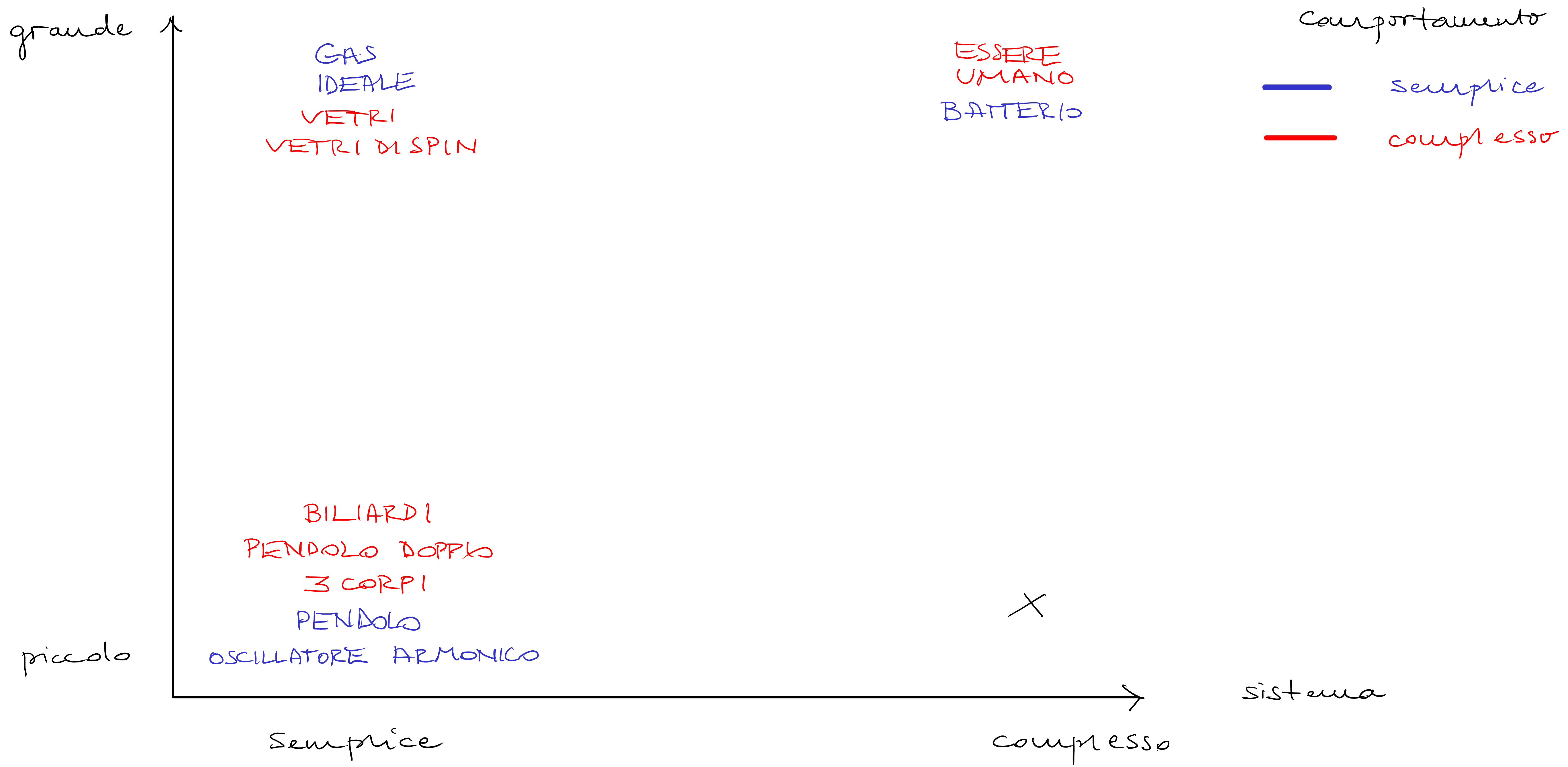
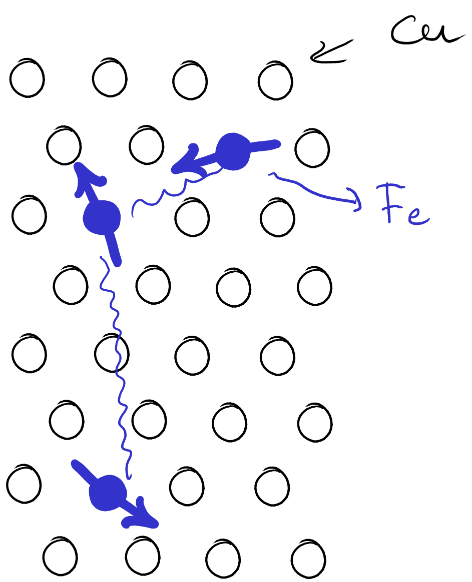

grande
piccolo
GAS IDEALE
VETRI
VETRI DI SPIN
ESSERE UMANO
BATTERIO
BILIARDI
PENDOLO DOPPIO
3 CORPI
PENDOLO
OSCILLATORE ARMONICO
sistema
semplice
complesso
comportamento
semplice
complesso

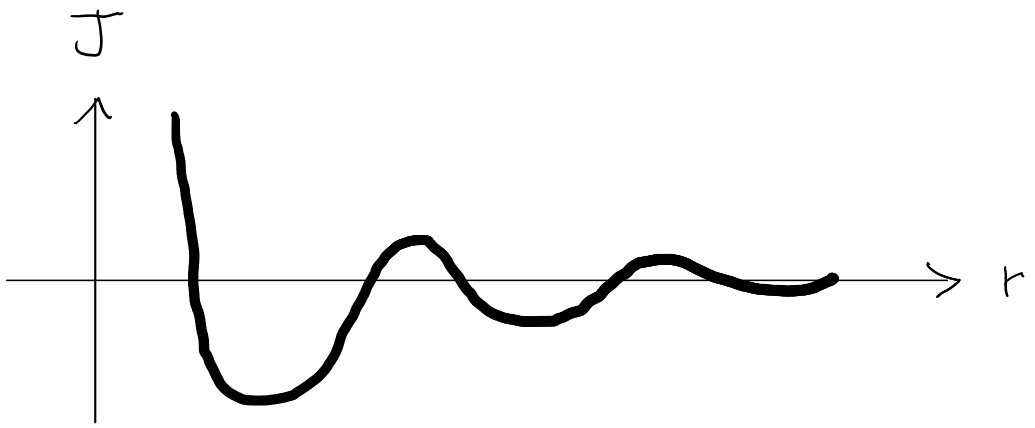
VETRI DI SPIN: INTRODUZIONE

Sistemi disordinati $\longrightarrow$ modelli semplificati : vetri di spin

Fenomenologia

Leghe metalliche : metalli [Au, Ag, Cu, Pt] + ioni metallici [Fe, Mn]





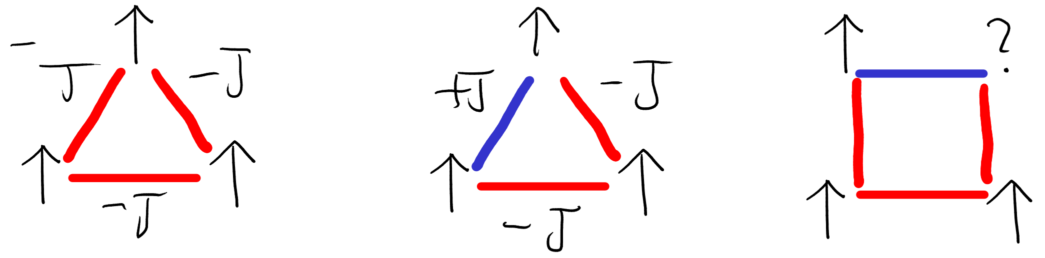
$$J \sim \frac{\cos(kr)}{(kr)^3} \sim \frac{1}{r^3}$$

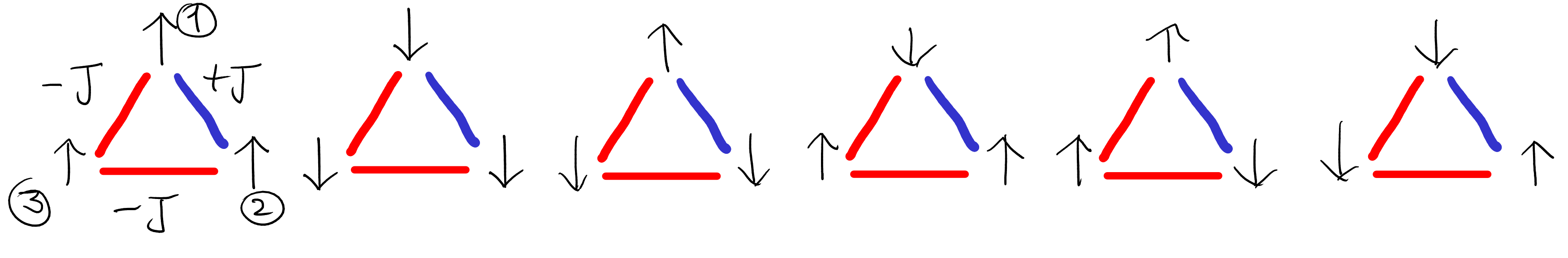
$\rightarrow$ disordine gelato

"quenched disorder"

Frustrazione: modello $\pm J$ $\quad \uparrow\uparrow \Rightarrow -J \quad \uparrow\uparrow \Rightarrow +J$

non tutti gli accoppiamenti sono "favorevoli"





