4 dicembre

Ricordiamo che ieri abbiamo calcolato che

$$R(x) = \frac{x^2+x+1}{(x-2)^4} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{A_4}{(x-2)^4}$$

$$A_i = \frac{1}{(4-i)!}\left(\frac{d}{dx}\right)^{4-i} R(x)\,(x-2)^4 \Big|_{x=2}$$

Teorema Sia $R(z) = \frac{P(z)}{Q(z)}$ una funzione razionale

con $\deg P < \deg Q$ e sia

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

con fattorizzazione

$$Q(z) = a_n (z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$$

dove $z_1, z_2, \dots, z_k$ sono le radici di Q

di molteplicità $m_1, m_2, \dots, m_k \in \mathbb{N}$

Allora esistono delle costanti

$A_{11}, \dots, A_{1m_1}$

$A_{21}, \dots, A_{2m_2}$

$\vdots$

$A_{k1}, \dots, A_{km_k}$

tali che

$$R(z) = \frac{P(z)}{Q(z)} = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{A_{j\ell}}{(z-z_j)^{\ell}}$$

Qui

$$A_{j\ell} = \frac{1}{(m_j-\ell)!}\left(\frac{d}{dz}\right)^{m_j-\ell} R(z)\,(z-z_j)^{m_j}\Big|_{z=z_j}$$

Esempio $R(z) = \frac{z^2+z+1}{(z-2)^4} = \sum_{\ell=1}^{4} \frac{A_\ell}{(z-2)^\ell}$

$$A_\ell = \frac{1}{(4-\ell)!}\left(\frac{d}{dz}\right)^{4-\ell} R(z)\,(z-2)^4\Big|_{z=2}$$

$$R(z) = \frac{z^2+z+1}{z^3 (z-1)^5 (z-2)^4} =$$

$$= \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3}$$

$$+ \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \frac{B_3}{(z-1)^3} + \frac{B_4}{(z-1)^4} + \frac{B_5}{(z-1)^5}$$

$$+ \frac{C_1}{(z-2)} + \frac{C_2}{(z-2)^2} + \frac{C_3}{(z-2)^3} + \frac{C_4}{(z-2)^4}$$

$$A_j = \frac{1}{(3-j)!}\left(\frac{d}{dz}\right)^{3-j} R(z)\,z^3\Big|_{z=0} \qquad j = 1,3$$

$$B_k = \frac{1}{(5-k)!}\left(\frac{d}{dz}\right)^{5-k} R(z)\,(z-1)^5\Big|_{z=1} \qquad k = 1,5$$

$$C_i = \frac{1}{(4-i)!}\left(\frac{d}{dz}\right)^{4-i} R(z)\,(z-2)^4\Big|_{z=2} \qquad i = 1,4$$

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2\,(z+i)^2} \qquad z^2+1 = (z-i)(z+i)$$

$$= \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{(z-i)^2} + \frac{D}{(z+i)^2} \qquad C = \frac{1}{(2-2)!}\left(\frac{d}{dz}\right)^{2-2} R(z)(z-i)^2 \Big|_{z=i}$$

$$C = R(z)\,(z-i)^2\Big|_{z=i} = \frac{1}{(z+i)^2}\Big|_{z=i} = \frac{1}{(2i)^2} = -\frac{1}{4}$$

$$D = R(z)\,(z+i)^2\Big|_{z=-i} = \frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{1}{(-2i)^2} = -\frac{1}{4}$$

$$A = \frac{1}{(2-1)!}\left(\frac{d}{dz}\right)^{2-1} R(z)\,(z-i)^2\Big|_{z=i} = \frac{d}{dz}\, R(z)\,(z-i)^2\Big|_{z=i}$$

$$= \frac{d}{dz}\,\frac{1}{(z+i)^2}\Big|_{z=i} = -\frac{2}{(z+i)^3}\Big|_{z=i} = \frac{-2}{(2i)^3} =$$

