December 5th

Exercise $\Omega \subseteq \mathbb{R}^d$ open

recall if $u \in L^p(\Omega)$ is s.t.

$$\int_\Omega u f \, dx = 0 \quad \forall f \in C_c^0(\Omega) \Rightarrow u = 0$$

Corollary $\forall \Omega \subseteq \mathbb{R}^d$ open then $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ $\forall 1 \leq p < \infty$

Proof If it is not true then let $Y = \overline{C_c^0(\Omega)}$ and so $Y \subsetneq L^p(\Omega) \Rightarrow \exists u \in L^{p'}(\Omega)$, $u \neq 0$ s.t.

$$\int_\Omega u f \, dx = 0 \quad \forall f \in C_c^0(\Omega) \Rightarrow u = 0$$

This gives a contradiction.

Prop $\Omega \subseteq \mathbb{R}^d$ open. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < +\infty$.

Pf start with $\Omega = \mathbb{R}^d$. take $\phi \in C_c^\infty(\mathbb{R}^d)$ $\operatorname{supp} \phi \subseteq D_{\mathbb{R}^d}(0,1)$

$\int_{\mathbb{R}^d} \phi \, dx = 1$ $\qquad \phi_\varepsilon = \varepsilon^{-d} \phi\left(\frac{\cdot}{\varepsilon}\right)$ $\qquad \varepsilon > 0$

Then $\forall \; g \in C_c^0(\mathbb{R}^d)$ we have

1) $\phi_\varepsilon * g \in C_c^\infty(\mathbb{R}^d)$ $\qquad \operatorname{supp} \phi_\varepsilon * g \subseteq \underbrace{\operatorname{supp} \phi_\varepsilon}_{\subseteq D(0,\varepsilon)} + \operatorname{supp} g$

2) $\phi_\varepsilon * g \xrightarrow{\varepsilon \to 0^+} g$ in $L^p(\mathbb{R}^d)$

$\Rightarrow \quad \overline{C_c^\infty(\mathbb{R}^d)} = \overline{C_c^0(\mathbb{R}^d)} = L^p(\mathbb{R}^d)$

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y) \, g(y) \, dy$$

If $f(x-y) g(y) \neq 0 \Rightarrow$ $x - y \in \operatorname{supp} f$, $y \in \operatorname{supp} g$

$x \in \operatorname{supp} f + y \in \operatorname{supp} f + \operatorname{supp} g$

If $\Omega \subsetneq \mathbb{R}^d$ $\qquad g \in C_c^0(\Omega)$ $\qquad K = \operatorname{supp} g$

$\operatorname{dist}(K, \partial\Omega) =: r > 0$

$\operatorname{supp} \phi_\varepsilon * g \subseteq \overline{D_{\mathbb{R}^d}(0,\varepsilon)} + K$

Notice that if $x \in$ $\overline{D_{\mathbb{R}^d}(0,\varepsilon)} + K$ $\Rightarrow$ $x = z + y$ $\quad y \in K$ and $|z| \le \varepsilon < r$

because $\Omega \supseteq \{ w \in \mathbb{R}^d : \operatorname{dist}(w, K) < r \}$

so for $0 < \varepsilon < r \Rightarrow \phi_\varepsilon * g \in C_c^\infty(\Omega)$

and $\phi_\varepsilon * g \xrightarrow{\varepsilon \to 0^+} g$ in $L^p(\mathbb{R}^d)$

$\Rightarrow \phi_\varepsilon * g \xrightarrow{\varepsilon \to 0^+} g \quad L^p(\Omega)$

Theorem (Kolmogorov-Riesz-Fréchet) Let $\mathcal{F} \subseteq L^p(\mathbb{R}^d)$ $\quad p<\infty$

1) $\mathcal{F}$ bounded in $L^p(\mathbb{R}^d)$: $\exists\, C>0$ s.t. $|f|_{L^p} \le C \quad \forall f \in \mathcal{F}$

2) $\forall \varepsilon>0\ \exists\, \delta(\varepsilon)>0$ s.t. $\underset{\mathbb{R}^d}{|h|}<\delta(\varepsilon) \Rightarrow |\tau_h f - f|_{L^p}<\varepsilon \ \forall f \in \mathcal{F}$

$\tau_h f = f(\cdot + h)$

Then $\forall\, \Omega \subseteq \mathbb{R}^d$ open and bounded

$$\mathcal{F}|_\Omega = \{ f|_\Omega : f \in \mathcal{F}\} \text{ is relatively compact in } L^p(\Omega)$$

($\overline{\mathcal{F}|_\Omega}$ is compact)

Pf. We will prove

$\forall \varepsilon>0 \quad \mathcal{F}|_\Omega$ is contained in a finite union of balls of radius ε in $L^p(\Omega)$.