$\rightarrow$ degenerazione: 6

$U = -2J$

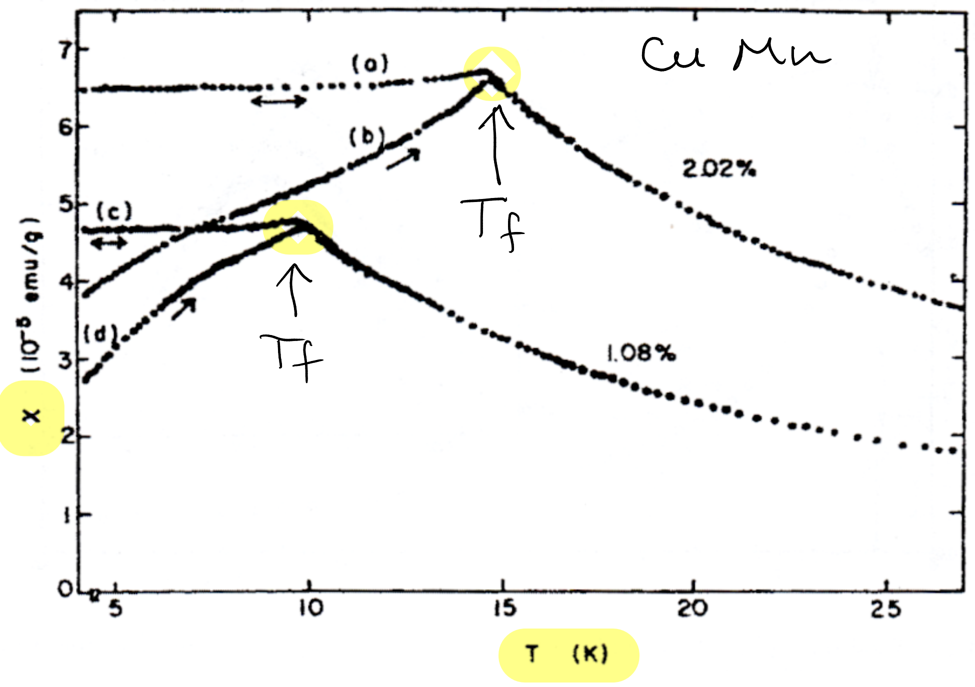
$\uparrow\uparrow\uparrow$

Transizione a vetro di spin: statica e dinamica

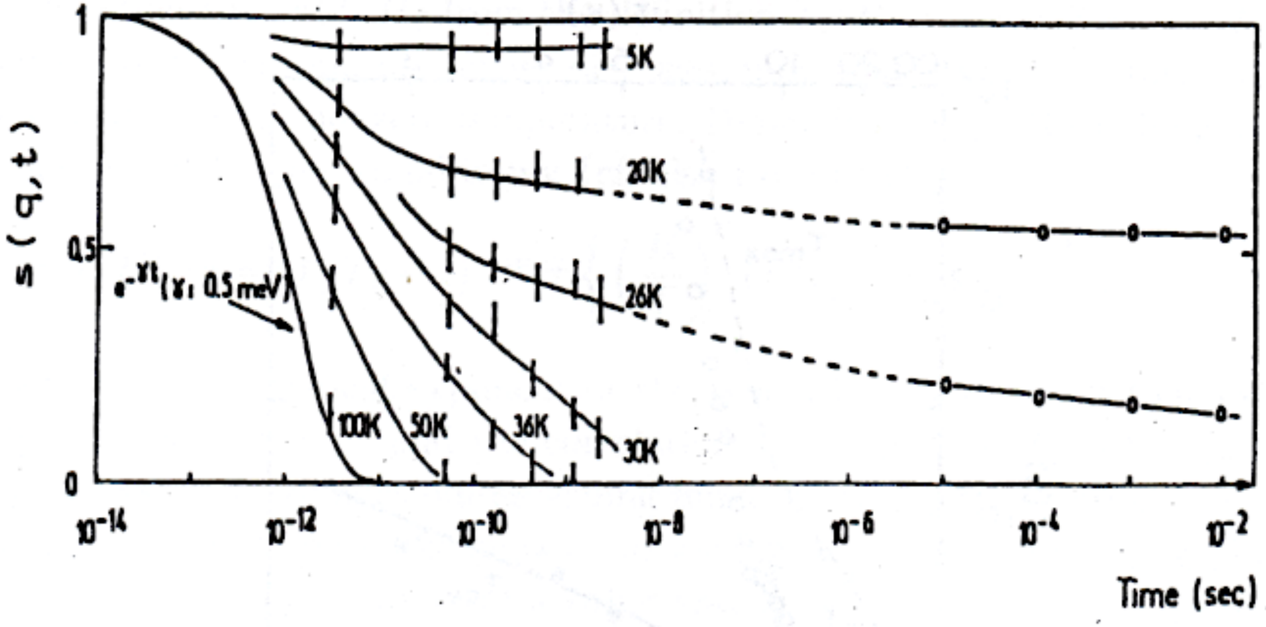
h = campo esterno $\qquad M$ = magnetizzazione $\qquad \chi$ = suscettibilità magnetica

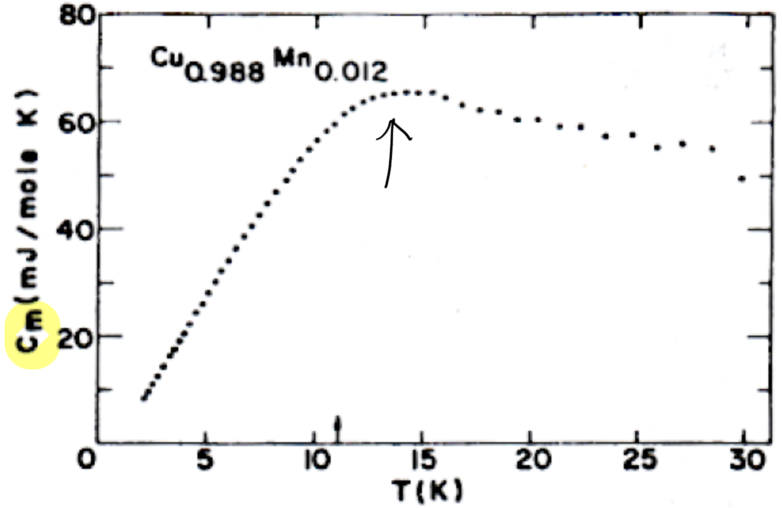
$\Delta M = \chi \, \Delta h \qquad M = \chi h \qquad \delta A = \chi \phi \qquad \chi = \frac{\partial M}{\partial h}$

Statica



dinamica





$$C = \frac{\partial U}{\partial T}$$

Modelli teorici

$J \to$ variabili aleatorie "gelate" (quenched) $\to \quad p(J)$

$\sigma_i \qquad i = 1, \dots, N \qquad \{\sigma_1, \dots, \sigma_N\} = \sigma^N$

Hamiltoniana:

$$\boxed{H = - \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij}\, \sigma_i \sigma_j}$$

J_{ij} aleatoria gelata $\qquad H = H[\sigma^N; J]$

- Spins: m componenti $\quad \sigma_i \qquad |\sigma_i| = 1$

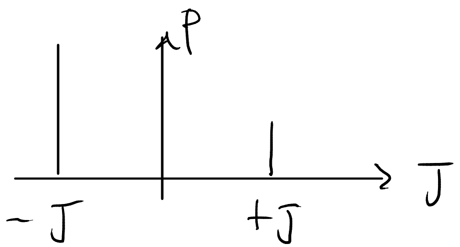
  $m=1$: Ising $\quad \sigma_i = \pm 1 \quad \uparrow\downarrow$

  $m=2$: XY $\quad |\sigma_i| = 1 \quad \to \theta_i$

  $m=3$: Heisenberg $\quad \to \theta_i, \varphi_i$

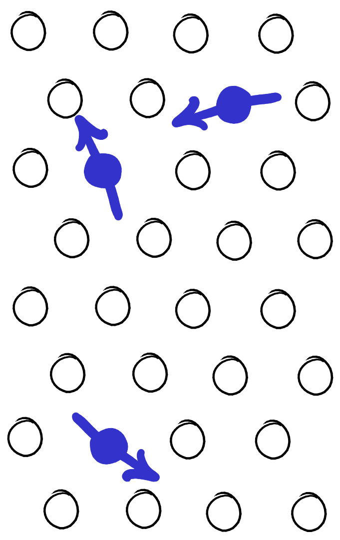
- Accoppiamenti: J_{ij}

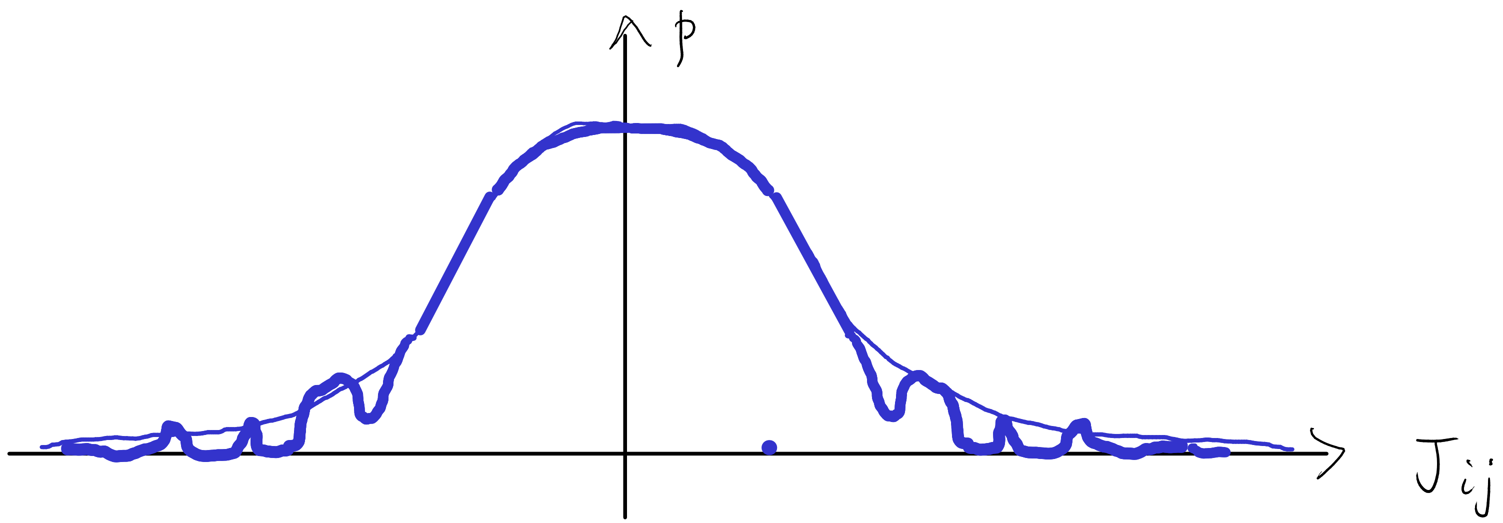
$\pm J$: $\quad p(J) = x\, \delta(J_{ij} - J) + (1-x)\, \delta(J_{ij} + J)$



gaussiana : $p(J) = \frac{1}{(2\pi \Delta J)^{1/2}} \exp\left[-\frac{(J-J_0)^2}{2\Delta J^2}\right]$ $\qquad \Delta J^2 = \frac{\tilde{\Delta J}^2}{N}$

$$p(J) = \frac{1}{(2\pi \Delta J^2/N)^{1/2}} \exp\left[-\frac{(J-J_0)^2}{2\Delta J^2/N}\right]$$





