg}^2\frac{x}{2}\right|+c$$

$$\int \frac{\mathrm{tg}^3+\mathrm{tg}x}{\mathrm{tg}x+4}dx=\mathrm{tg}x-4\log|\mathrm{tg}x+4|+c$$

$$\int \frac{\mathrm{tg}x}{\sin^2 x+1}dx=\frac{1}{4}\log(1+2\mathrm{tg}^2 x)+c$$

$$\int x\log(1-2x-3x^2)dx=\frac{1}{2}x^2\log(1-2x-3x^2)-\frac{1}{2}x^2+\frac{1}{3}x-\frac{1}{18}\log\left|x-\frac{1}{3}\right|-\frac{1}{2}\log|x+1|+c.$$