5 dicembre

$f \in L_{loc}([a,b))$ è integrabile (sommabile)

se $\lim_{R \to b^-} \int_a^R f(x)\,dx = L \in \mathbb{R}$

e si pone $L = \int_a^b f(x)\,dx$ $\qquad f \in L([a,b))$

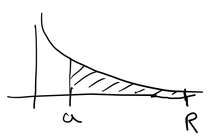
Esempio $[a,+\infty)$ $\boxed{a>0}$, x^{-p}

Risulta $x^{-p} \in L([a,+\infty)) \iff p>1$

Verifichiamolo. Sia $p \neq 1$. Notare

che $x^{-p} \in C^0([a,+\infty))$ $\forall\, p \in \mathbb{R}$

x^{-2}



$x^{-p} \in C^0([a,+\infty)) \Rightarrow x^{-p} \in C^0([\alpha,\beta])$ $\forall$

$[\alpha,\beta] \in [a,+\infty) \Rightarrow x^{-p} \in L([\alpha,\beta])$

$\Rightarrow x^{-p} \in L_{loc}([a,+\infty))$.

$x^{-p} \in L([a,+\infty)) \iff \lim_{R\to+\infty} \int_a^R x^{-p}\,dx \in \mathbb{R}$

$p \neq 1$ $\qquad \int_a^R x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right]_a^R =$

$$= \frac{R^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}$$

$$\lim_{R\to+\infty} \int_a^R x^{-p}\,dx = \lim_{R\to+\infty} \frac{R^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}$$

$$= \begin{cases} +\infty & \text{se } 1-p>0 \iff p<1 \\ -\dfrac{a^{1-p}}{1-p} & \text{se } 1-p<0 \iff p>1 \end{cases}$$

$\qquad = \dfrac{a^{1-p}}{p-1}$ $\qquad p>1$

$x^{-1} \notin L([a,+\infty))$ $\qquad a>0$

$$\lim_{R\to+\infty} \int_a^R x^{-1}\,dx = \lim_{R\to+\infty} \lg R - \lg a$$

$= +\infty$

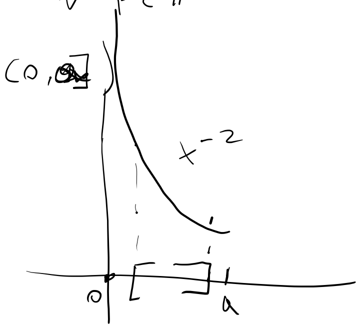
$a > 0$

Esempio $(0,a]$ $\qquad x^{-p} \in L((0,a])$

se solo se $\quad p < 1$

$x^{-p} \in C^0((0,a]) \quad \forall\, p \in \mathbb{R}$

$\Rightarrow \quad x^{-p} \in L_{loc}((0,a])$



$x^{-p} \in L((0,a]) \Leftrightarrow$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^a x^{-p}\,dx \in \mathbb{R}$$

$p \neq 1$

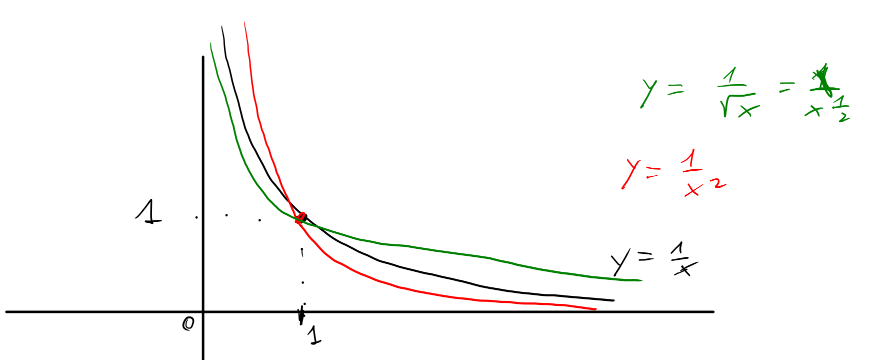
$$\int_\varepsilon^a x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right]_\varepsilon^a =$$

$$= \frac{a^{-p+1}}{-p+1} - \frac{\varepsilon^{-p+1}}{-p+1} \xrightarrow{\varepsilon \to 0^+} \begin{cases} \dfrac{a^{-p+1}}{1-p} & \text{se } 1-p>0 \quad 1>p \\ +\infty & \text{se se } 1-p<0 \quad 1<p \end{cases}$$

se $\quad p \neq 1 \quad x^{-p} \in L((0,a]) \Leftrightarrow \quad p<1$

$x^{-1} \notin L((0,a])$

$$\int_\varepsilon^a x^{-1}\,dx = \log a - \log \varepsilon \xrightarrow{\varepsilon \to 0^+} +\infty$$



Esercizio   stabilire quando $\frac{1}{\lg^p x} \in L((1,2])$
al variare di p

Proprietà.