- Range

  - fully - connected
  - corto - raggio $\qquad J_{ij} = 0$ se i,j non sono primi vicini

$$H = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j$$

Edwards - Anderson

Self-averaging

$p(J)$

SA : proprietà che non dipende da J_{ij} per $N \to \infty$, $A(N;J)$

$$\lim_{N\to\infty} A(N;J) = \lim_{N\to\infty} \int dJ\, p(J)\, A(N;J)$$

$$\overline{A} = \int dJ\, p(J)\, A(J)$$ media sul disordine $\overline{\cdots}$ $([\ldots])$

$$\langle A \rangle = \mathrm{Tr}\left[e^{-\beta H[\sigma^N;J]} A(\sigma^N;J) \right]$$ media termica $\langle \ldots \rangle$

$$= \int d\sigma^N e^{-\beta H[\sigma^N;J]} A(\sigma^N;J)$$

Es.: energia libera

$$F(N;J) = -k_B T \ln\left[\mathrm{Tr}\left[e^{-\beta H}\right]\right] = -k_B T \ln\left(Z(N;J)\right)$$

$$f(N;J) = \frac{F(N;J)}{N}$$ densità

$$-\beta f = \lim_{N\to\infty} \frac{1}{N} \overline{\ln\left(Z(N;J)\right)}$$

Parametri di ordine

Magnetizzazione media : $m = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle$ OK per Ising

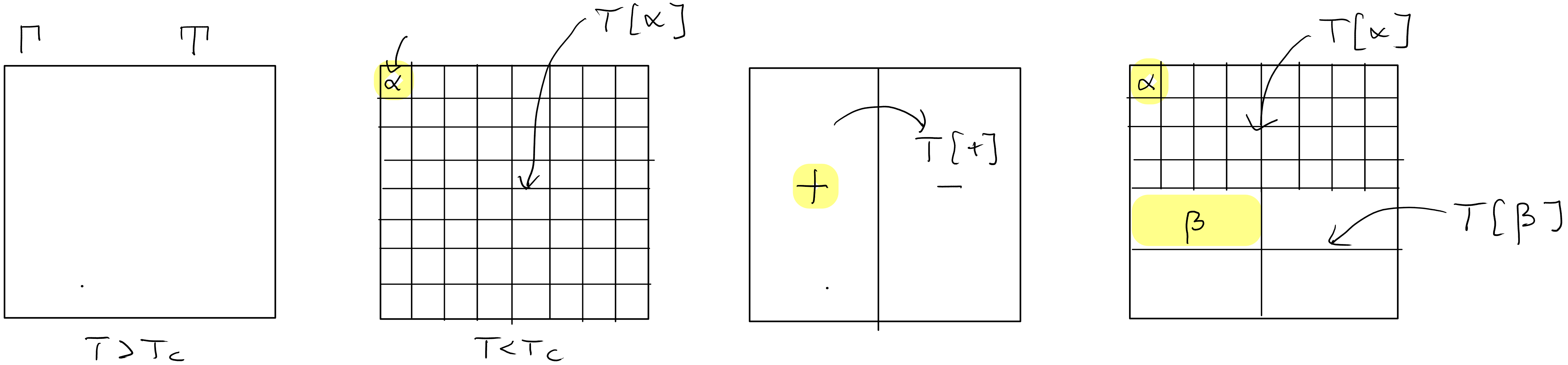
Edwards-Anderson , $p(J_{ij}) = p(-J_{ij})$ $J_0 = 0$

$m = 0$ anche a bassa T a causa del disordine gelato

Parametro d'ordine di Edwards-Anderson

$$q_{EA} = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2} \rightsquigarrow \text{overlap}$$

Rottura di simmetria e di ergodicità

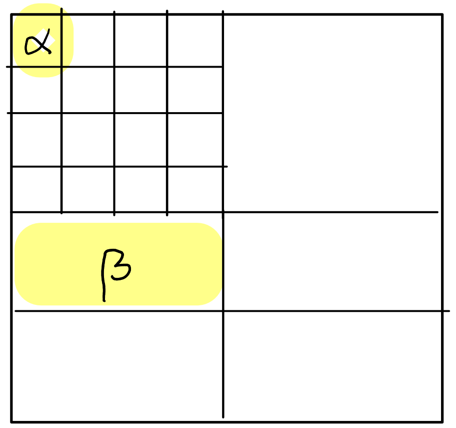


Es. : cristallizzazione, ferromagnetismo

Stato : $\langle \ldots \rangle = \sum_\alpha w_\alpha \langle \ldots \rangle_\alpha$

peso di α — media vincolata allo stato α

T<Tc



$$\langle A \rangle = \frac{1}{Z}\int d\sigma^N e^{-\beta H[\sigma^N]} A(\sigma^N) \qquad Z = \int d\sigma^N e^{-\beta H[\sigma^N]} A(\sigma^N)$$

$$= \frac{1}{Z}\sum_\alpha \int_\alpha d\sigma^N e^{-\beta H} A \qquad Z_\alpha = \int_\alpha d\sigma^N e^{-\beta H[\sigma^N]} A(\sigma^N)$$

$$= \sum_\alpha \frac{Z_\alpha}{Z}\,\frac{1}{Z_\alpha}\int_\alpha d\sigma^N e^{-\beta H} A \qquad \square$$

Es.: Ising $T<T_c$ $\pm$ $\quad \langle \ldots \rangle = \frac{1}{2}\langle \ldots \rangle_+ + \frac{1}{2}\langle \ldots \rangle_-$

$$\langle \sigma_i \rangle = \frac{1}{2}\langle \sigma_i \rangle_+ + \frac{1}{2}\langle \sigma_i \rangle_- = \emptyset$$

Overlap $\to$ metrica di somiglianza

- Configurazioni

$\sigma^N, \sigma'^N \qquad q_{\sigma\sigma'} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma_i' \quad \begin{cases} 1 & \text{identiche} \\ -1 & \text{anticorrelate} \\ 0 & \text{scorrelate} \end{cases}$

Self-overlap: $q_{\sigma\sigma} = 1$

- Stati

$\alpha, \beta \qquad q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{Z_\alpha}\int_\alpha d\sigma^N e^{-\beta[\sigma^N]} \sigma_i \frac{1}{Z_\beta}\int_\beta d\sigma'^N e^{-\beta H[\sigma'^N]} \sigma_i'$

$$= \frac{1}{Z_\alpha}\frac{1}{Z_\beta}\int_\alpha d\sigma^N \int_\beta d\sigma'^N \underbrace{e^{-\beta H[\sigma]} e^{-\beta H[\sigma']}}_{\text{peso stat.}} \underbrace{\frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'}_{q_{\sigma\sigma'}}$$

Self-overlap: $q_{\alpha\alpha} = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha^2$

es: paramagnetico $q_{\alpha\alpha} \to 0$, $T \to 0$ $q_{\alpha\alpha} \to 1$

$q_{\alpha\alpha} \nearrow$ taglia $\searrow$

Nota: $q_{\alpha\beta} = \left| \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta \right|$

Distribuzione degli overlap

2 copie del sistema per J_{ij} fissato : "repliche"

$$P(q) = \frac{1}{Z^2} \int d\sigma^N \int d\sigma'^N \, e^{-\beta H[\sigma]} \, e^{-\beta H[\sigma']} \, \delta(q - q_{\sigma\sigma'})$$

$$= \sum_\alpha \sum_\beta w_\alpha w_\beta \frac{1}{Z_\alpha} \int_\alpha d\sigma^N \frac{1}{Z_\beta} \int_\beta d\sigma'^N \, e^{-\beta H[\sigma^N]} \, e^{-\beta H[\sigma'^N]} \, \delta(q - q_{\sigma\sigma'})$$

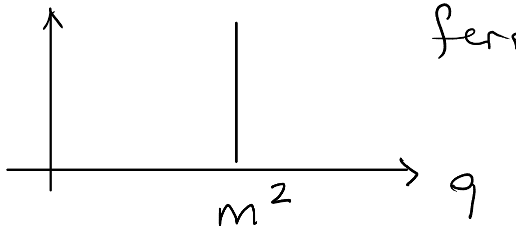
$$P(q) = \sum_\alpha \sum_\beta w_\alpha w_\beta \, \delta(q - q_{\alpha\beta})$$

Es.: Ising

$$q_{++} = \frac{1}{N} \sum_{i=1}^N \langle \sigma_i \rangle_+^2 = q_{--} = m^2$$

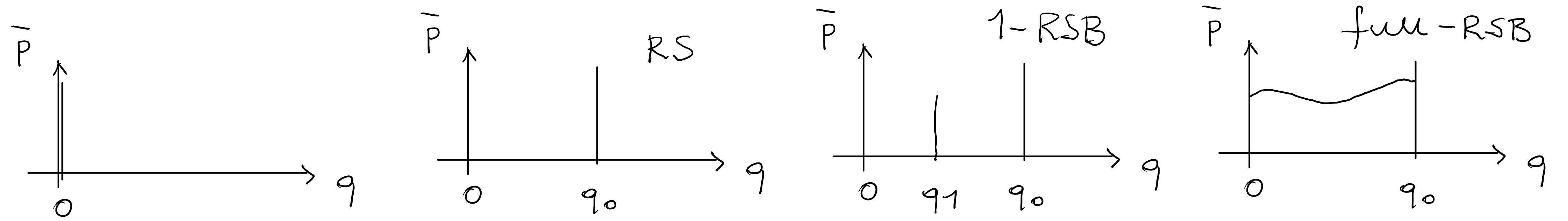
$$q_{+-} = \frac{1}{N} \sum_{i=1}^N \langle \sigma_i \rangle_+ \langle \sigma_i \rangle_- = q_{-+} = -m^2$$

$T < T_c$ ferro



$T > T_c$ para

Con disordine gelato: $\overline{P(q)} = \int dJ \, p(J) \, P(q;J)$



1 solo stato

\+ modi di ordinarsi

Metodo delle repliche

$$F = -K_B T\, \overline{\log Z(J)} = -K_B T \int dJ\, p(J) \log Z(J) \qquad \text{quenched} \qquad \checkmark$$

$$F_a = -K_B T \log \overline{Z(J)} = -K_B T \log \left[\int dJ\, p(J)\, Z(J) \right] \qquad \text{annealed} \rightarrow \text{alta } T \qquad \times$$

Trucco delle repliche ("replica trick")

$$Z^n = \exp\left[\log Z^n\right] = \exp\left[n \log Z\right] \approx 1 + n \log Z + O(n^2)$$

$$\lim_{n\to 0} \frac{Z^n - 1}{n} = \log Z$$

$$\boxed{\overline{\log Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}}$$

$$Z^n = \underbrace{Z \cdots Z}_{n}$$

f. partizione n repliche non-interagenti con J fissato

$$Z^n = \prod_{a=1}^{n} Z_a = \int d\sigma_1^N \cdots \int d\sigma_n^N\, e^{-\beta H[\sigma_1^N]}\, e^{-\beta H[\sigma_2^N]} \cdots e^{-\beta H[\sigma_n^N]}$$

$$\overline{Z^n} = \overline{\prod_{a=1}^{n} Z_a} = \overline{\int d\sigma_1^N \dots \int d\sigma_n^N \; e^{-\beta H[\sigma_1^N]} \dots e^{-\beta H[\sigma_n^N]}}$$

$$= \underbrace{\int d\sigma_1^N \dots \int d\sigma_n^N}_{\mathrm{Tr}\,\sigma_n} \; \overline{e^{-\beta \sum_{a=1}^{n} H[\sigma_a^N]}} \qquad \sigma_1^N = \{\sigma_{1i}\}_{i=1,\dots N}$$

$$\overline{\exp\left[-\beta \sum_{a=1}^{n} H[\sigma_a^N]\right]} = \exp\left[-\beta H^{eff}\left[\underbrace{\{\sigma_1^N, \dots, \sigma_n^N\}}_{n \times N}\right]\right]$$

⚠ H^{eff} accoppiato le repliche!