$$= \frac{\cancel{-}2}{8\,(\cancel{-}i)} = \frac{1}{4i} = -\frac{i}{4}$$

$$B = \frac{d}{dz}\, R(z)\,(z+i)^2\Big|_{z=-i} = \frac{d}{dz}\,\frac{1}{(z-i)^2}\Big|_{z=-i}$$

$$= -\frac{2}{(z-i)^3}\Big|_{z=-i} = -\frac{2}{(-2i)^3} = \frac{\cancel{-}2}{\cancel{-}8\,(-i)} = -\frac{1}{4i} =$$

$$= \frac{i}{4}$$

$$\frac{1}{(z^2+1)^2} = -\frac{i}{4}\,\frac{1}{z-i} + \frac{i}{4}\,\frac{1}{z+i} - \frac{1}{4}\,\frac{1}{(z-i)^2} - \frac{1}{4}\,\frac{1}{(z+i)^2}$$

$$= -\frac{i}{4}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] - \frac{1}{4}\left[\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2}\right]$$

$$= -\frac{i}{4}\,\frac{\cancel{z}+i-(\cancel{z}-i)}{z^2+1} - \frac{1}{4}\,\frac{(z+i)^2+(z-i)^2}{(z^2+1)^2}$$

$$= -\frac{i}{4}\,\frac{2i}{z^2+1} - \frac{1}{4}\,\frac{z^2+\cancel{2zi}-1+z^2-\cancel{2zi}-1}{(z^2+1)^2}$$

$$= \frac{1/2}{z^2+1} - \frac{1}{4}\,\frac{2z^2-2}{(z^2+1)^2}$$

$$= \frac{1/2}{z^2+1} - \frac{1}{2}\,\frac{z^2-1}{(z^2+1)^2}$$

$$\frac{1}{z^2+1} = -\frac{i}{4}\frac{1}{z-i} + \frac{i}{4}\frac{1}{z+i} - \frac{1}{4}\frac{1}{(z-i)^2} - \frac{1}{4}\frac{1}{(z+i)^2}$$

$$= \frac{1}{2}\frac{1}{z^2+1} + \frac{1}{4}\frac{d}{dz}\frac{1}{z-i} + \frac{1}{4}\frac{d}{dz}\frac{1}{z+i}$$

$$= \frac{1}{2}\frac{1}{z^2+1} + \frac{d}{dz}\left[\frac{1}{4}\left(\frac{1}{z-i} + \frac{1}{z+i}\right)\right]$$

$$= \frac{1}{2}\frac{1}{z^2+1} + \frac{d}{dz}\left[\frac{1}{4}\frac{z+\cancel{i}+z-\cancel{i}}{z^2+1}\right]$$

$$= \frac{1}{2}\frac{1}{z^2+1} + \frac{d}{dz}\frac{1}{\cancel{4}2}\,\cancel{2}\cdot\frac{z}{z^2+1}$$

$$= \frac{1}{2}\frac{1}{z^2+1} + \frac{d}{dz}\frac{1}{2}\frac{z}{z^2+1}$$

$$\lim_{x\to 0^-} \frac{\sqrt{x^2+2x^3}+x}{\sin^2 x} = \lim_{x\to 0^-} \frac{\sqrt{x^2+2x^3}+x}{x^2}$$

dal $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\frac{\sqrt{x^2+2x^3}+x}{x^2} = \frac{\sqrt{x^2(1+2x)}+x}{x^2} =$$

$$= \frac{\sqrt{x^2}\sqrt{1+2x}+x}{x^2}$$

$\sqrt{x^2} = -x$

$$= \frac{1-\sqrt{1+2x}}{x^{\cancel{2}}} \cancel{x} = -\frac{(1+2x)^{\frac{1}{2}}-1}{2x} \; 2$$