1) Se f e g sono integrabili in $[a,b)$

$\Rightarrow f+g \in L([a,b))$ con

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2) $f \in L([a,b))$ e $c \in \mathbb{R}$ allora $c\,f(x) \in L([a,b))$

con $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

3) Se $f, g \in L([a,b))$ con $f(x) \le g(x)$ $\forall x \in [a,b)$

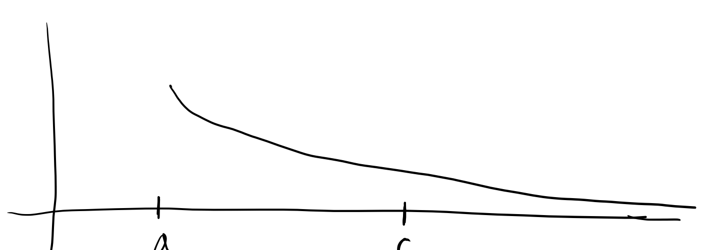
$\Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

4) Se $f \in L_{loc}([a,b))$ e $c \in (a,b)$ allora

$f \in L([a,b)) \iff f \in L([c,b))$ e vale

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

x^{-p} $[a,+\infty)$ $p>0$



$$\lim_{x\to+\infty} \frac{x^2 \lg\left(\frac{5+x^2}{3+x^2}\right)-2}{e^{\sin\frac{5}{x}}-1-\frac{5}{x}} = \lim_{x\to+\infty} \frac{2\ \text{num}}{\left(\frac{5}{x}\right)^2}$$

$$e^y = 1+y+\frac{y^2}{2}+o(y^2)$$

$$\text{denom} = e^{\sin\frac{5}{x}} - 1 - \frac{5}{x} = \cancel{1} + \sin\frac{5}{x} - \cancel{1} - \frac{5}{x} + \frac{\sin^2\frac{5}{x}}{2} + o\left(\sin^2\frac{5}{x}\right)$$

$$\sin y = y - \frac{y^3}{3!} + o(y^3)$$

$$\sin\frac{5}{x} = \frac{5}{x} - \frac{\left(\frac{5}{x}\right)^3}{3!} + o\left(\left(\frac{5}{x}\right)^3\right) = \frac{5}{x} + o\left(\left(\frac{5}{x}\right)^2\right)$$

$$= \frac{\sin^2\frac{5}{x}}{2} + \underbrace{o\left(\left(\frac{5}{x}\right)^2\right) + o\left(\sin^2\left(\frac{5}{x}\right)\right)}_{o\left(\left(\frac{5}{x}\right)^2\right) = o(x^{-2})}$$

$$\text{num} = x^2\Big[\lg(\underbrace{5+x^2}_{x^2(1+5x^{-2})}) - \lg(\underbrace{3+x^2}_{x^2(1+3x^{-2})})\Big] - 2 =$$

$$= x^2\left[\cancel{\lg x^2} + \lg\left(1+5x^{-2}\right) - \lg\left(1+3x^{-2}\right)\right] - 2$$

$$\lg(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3)$$

$$-2 + \lg(1+5x^{-2}) - \lg(1+3x^{-2}) = x^2\left[2x^{-2} - \frac{(5x^{-2})^2}{2} + \frac{(3x^{-2})^2}{2} + o(x^{-4})\right] - 2$$

$$= \frac{1}{2}\left[\frac{1}{9} - \frac{1}{25}\right]x^{-2} + o(x^{-2})$$

$$\lim_{x\to+\infty} \frac{\cancel{2}\,\frac{1}{\cancel{2}}\left[\frac{1}{9}-\frac{1}{25}\right]\cancel{x^{-2}}}{5^2\,\cancel{x^{-2}}} = \frac{\frac{1}{9}-\frac{1}{25}}{25}$$

Osservazione Non è vero in generale che $f, g \in L([a,b))$

$\Longrightarrow \quad f \cdot g \in L([a,b))$

Esempio $\quad x^{-\frac{2}{3}} \in L((0,1]))$ perché

$\frac{2}{3} < 1$ $\quad$ ma $\quad \left(x^{-\frac{2}{3}}\right)^2 = x^{-\frac{4}{3}} \notin L((0,1]))$