☺ disaccopio i siti $i = 1, \dots, N$

① $n \to 0$ $\quad (N \gg 1)$

② $N \to \infty \quad \Rightarrow \quad$ LT

$$-\beta f = \lim_{N \to \infty} \frac{\overline{F(J)}}{N} = \lim_{N \to \infty} \lim_{n \to 0} \frac{\overline{Z^n} - 1}{Nn} \longrightarrow q_{ab} \begin{cases} \text{RS} \\ \text{1-RSB} \\ \text{full-RSB} \end{cases}$$

Modelli prototipo fully-connected

→ campo medio (mean-field)

1) Sherrington-Kirkpatrick (SK) in campo esterno h

$$H = - \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \qquad p(J_{ij}) \sim \exp\left[- \frac{(J_{ij} - J_0)^2}{2 \Delta J^2} \right]$$

full-RSB $\qquad \rightarrow \quad \sim \exp\left[- \frac{J_{ij}^2}{2 \Delta J^2 / N} \right]$

2) p-spin

$$N = \sum_{i=1}^{N} \sigma_i^2$$

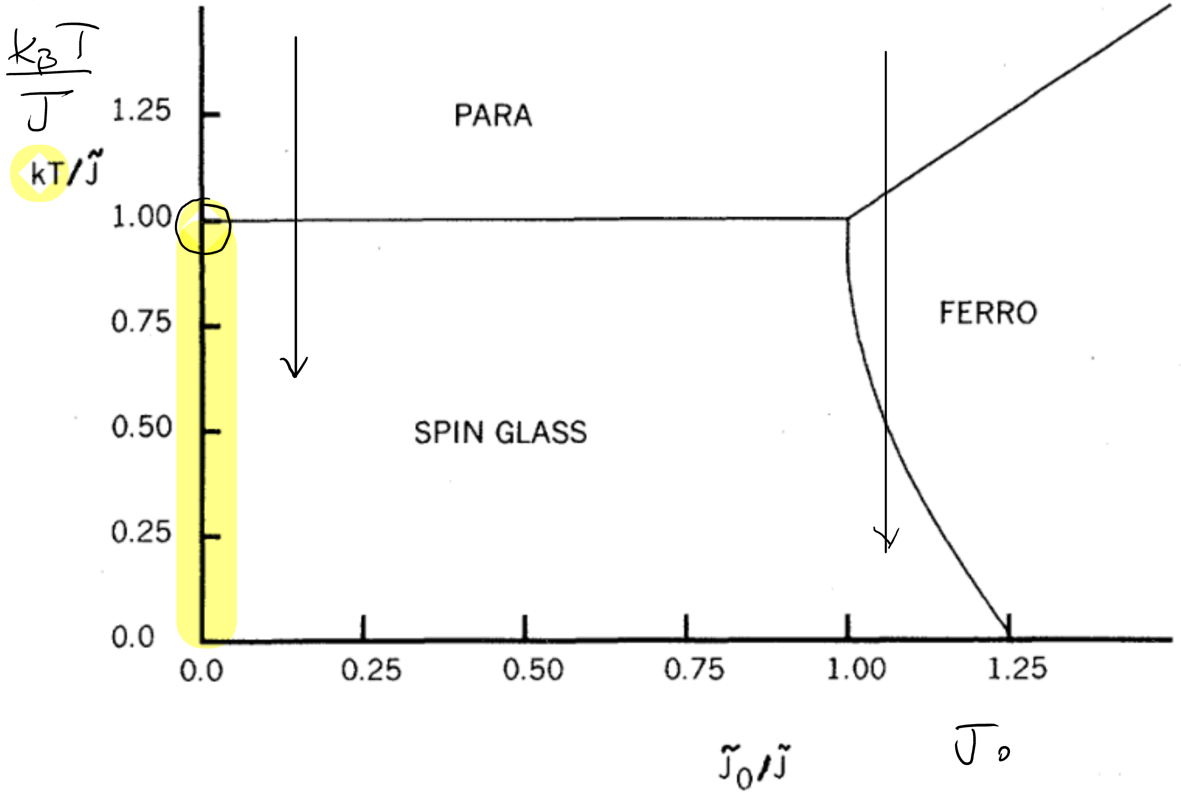
$$H = - \underbrace{\sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_p=1}^{N}}_{\times p} J_{i_1 \ldots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \qquad p = 3 \quad H = - \sum_{i_1, i_2 i_3} J_{i_1 i_2 i_3} \sigma_{i_1} \sigma_{i_2} \sigma_{i_3}$$

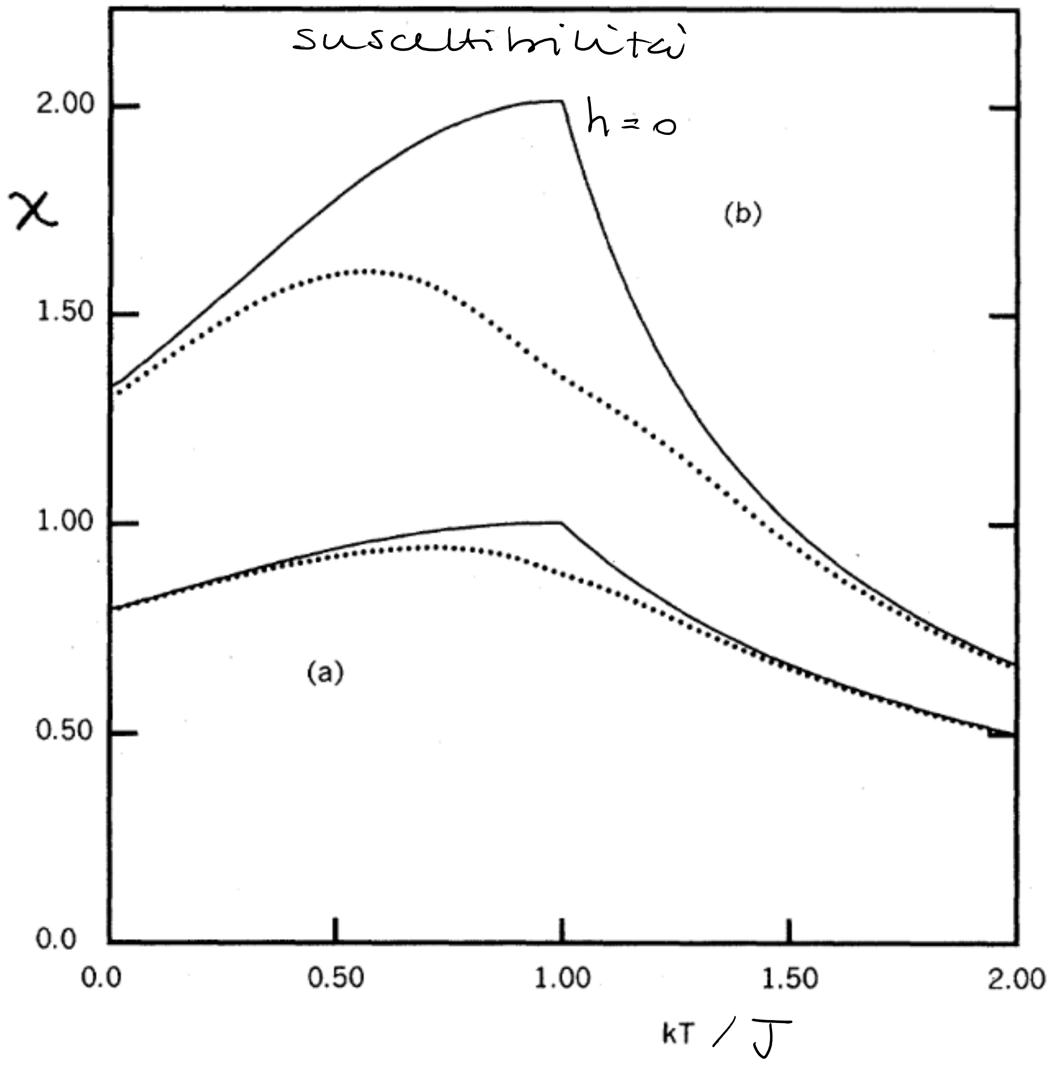
1-RSB $\quad p \geq 3 \qquad p = 2$: RS $\qquad p(J_{i_1 \ldots i_p}) \sim \exp\left[- \frac{N^{p-1}}{p!} J_{i_1 \ldots i_p}^2 \right]$

Modello di Sherrington-Kirkpatrick (1975)

$$H = -\sum_{i=1}^{N}\sum_{j>i}^{N} J_{ij}\,\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \qquad p(J_{ij}) = \frac{1}{\sqrt{2\pi J^2/N}}\cdot\exp\left(-\frac{(J_{ij}-J_0/N)^2}{2J^2/N}\right)$$

$h=0$, $J_0=0$





[V] transizione vetro di spin

[X] soluzione SK è instabile per $T<T_f$

① replica trick : $-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \dfrac{\overline{Z^n} - 1}{nN}$

② integrali gaussiani : $e^{\lambda a^2/2} = \sqrt{\dfrac{\lambda}{2\pi}} \displaystyle\int_{-\infty}^{\infty} dx \exp\left(-\frac{\lambda x^2}{2} + \lambda x a\right)$