$\frac{1}{2}$

$\to -1$

dove ho usato

$$\lim_{y\to 0} \frac{(1+y)^a-1}{y} = a$$

$$\frac{1-\sqrt{1+2x}}{x} = \frac{1-(1+2x)^{\frac{1}{2}}}{x} = \frac{\cancel{1}-[\cancel{1}+\frac{1}{2}\cancel{2}x+o(x)]}{x}$$

$$= \frac{-x+o(x)}{x} \doteq \frac{-\cancel{x}(1+o(1))}{\cancel{x}} = -1+o(1) \xrightarrow{x\to 0^-} -1$$

$$(1+y)^a = 1 + ay + \binom{a}{2}y^2 + \binom{a}{3}y^3 + o(y^3)$$

$$\binom{a}{1} = a$$

$$R(x) = \frac{x^2+x+1}{(x-2)^4} = \frac{1}{(x-2)^2} + \frac{5}{(x-2)^3} + \frac{7}{(x-2)^4} =$$

$$= \frac{d}{dx}\left[\frac{-1}{x-2} - \frac{5/2}{(x-2)^2} - \frac{7/3}{(x-2)^3}\right]$$

Se $R(z) = \dfrac{P(z)}{Q(z)}$

$\deg Q > \deg P$ e $Q(z) = a_n (z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$

$m_1 + \cdots + m_k = n$

allora esistono dei coefficienti $\alpha_1, \dots, \alpha_k$ e una funzione razionale $\tilde{R}(z)$ t.c.

$$R(z) = \sum_{j=1}^{k} \frac{\alpha_j}{z-z_j} + \frac{d}{dz}\tilde{R}(z)$$

$$\boxed{\alpha_j = A_{j1}}$$

Inoltre se $P(z)$ e $Q(z)$ $\left(R(z) = \dfrac{P(z)}{Q(z)}\right)$

se $Q(z)$ è un polinomio a coefficienti reali

se z_1 è complesso non reale, è una radice di $Q(z)$

anche $\bar{z}_1$ complesso coniugato è una radice di $Q(z)$

In questo caso

$$\frac{A}{z-z_1} + \frac{B}{z-\bar{z}_1} = \frac{(z-\bar{z}_1)A + (z-z_1)B}{z^2-(z_1+\bar{z}_1)z+|z_1|^2} =$$

$$= \frac{(A+B)z - (\bar{z}_1 A + z_1 B)}{z^2-(z_1+\bar{z}_1)z+|z_1|^2}$$

$$= \frac{\alpha z + \beta}{z^2-(z_1+\bar{z}_1)z+|z_1|^2}$$

Esempio

$$R(x) = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$A = R(x)x\Big|_{x=0} = \frac{1}{x^2+1}\Big|_{x=0} = 1$$

so subito che $B = -1$

$$R(x) = \frac{1}{x} + \frac{Bx+C}{x^2+1} = \frac{x^2+1+Bx^2+Cx}{x(x^2+1)} =$$

$$\frac{1}{x\cancel{(x^2+1)}} = \frac{x^2(1+B)+Cx+1}{x\cancel{(x^2+1)}}$$

$$1 = x^2(1+B) + Cx + 1 \;;$$

$1+B = 0 \qquad B = -1$

$C = 0 \qquad C = 0$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \qquad \cdot x$$

$$\frac{1}{x^2+1} = A + \frac{Bx^2}{x^2+1} + \frac{Cx}{x^2+1}$$

$\downarrow x \to +\infty$

$$0 = A + B$$

Sia ora $R(z) = \dfrac{P(z)}{Q(z)}$ con $\deg P \geq \deg Q$

Si divide P per Q

Esistono ~~un~~ $q(z)$ due polinomi, $q(z)$ il quoziente

~~ed~~ ed $r(z)$ il resto t.c.

$$P(z) = q(z)\, Q(z) + r(z)$$

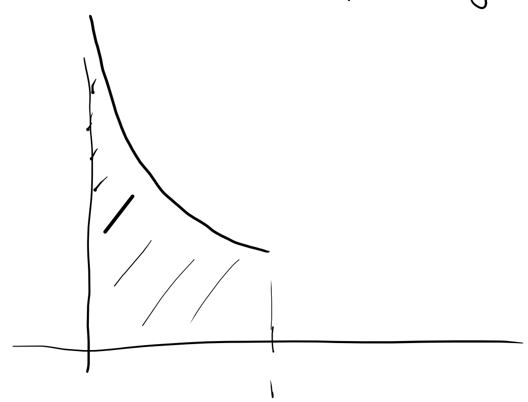
con $\deg r(z) < \deg Q(z)$

$$\frac{P(z)}{Q(z)} = \frac{q(z)\, Q(z) + r(z)}{Q(z)} = q(z) + \frac{r(z)}{Q(z)}$$