perché $\quad \frac{4}{3} > 1$

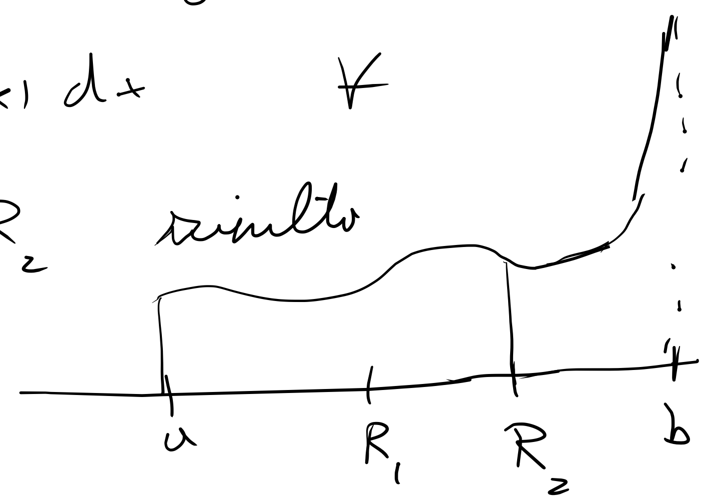
Teor (Aut-aut) Sia $f \in L_{loc}([a,b))$ con $f(x) \geq 0$

Allora $\lim\limits_{R \to +\infty} \int_a^R f(x)\,dx$ esiste ed e'

o un numero reale ≥ 0 o è uguale a $+\infty$.

Dim Sia $F(R) = \int_a^R f(x)\,dx \quad \forall$

$a \leq R < b$. Se $R_1 < R_2$ risulta



$$F(R_2) = \int_a^{R_2} f(x)\,dx =$$

$$= \int_a^{R_1} f(x)\,dx + \underbrace{\int_{R_1}^{R_2} f(x)\,dx}_{\geq 0} \geq \int_a^{R_1} f(x)\,dx =$$

$$= F(R_1)$$

Abbiamo dimostrato

$R_1 < R_2 \Rightarrow F(R_1) \leq F(R_2) \Rightarrow$ F e' crescente

Allora

$$\lim_{R \to b^-} \int_a^R f(x)\,dx = \lim_{R \to b^-} F(R) = \sup F([a,b))$$

$$= \sup \{ F(R) : a \leq R < b \}$$

Teorema (confronto) Siano $f, g \in L_{loc}([a,b))$ con

$0 \le f(x) \le g(x)$ $\forall x \in [a,b)$.

Allora valgono

1) Se $g \in L([a,b)) \Rightarrow f \in L([a,b))$

2) $f \notin L([a,b)) \Longrightarrow g \notin L([a,b))$.

$$\overset{P_1}{} \qquad \overset{P_2}{}$$

$$(P_1 \text{ vero} \Rightarrow P_2 \text{ vero}) \Longleftrightarrow (P_2 \text{ falso} \Rightarrow P_1 \text{ falso})$$

Dim 1 $\quad 0 \le f(x) \le g(x)$. con $g \in L([a,b))$

$$\Rightarrow \lim_{R \to b^-} \int_a^R g(x)\,dx = \int_a^b g(x)\,dx \in \mathbb{R}$$

Voglio verificare che $\lim_{R \to b^-} \int_a^R f(x)\,dx$ esiste ed è finito.

Il limite esiste grazie all'Aut cont. Per la monotonia tra integrali

$$\int_a^R f(x)\,dx \le \int_a^R g(x)\,dx \qquad \forall a \le R < b$$

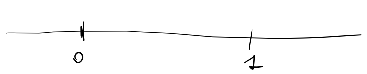
Per il teorema del confronto tra i limiti

$$0 \le \lim_{R \to b^-} \int_a^R f(x)\,dx \le \lim_{R \to b^-} \int_a^R g(x)\,dx = \int_a^b g(x)\,dx < +\infty$$

$$\Rightarrow \lim_{R \to b^-} \int_a^R f(x)\,dx \in \mathbb{R} \Rightarrow f \in L([a,b))$$

Esempio $\quad \frac{x^3}{1+x^4}$ in $[0,+\infty)$. Verifichiamo che non è integrabile in $[0,+\infty)$

Mi concentro su $[1,+\infty)$



$$\frac{x^3}{1+x^4} = \frac{x^3}{x^4(1+\frac{1}{x^4})} = \frac{1}{x}\,\frac{1}{1+\frac{1}{x^4}} \ge \frac{1}{x}\,\frac{1}{1+1} = \frac{1}{2}\,\frac{1}{x} \qquad \frac{1}{x} \notin L([1,\infty))$$

in $[1,+\infty)$ $\quad 0 < \frac{1}{x^4} \le 1$

$$\xRightarrow{\text{confronto}} \frac{x^3}{1+x^4} \notin L([1,+\infty)) \Rightarrow \frac{x^3}{1+x^4} \in L([0,+\infty))$$