③ approx punto sella : $\displaystyle\int_{-\infty}^{\infty} dx\, e^{Nf(x)} \approx \sqrt{\frac{2\pi}{N|f''(x_0)|}} \exp\left(N f(x_0)\right)$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \prod_{j>i}^{N} dJ_{ij}\, p(J_{ij})\, \mathrm{Tr}\left[\exp\left(\beta \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \sigma_{i\alpha} \sigma_{j\alpha}\right)\right] \qquad \mathrm{Tr} \to n, N \to \sigma_{i\alpha}$$

$$= \mathrm{Tr}\left[\int_{-\infty}^{\infty} \prod_{i=1}^{N} \prod_{j>i}^{N} dJ_{ij}\, p(J_{ij}) \exp\left(\beta \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \underbrace{\sum_{\alpha=1}^{n} \sigma_{i\alpha} \sigma_{j\alpha}}_{\phi^{(n)}}\right)\right]$$

$$\frac{1}{\sqrt{2\pi J^2/N}} \int_{-\infty}^{\infty} dJ_{ij} \exp\left(-\frac{J_{ij}^2}{2J^2/N} + \frac{\beta\phi^{(n)} J^2/N}{J^2/N} J_{ij}\right) \qquad x = J_{ij} \quad \lambda = \frac{1}{J^2/N} \quad a = \beta J^2/N\, \phi^{(n)}$$

$$= \exp\left(\frac{\beta^2 (J^2/N)^2\, \phi^{(n)2}}{2\,(J^2/N)}\right) = \exp\left(\frac{\beta^2 J^2}{2N} \phi^{(n)2}\right) = \exp\left(\frac{\beta^2 J^2}{2N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \sigma_{i\alpha} \sigma_{i\beta} \sigma_{j\alpha} \sigma_{j\beta}\right)$$

$$\overline{Z^n} = \mathrm{Tr}\Big[\prod_{i=1}^{N} \prod_{j>i}^{N} \exp\Big(\frac{\beta^2 J^2}{2N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \sigma_{i\alpha}\sigma_{j\alpha}\sigma_{i\beta}\sigma_{j\beta} \Big)\Big] \qquad \sigma_{i\alpha}^2 = 1$$

$$= \mathrm{Tr}\Big[\exp\Big(\frac{\beta^2 J^2}{2N} \sum_{i=1}^{N} \sum_{j>i}^{N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \underbrace{\sigma_{i\alpha}\sigma_{j\alpha}\sigma_{i\beta}\sigma_{j\beta}}_{\ldots} \Big)\Big]$$

$$\sum_{i=1}^{N} \sum_{j>i}^{N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \ldots = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \Big(\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \ldots - \underbrace{\sum_{i=1}^{N} \sigma_{i\alpha}^2 \sigma_{i\beta}^2}_{N} \Big) \qquad \sum{}' \to \alpha \neq \beta$$

$$= \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n}{}' \Big(\underbrace{\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \ldots}_{\text{ok}} - \underbrace{N}_{\text{no}} \Big) + \underbrace{\sum_{\alpha=1}^{n} \Big(\frac{1}{2} \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i\alpha}^2 \sigma_{j\alpha}^2}_{N^2} - N \Big)}_{\underbrace{\frac{1}{2} n N^2}_{\text{ok}} - \underbrace{nN}_{\text{no}}}$$

$$\overline{Z^n} = \mathrm{Tr}\Big[\exp\Big(\frac{\beta^2 J^2}{4N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n}{}' \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i\alpha}\sigma_{j\alpha}\sigma_{i\beta}\sigma_{j\beta} + \frac{\beta^2 J^2 N n}{4} \Big)\Big]$$

$$= \exp\left(\frac{N\beta^2 J^2 n}{4}\right) \mathrm{Tr}\left[\exp\left(\frac{\beta^2 J^2}{2N}\sum_{\alpha=1}^{n}\sum_{\beta>\alpha}^{n} \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{i\alpha}\sigma_{j\alpha}\sigma_{i\beta}\sigma_{j\beta}}_{\left(\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)^2}\right)\right]$$

Linearizzo gli argomenti dell'esponenziale

$$e^{\lambda a^2/2} = \sqrt{\frac{\lambda}{2\pi}}\int_{-\infty}^{\infty} dx \, \exp\left(-\frac{\lambda x^2}{2} + \lambda x a\right) \qquad x = q_{\alpha\beta} \quad a = \sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta} \quad \lambda = \frac{\beta^2 J^2}{N}$$

$$\exp\left[\frac{\beta^2 J^2}{2N}\left(\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)^2\right] = \sqrt{\frac{\beta^2 J^2}{2\pi N}}\int_{-\infty}^{\infty} dq_{\alpha\beta}\,\exp\left(-\frac{\beta^2 J^2}{2N} q_{\alpha\beta}^2\right)\exp\left(\frac{\beta^2 J^2}{N} q_{\alpha\beta}\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)$$

$$\overline{Z^n} = \exp\left(\frac{N\beta^2 J^2 n}{4}\right)\int_{-\infty}^{\infty}\prod_{\alpha=1}^{n}\prod_{\beta>\alpha}^{n} dq_{\alpha\beta}\,\frac{\beta J}{\sqrt{2\pi N}}\exp\left(-\frac{\beta^2 J^2}{2}\sum_\alpha\sum_{\beta>\alpha} N q_{\alpha\beta}^2\right)\mathrm{Tr}\left[\exp\left(\beta^2 J^2 \sum_\alpha\sum_{\beta>\alpha} q_{\alpha\beta}\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)\right]$$

$$\mathrm{Tr}\left[\exp\left(\beta^2 J^2\sum_{\alpha=1}^{n}\sum_{\beta>\alpha}^{n} q_{\alpha\beta}\sum_{i=1}^{N}\sigma_{i\alpha}\sigma_{i\beta}\right)\right] = \left\{\mathrm{Tr}_{\sigma_\alpha}\left[\exp\left(\beta^2 J^2\sum_{\alpha=1}^{n}\sum_{\beta>\alpha}^{n} q_{\alpha\beta}\,\sigma_\alpha\sigma_\beta\right)\right]\right\}^N$$

$$= \exp\left[N\log\left(\mathrm{Tr}_{\sigma_\alpha}\left[\exp\left(\beta^2 J^2\sum_{\alpha=1}^{n}\sum_{\beta>\alpha}^{n} q_{\alpha\beta}\,\sigma_\alpha\sigma_\beta\right)\right]\right)\right]$$

$$= \exp\left[N \log\left(\mathrm{Tr}_{\sigma_\alpha} \left[e^{L} \right] \right) \right]$$

$$L = \beta^2 J^2 \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}\, \sigma_\alpha \sigma_\beta$$

$$\overline{Z^n} = \exp\left(\frac{N\beta^2 J^2 n}{4} \right) \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\beta>\alpha}^{n} dq_{\alpha\beta} \frac{\beta J}{\sqrt{2\pi N}} \exp\left(-\frac{N\beta^2 J^2}{2} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + N \log \mathrm{Tr}_{\sigma_\alpha} e^{L} \right)$$

$$= \frac{\beta J}{\sqrt{2\pi N}} \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\beta>\alpha}^{n} dq_{\alpha\beta} \exp\left(\frac{N\beta^2 J^2 n}{4} - \frac{N\beta^2 J^2}{2} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + N \log \mathrm{Tr}_{\sigma_\alpha} e^{L} \right)$$

Approx ~~punto~~ sella $N \gg 1$

$$\overline{Z^n} \approx \exp\left(\frac{N\beta^2 J^2 n}{4} - \frac{N\beta^2 J^2}{2} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + N \log \mathrm{Tr}_{\sigma_\alpha} e^{L} \right)$$

$$= \exp\left[Nn \left(\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + \frac{1}{n} \log \mathrm{Tr}_{\sigma_\alpha} e^{L} \right) \right]$$

Limite $n \to 0$

$$\overline{Z^n} \approx 1 + Nn \left(\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + \frac{1}{n} \log \mathrm{Tr}_{\sigma_\alpha} e^{L} \right)$$

Energia libera

$$-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \frac{\overline{Z^n} - 1}{nN} = \lim_{N\to\infty} \lim_{n\to 0} \frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 + \frac{1}{n} \log \mathrm{Tr}_{\sigma_\alpha} e^L$$

Condizione di estremizzazione dell'argomento

$$\frac{\partial}{\partial q_{\alpha\beta}} \left[-\frac{N\beta^2 J^2}{2} \sum_{\alpha=1}^{n} \sum_{\beta>\alpha}^{n} q_{\alpha\beta}^2 \right] + \frac{\partial}{\partial q_{\alpha\beta}} \left[N \log \mathrm{Tr}_{\sigma_\alpha} e^L \right] = 0$$

$$\beta^2 J^2 q_{\alpha\beta} = \frac{\partial}{\partial q_{\alpha\beta}} \left[\log \mathrm{Tr}_{\sigma_\alpha} e^L \right]$$

$$q_{\alpha\beta} = \frac{1}{\beta^2 J^2} \frac{\partial}{\partial q_{\alpha\beta}} \left[\log \mathrm{Tr}_{\sigma_\alpha} e^L \right] \qquad L = \beta^2 J^2 \sum_{\alpha=1}^{n} \sum_{>\alpha}^{n} q_{\alpha\beta} \sigma_\alpha \sigma_\beta$$

$$= \frac{1}{\beta^2 J^2} \frac{\partial}{\partial q_{\alpha\beta}} \left[\log \mathrm{Tr}_{\alpha_\alpha} \exp\left(\beta^2 J^2 \sum_{\alpha=1}^{n} \sum_{>\alpha}^{n} q_{\alpha\beta} \sigma_\alpha \sigma_\beta \right) \right]$$

$$= \frac{\mathrm{Tr}_{\sigma_\alpha} [\sigma_\alpha \sigma_\beta \exp(L)]}{\mathrm{Tr}_{\sigma_\alpha} [\exp(L)]}$$

Soluzione RS : $q_{ab} = q$ Nota: da adesso a, b indici di repliche

$$\begin{cases} -\beta f = \dfrac{\beta^2 J^2}{2}(1-q)^2 + \displaystyle\int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \log\left[2\cosh(\beta J \sqrt{q}\, z)\right] \\ \dfrac{\beta^2 J^2}{2}(q-1) + \displaystyle\int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \tanh(\beta J \sqrt{q}\, z) \frac{\beta J}{2\sqrt{q}} z = 0 \end{cases}$$