Integrale improprio

$\frac{1}{\sqrt{x}}$ non è integrabile per Riemann in $[0,1]$

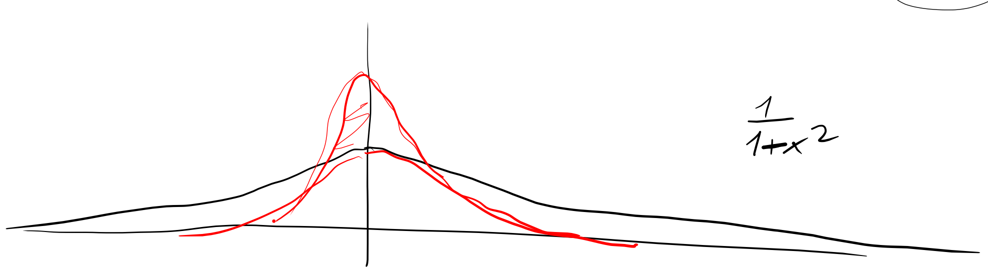
ma è integrabile in un senso più generalizzato



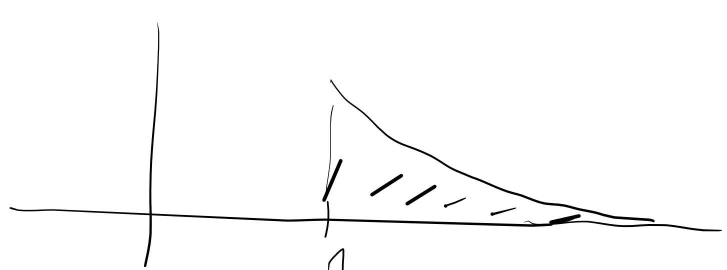
e^{x^2}

$\frac{1}{\pi}$ $\frac{1}{1+x^2}$ è integrabile in $\mathbb{R}$

e^{-x^2}



$\frac{1}{x^3}$ in $[1,+\infty)$



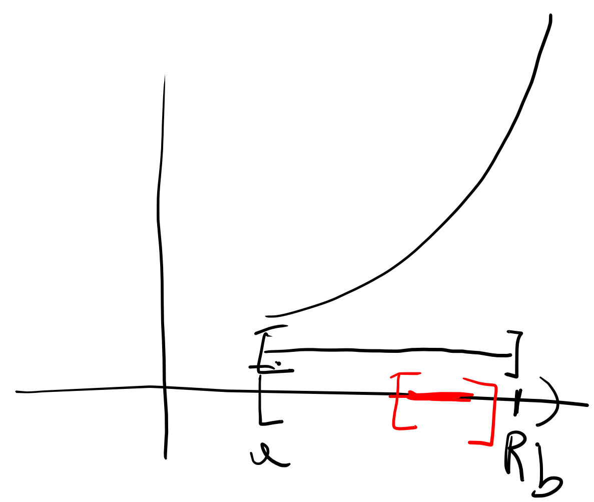
Integrale improprio

Def Una funzione $f: [a, b) \longrightarrow \mathbb{R}$ si dice integrabile in senso improprio in $[a,b)$ se

1) $f \in L_{loc}[a,b)$

2) $\displaystyle\lim_{R \to b^-} \int_a^R f(x)\,dx$ esiste

ed è un numero reale



Se 1) e 2) sono soddisfatte denoto con

$\displaystyle\int_a^b f(x)\,dx$ il limite e lo chiamo

integrale improprio di f in $[a, b)$

Osservazione Se f è integrabile per Riemann in $[a,b]$ allora è integrabile in senso improprio in $[a, b)$ e i due integrali sono uguali.