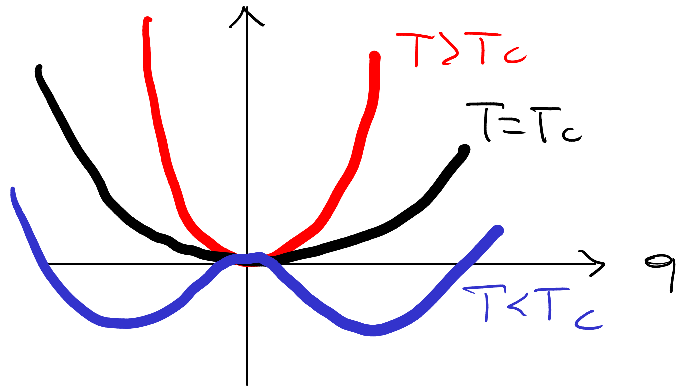
(es.)

$$\Rightarrow \quad q = \int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \tanh^2(\beta J \sqrt{q}\, z)$$

sviluppo in serie di Taylor per piccoli q (es.)

$$\beta f = -\frac{1}{4}\beta^2 J^2 - \log 2 - \frac{\beta^2 J^2}{4}(1-\beta^2 J^2)\, q^2 + O(q^3)$$

$T = T_c$: $\beta_c^2 J^2 = 1$ $\quad k_B T_c = J \quad T_c = J/k_B$



Patologie:

0) Entropia: $T \to 0$, $S < 0$

1) $T > T_c$: $f(q)$ ha un massimo in $q = 0$! $f(q) \to -f(q)$!

$$\sum_{a=1}^{n} \sum_{b>a}^{n} q^2 = \frac{n(n-1)}{2} q^2 \qquad n \to 0$$

2) Soluzione RS è instabile per $T < T_c$

$$-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \frac{\text{cost}}{nN} \log \int_{-\infty}^{\infty} \prod_{a=1}^{n} \prod_{b>a}^{n} dq_{ab} \exp(-N\, S(q_{ab}))$$

The second point we have to pay attention to is what we actually mean by the 'minimum' of S. The problem here is that the number of independent elements of Q_{ab} is $n(n-1)/2$, which becomes negative in the limit $n \to 0$. It is hard to say what is the minimum of a function with a negative number of variables! There is however a criterion we can use to select the correct saddle point: the corrections to the saddle point result are given by the Gaussian integration around the saddle point itself. This integration gives as a result the square root of the determinant of the second-derivative matrix of S, and thus, in order to have a sensible result, we must have all the eigenvalues of this matrix positive. Summarizing, we have to select saddle points with a positive-definite second derivative of S [12].

Castellani, Cavagna

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} dx \exp(N f(x)) = \sqrt{\frac{2\pi}{N\,|f''(x_0)|}} \exp(N f(x_0))$$

Rottura di simmetria delle repliche

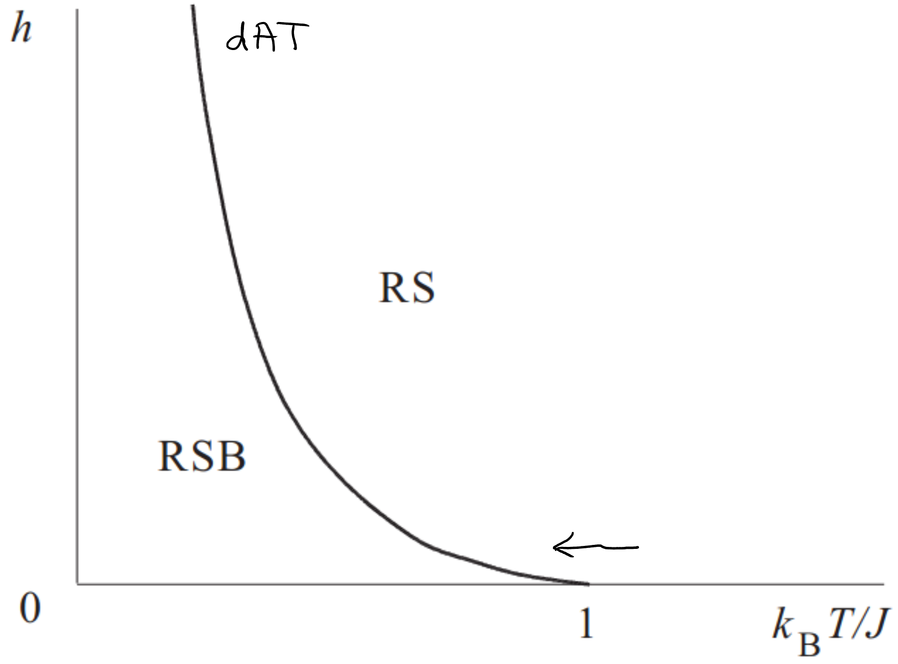
1975 : Sherrington Kirkpatrick

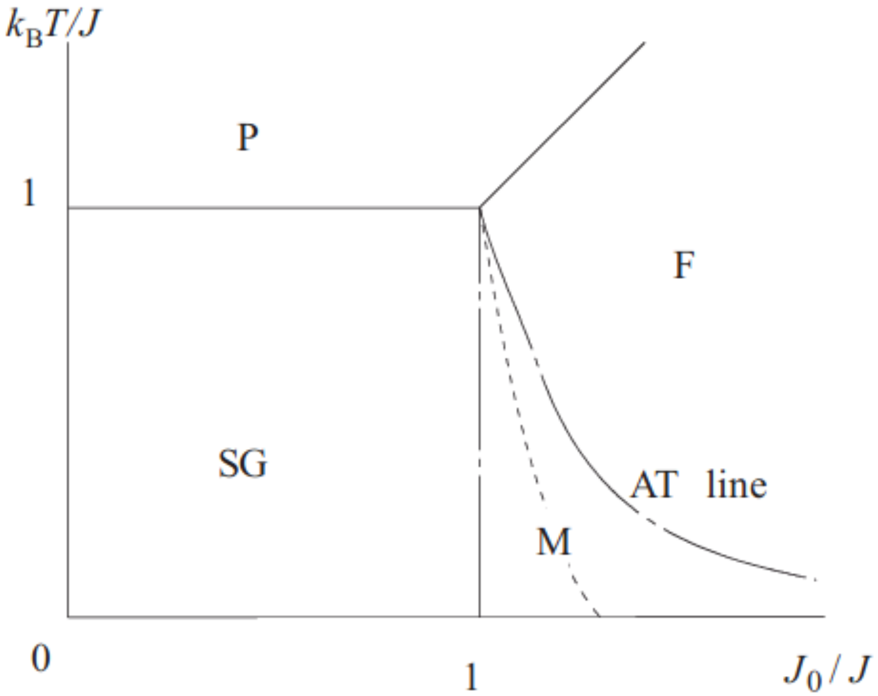
1978 : de Almeida - Thouless → linea dAT

1979 - 1980 : Parisi → RSB

2006 : Talagrand

G. Parisi
Nobel 2021





Nishimori

"Connessione" tra le repliche e la fisica

$$q_{ab} = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_{ia}\sigma_{ib}\rangle}$$

$$q_{ab} = \frac{Tr_{\sigma_a}\left[\sigma_a \sigma_b \, e^{L}\right]}{Tr_{\sigma_a}\left[e^{L}\right]} \qquad (\text{estremo})$$

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_{ia}\sigma_{ib}\rangle} = \overline{\left(\frac{Tr\left[\sigma_{ia}\sigma_{ib}\exp(-\beta H^{(n)})\right]}{Tr\left[\exp(-\beta H^{(n)})\right]}\right)} \qquad H^{(n)} = \sum_{r=1}^{n} H_r \qquad n\to 0$$

$$q_{ab} = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_{ia}\rangle\langle\sigma_{ib}\rangle}$$

Se a e b non sono distinguibili

$$q_{ab} = \frac{1}{N}\sum_{i=1}^{N}\overline{\langle \sigma_i\rangle^2} = q_{EA}$$

Espressione simmetrizzata:

$$\frac{1}{N}\sum_{i=1}^{N}\overline{\langle \sigma_i\rangle^2} = \lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a=1}^{n}\sum_{b>a}^{n} q_{ab}$$

$$q_{EA} = \frac{1}{N}\sum_{i=1}^{N}\sum_{\alpha}\sum_{\beta}\overline{w_\alpha w_\beta \langle\sigma_i\rangle_\alpha \langle\sigma_i\rangle_\beta} = \sum_\alpha\sum_\beta \overline{w_\alpha w_\beta\, q_{\alpha\beta}}$$

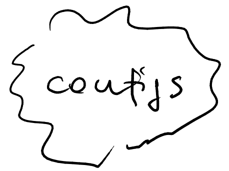
$$= \int_{-\infty}^{\infty} dq\, \underbrace{\sum_\alpha\sum_\beta \overline{w_\alpha w_\beta\, \delta(q-q_{\alpha\beta})}}_{\overline{P(q)}}\, q = \int_{-\infty}^{\infty} dq\, \overline{P(q)}\, q = q^{(1)} = \frac{1}{N}\sum_{i=1}^{N}\overline{\langle\sigma_i\rangle^2}$$

$$q^{(k)} = \frac{1}{N^k}\sum_{i_1\dots i_k}^{N}\overline{\langle\sigma_{i_1}\cdots\sigma_{i_k}\rangle^2} = \lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a=1}^{n}\sum_{b>a}^{n} q_{ab}^{k} = \int_{-\infty}^{\infty} dq\, \overline{P(q)}\, q^k$$

$$\int_{-\infty}^{\infty} dq\, f(q)\, \overline{P(q)} = \lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a=1}^{n}\sum_{b>a}^{n} f(q_{ab}) \qquad f(q) = \delta(q-q')$$

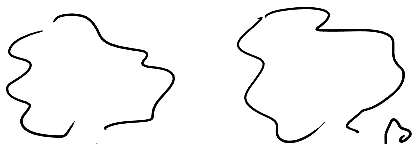
$$\overline{P(q)} = \lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a=1}^{n}\sum_{b>a}^{n}\delta(q-q_{ab})$$

La prob. media dell'overlap q tra stati è la frazione di elementi di q_{ab} con valore q



α

$q_{\alpha\beta} \to q_{\alpha\alpha}$ self-overlap



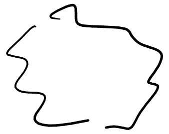
α β

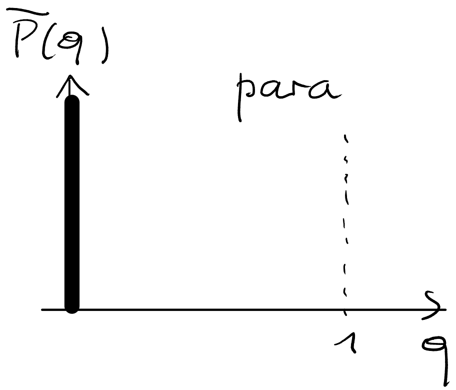
$q_{\alpha\beta}$ $\alpha \neq \beta$

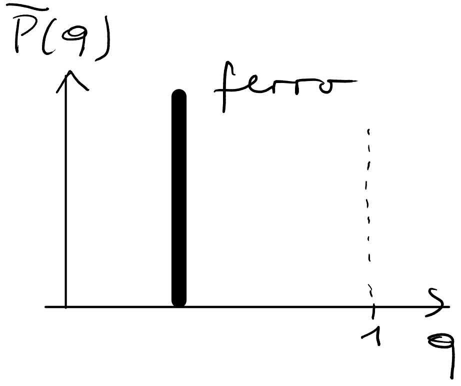
Parametrizzazione della matrice q_{ab}

– **RS : simmetria tra le repliche** $\quad a \neq b \qquad (q_{aa} = 1)$

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & q & & \\ & & 0 & & & \\ & & & 0 & & \\ & q & & & 0 & \\ & & & & & 0 \end{pmatrix} \qquad q_{ab} = q \qquad \text{self-overlap dello stato}$$

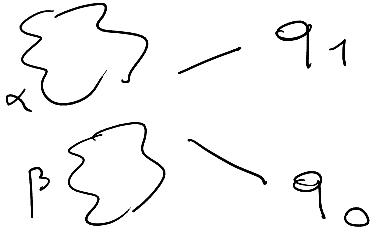






– **1–RSB : rottura a uno step**

$$\begin{array}{cc} & \overleftrightarrow{\quad\quad n \quad\quad} \\ m \updownarrow & \left(\begin{array}{ccc|ccc} 0 & q_1 & q_1 & & & \\ q_1 & 0 & q_1 & & q_0 & \\ q_1 & q_1 & 0 & & & \\ \hline & & & 0 & q_1 & q_1 \\ & q_0 & & q_1 & 0 & q_1 \\ & & & q_1 & q_1 & 0 \end{array}\right) . \end{array}$$



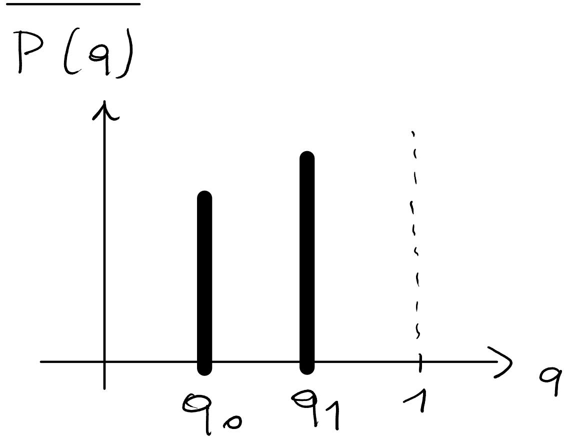
$$0 \leq q_0 < q_1 \leq 1$$

configs di stati $\neq$ $\qquad$ configs di uno stesso stato

$$\overline{P(q)} = \lim_{n \to 0} \frac{2}{n(n-1)} \sum_a \sum_{b > a} \delta(q - q_{ab})$$

$$= \frac{m-1}{n-1}\, \delta(q - q_1) + \frac{n-m}{n-1}\, \delta(q - q_0)$$

$1 \leq m \leq n \qquad n \to 0 \quad !!$



$0 \leq m \leq 1$: $\overline{P(q)} = (1-m)\,\delta(q-q_1) + m\,\delta(q-q_0)$

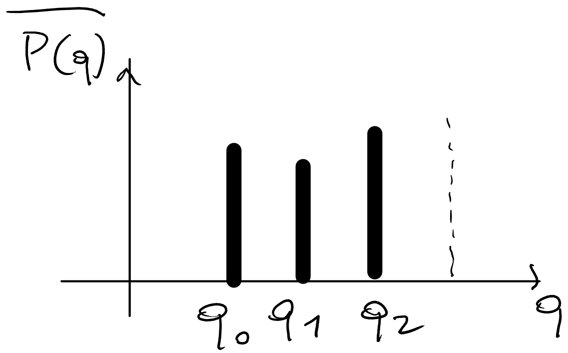
$$\begin{cases} q_{ab} = q_{ab}(q_0, q_1, m) \\ 0 \leq q_0 \leq q_1 \leq 1 \\ 0 \leq m \leq 1 \end{cases} \Rightarrow \text{soluzione } \underline{\text{1-RSB}} \text{ SK è instabile !}$$

- $\underline{2-RSB}$

$$m_2 \updownarrow \begin{pmatrix} \begin{array}{c|c} \begin{matrix} 0 & q_2 & q_2 \\ q_2 & 0 & q_2 \\ q_2 & q_2 & 0 \end{matrix} & q_1 \\ \hline q_1 & \begin{matrix} 0 & q_2 & q_2 \\ q_2 & 0 & q_2 \\ q_2 & q_2 & 0 \end{matrix} \end{array} & \Big| \; q_0 \\ \hline q_0 & \begin{array}{c|c} \begin{matrix} 0 & q_2 & q_2 \\ q_2 & 0 & q_2 \\ q_2 & q_2 & 0 \end{matrix} & q_1 \\ \hline q_1 & \begin{matrix} 0 & q_2 & q_2 \\ q_2 & 0 & q_2 \\ q_2 & q_2 & 0 \end{matrix} \end{array} \end{pmatrix} \updownarrow m_1$$

$0 \leq q_0 \leq q_1 \leq q_2 \leq 1$

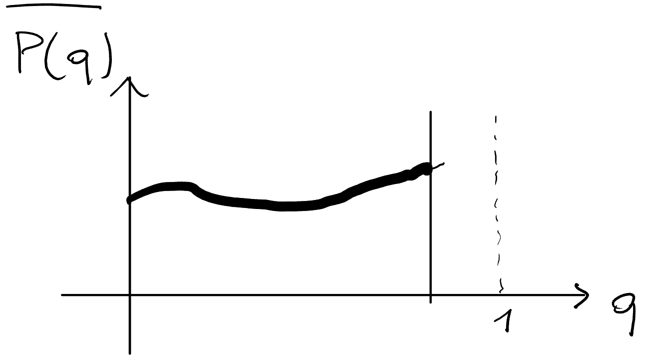
$q_{ab} = q_{ab}(q_0, q_1, q_2, m_1, m_2)$



- k-RSB

- **full-RSB** $k \to \infty$ $q(x)$ $x \in [0,1]$ $\Rightarrow$ OK per SK!

$$\frac{1}{n}\sum_a \sum_{b\neq a} q_{ab}^k \longrightarrow -\int_0^1 dx\, q^k(x)$$



Ultrametricità

$$P(q_1,q_2,q_3;J) = \sum_\alpha \sum_\beta \sum_\gamma w_\alpha w_\beta w_\gamma \,\delta(q_1-q_{\alpha\beta})\,\delta(q_2-q_{\beta\gamma})\,\delta(q_3-q_{\alpha\gamma})$$

$\overline{P(q_1,q_2,q_3)}$ $\longrightarrow$ full-RSB

$\overline{P}\neq 0$ $\begin{cases} q_1 = q_2 = q_3 & \rightarrow \text{triangolo equilatero} \\ q_1 > q_2 \quad q_2 = q_3 & \rightarrow \text{triangolo isoscele} \end{cases}$

Modelli p-spin sferici

σ_i continue

variabile aleatoria

$$\sum_{i=1}^{N} \sigma_i^2 = N$$

$$H = - \sum_{i_1=1}^{N} \sum_{i_2>i_1}^{N} \dots \sum_{i_p>i_{p-1}}^{N} J_{i_1 \dots i_p} \, \sigma_{i_1} \dots \sigma_{i_p} \qquad p \geq 3 \quad \Rightarrow \quad 1RSB$$

q_{ab} 1RSB $\Rightarrow \overline{P}(q)$

$q_{ab} \approx q_{ab}(q_0, q_1, m) \Rightarrow f = f(q_0, q_1, m)$

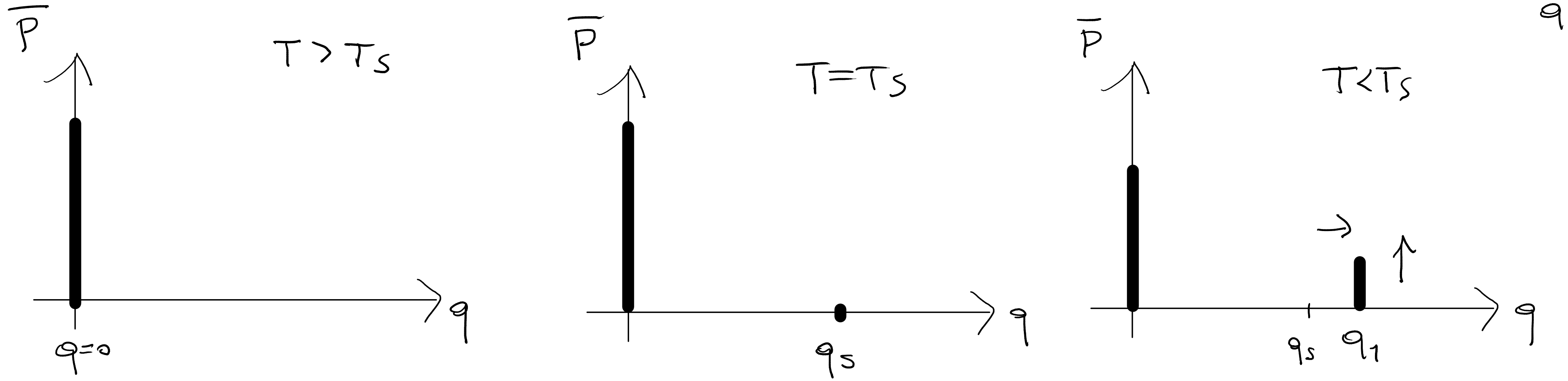
$$\begin{cases} \dfrac{\partial f}{\partial q_0} = 0 \\ \begin{cases} \dfrac{\partial f}{\partial q_1} = 0 \\ \dfrac{\partial f}{\partial m} = 0 \end{cases} \end{cases}$$

$\frac{\partial f}{\partial q_0} = 0 \rightarrow q_0 = 0$ overlap mutuali

$\rightarrow$ alta T: $q_1 = 0$, m indeterminato

bassa T: $m = 1$, $\exists\, T_s$ t.c. $q_1 \neq 0$

$q_1 = q_s$, $T = T_s$



Dinamica

Langevin sovrasmorzata

$$\frac{\partial \sigma_i}{\partial t} = -\frac{\partial H}{\partial \sigma_i} + \mu(t)\sigma_i(t) + \xi_i(t) \qquad \langle \xi_i(t)\,\xi_i(t')\rangle = 2T\,\delta(t-t') \qquad \sum_{i=1}^{N}\sigma_i^2 = N$$

$$\frac{\partial H}{\partial \sigma_i} \sim \sum_j \sum_l J_{ijl}\,\sigma_j\sigma_l \qquad p=3$$

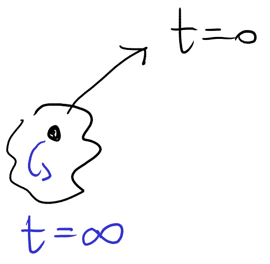
generating functional

$$\frac{\partial \sigma}{\partial t} = -\mu(t)\sigma(t) + \frac{1}{2}p(p-1)\int_{-\infty}^{t} ds\; \underbrace{R(t,s)}_{\text{funzione di risposta}}\; C(t,s)^{p-2}\,\sigma(s) + \xi(t)$$

$$C(t,s) = \langle \sigma(t)\,\sigma(s)\rangle$$

- stazionarietà: $C(t)$
- fluttuazione-dissipazione: $R(t) = -\beta \frac{dC}{dt}$

$$\frac{dC}{dt} + T\,C(t) + \frac{p}{2T}\int_0^t ds\; C^{p-1}(t-s)\,\frac{dC}{ds}(s) = 0 \quad \Rightarrow \quad C(t)$$



$$\frac{dC}{dt} + T\,C(t) + \frac{3}{2T}\int_0^t ds\; C^{2}(t-s)\,\frac{dC}{ds}(s) = 0 \qquad p=3$$

→ versione schematica di MCT! [Kirkpatrick - Thirumalai PRB 1987]

→ plateau $C(t^*) \to q_*$

$T \to T_d$ transizione dinamica $\tau_\alpha \sim \frac{1}{|T - T_d|^\gamma}$

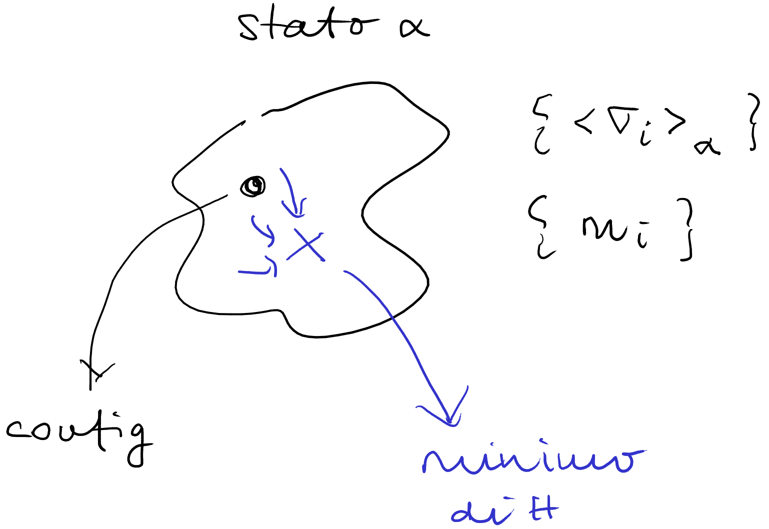
$T = T_d$ $\tau_\alpha = \infty$

$$T_d = \left(\frac{p(p-2)^{p-2}}{2(p-1)^{p-1}} \right)^{1/2} > T_s$$

↑

stati metastabili

Connessione con l'energy landscape

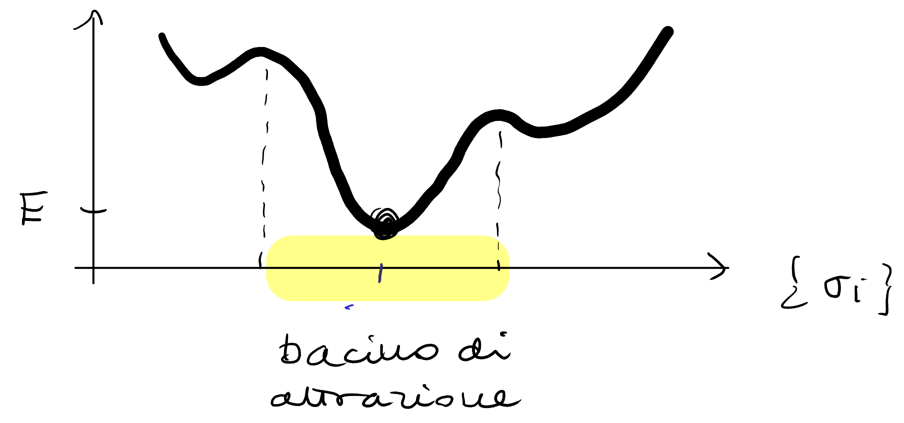


$f = f(\{m_i\})$ minimi ⇒ stati ⇒ TAP free energy

Es.:

- Ising $T < T_c$ $\quad q = \pm m^2 \;\to\; |q| = m^2$
- vetri di spin $T < T_s$ $\quad \infty$ stati

$H = H(\{\sigma_i\})$ minimi ⇒ $f = f(E,T)$



Energy landscape → entropia configurazionale S_c
complexity Σ

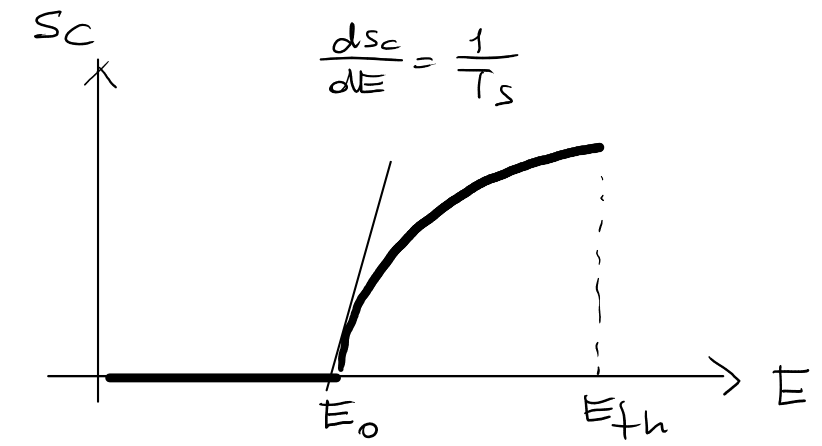
$$S_c = \frac{1}{N}\,\overline{\log \mathcal{N}(E)} \qquad N \to \infty$$

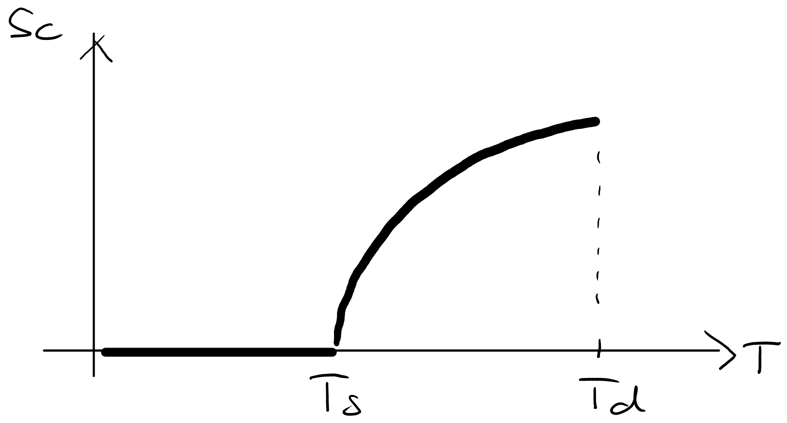
↑ n. minimi con energia E

$$S_c \simeq \frac{1}{N} \log \overline{\mathcal{N}(E)}$$

$\overline{\mathcal{N}(E)} \sim \exp(S_c N)$ ⇒ $S_c > 0$ $\qquad$ se $\overline{\mathcal{N}(E)}$ è "sotto esponenziale" $\mathcal{N}(E) \sim c N^a$ ⇒ $S_c = 0$

Transizione a vetro ideale!





$E = E_0$ $\quad s_c = 0$ $\quad$ @ T_s transizione statica

$E > E_0$ $\quad s_c > 0$

$E = E_{th}$ $\quad s_c$ immaginaria! @ T_d transizione dinamica

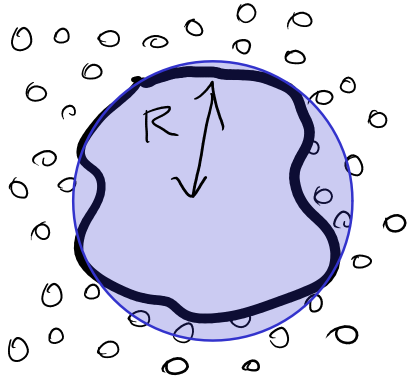
$E > E_{th}$ $\quad$ punti sella

vetri strutturali :

- $T_{MCT} \to T_d$
- $T_K \to T_s$ (1RSB)

RFOT : random first order transition

α



Biroli- Bouchaud

$$s_c = \frac{S_c}{V}$$

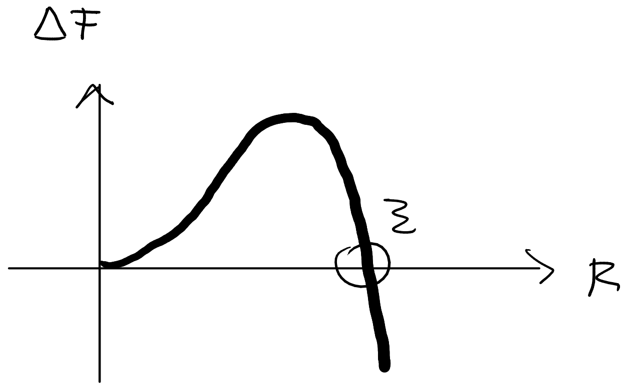
$$\Delta F = \Delta F_{cost} + \Delta F_{gain}$$

$$\Delta F = Y R^{\theta} - T s_c R^{d}$$

$$\theta = 2$$

$$\Delta F(\xi) = 0 \Rightarrow Y \xi^{\theta} = T s_c \xi^{d}$$

$$\xi = \left(\frac{Y}{T s_c}\right)^{\frac{1}{\theta - d}}$$



nucleazione

$$\tau_\alpha = \tau_0 \exp\left(A \frac{\xi^{\psi}}{T}\right) = \tau_0 \exp\left[\frac{A}{T}\left(\frac{Y}{T s_c}\right)^{\frac{\psi}{\theta - d}}\right] \sim AG \quad \square$$