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Claim: $\forall \varepsilon>0\ \exists\, \omega \subset\subset \Omega$ s.t. $|f|_{L^p(\Omega\setminus\omega)} \le \frac{\varepsilon}{\cancel{3}} \quad \forall f \in \mathcal{F}$

$\forall\, a,b \in \mathbb{R}_+$ let

$$T(a,b) = \{ f \in C^1(\mathbb{R}^d) : |f|_{L^\infty} \le a,\ |\nabla f|_{L^\infty} \le b \}$$

Then if $\omega \subset\subset \mathbb{R}^d$ $\quad T(a,b)|_\omega$ is relatively compact in $C^0(\omega,\mathbb{R})$ (Ascoli-Arzelà)

$\rho \in C_c^\infty(D_{\mathbb{R}^d}(0,1),[0,1]) \qquad \int_{\mathbb{R}^d} \rho\, dx = 1$

$\rho_n = n^d \rho(n\,\cdot) \qquad \operatorname{supp} \rho_n \subseteq D_{\mathbb{R}^d}(0,\tfrac{1}{n})$

$$|\rho_n * f - f|_{L^p(\mathbb{R}^d)} = \left| \int_{\mathbb{R}^d} \rho_n(y)\big(f(x-y)-f(x)\big)\,dy \right|_{L^p(\mathbb{R}^d)}$$

$$\le \int_{|y|<\frac{1}{n}} \rho_n(y)\, |\tau_{-y} f - f|_{L^p(\mathbb{R}^d)}\, dy < \frac{\varepsilon}{4} \underbrace{\int_{\mathbb{R}^d} \rho_n(y)\,dy}_{1}$$

$n > \frac{1}{\delta(\frac{\varepsilon}{4})} \qquad 0 < \frac{1}{n} < \delta(\frac{\varepsilon}{4}) \qquad \forall f \in \mathcal{F}$

$$|\rho_n * f - f|_{L^p(\mathbb{R}^n)} < \frac{\varepsilon}{4} \qquad \forall\, n > \frac{1}{\delta(\frac{\varepsilon}{4})} \quad \forall f \in \mathcal{F}.$$

$\forall\, n \in \mathbb{N}$

$$|\rho_n * f(x)| \le \int_{\mathbb{R}^d} \rho_n(x-y)\,|f(y)|\,dy \le |\rho_n(x-\cdot)|_{L^{p'}(\mathbb{R}^d)} \underbrace{|f|_{L^p(\mathbb{R}^d)}}_{\le C\ \forall f \in \mathcal{F}}$$

$\le a_n \qquad \forall f \in \mathcal{F}$

$$|\rho_n(x-\cdot)|_{L^{p'}} = |n^d \rho(n\,\cdot)|_{L^{p'}} = n^d |\rho(n\,\cdot)|_{L^{p'}} = n^d\, n^{-\frac{d}{p'}} |\rho|_{L^{p'}} = n^{\frac{d}{p}} |\rho|_{L^{p'}}$$

$a_n = C\, n^{\frac{d}{p}} |\rho|_{L^{p'}}$

$|\rho_n * f(x)| \le a_n \qquad \forall f \in \mathcal{F}$

$|\nabla \rho_n * f(x)| \le n^{\frac{d}{p}+1} |\nabla\rho|_{L^{p'}}\, C = b_n$

$\Rightarrow \{\rho_n * f : f \in \mathcal{F}\} \subseteq T(a_n,b_n) \qquad \forall n$

$T(a_n,b_n)|_\omega$ is relatively compact in $C_c^0(\omega, \cancel{\mathbb{R}})$ $\subseteq L^\infty(\omega) \subset L^p(\omega)$

Fix $n_0 > \frac{1}{\delta(\frac{\varepsilon}{4})}$

$\Rightarrow T(a_{n_0},b_{n_0}) \subseteq \bigcup_{j=1}^N D_{L^p(\omega)}(u_j,\tfrac{\varepsilon}{4})$

We claim $\mathcal{F}|_\Omega \subseteq \bigcup_{j=1}^N D_{L^p(\Omega)}(u_j,\varepsilon)$

$f \in \mathcal{F} \qquad \rho_{n_0} * f \in T(a_{n_0},b_{n_0}) \Rightarrow \exists\, u_j \in C_c^0(\omega)$

s.t. $|\rho_{n_0} * f - u_j|_{L^p(\omega)} < \frac{\varepsilon}{4}$

$$|f-u_j|_{L^p(\Omega)} \le \underbrace{|f-\overset{0}{u_j}|_{L^p(\Omega\setminus\omega)}}_{<\frac{\varepsilon}{4}} + |f-u_j|_{L^p(\omega)}$$

$$\le \frac{\varepsilon}{4} + \underbrace{|\rho_{n_0} * f - f|_{L^p(\mathbb{R}^d)}}_{\frac{\varepsilon}{4}} + \underbrace{|\rho_{n_0} * f - u_j|_{L^p(\omega)}}_{<\frac{\varepsilon}{4}} < \varepsilon.$$

Proof of the Claim: $f \in \mathcal{F}$

$$|f|_{L^p(\Omega\setminus\omega)} \le \underbrace{|\rho_{n_0} * f - f|_{L^p(\mathbb{R}^d)}}_{<\frac{\varepsilon}{4}} + |\rho_{n_0} * f|_{L^p(\Omega\setminus\omega)}$$

$$< \frac{\varepsilon}{4} + \underbrace{a_{n_0} |\Omega\setminus\omega|^{\frac{1}{p}}}_{\text{we choose } \omega \text{ s.t.}}$$

$a_{n_0} |\Omega\setminus\omega|^{\frac{1}{p}} < \frac{\varepsilon}{2}$

$\Rightarrow |\Omega\setminus\omega| < \frac{(\frac{\varepsilon}{2})^p}{a_{n_0}^p}$

Def A Pre Hilbert space on $\mathbb{R}$ is a space H on $\mathbb{R}$ with a symmetric bilinear form $(u,v)_H$ which is s.t. $(u,u) > 0 \ \forall u \neq 0$

Then $\|u\| = \sqrt{(u,u)}$ defines a norm and if H is complete for the norm than H is a Hilbert space.

Def Let H be $\mathbb{C}$ vector space a map $B : H \times H \to \mathbb{C}$

$$B(\lambda x + \mu y, z) = \lambda B(x,z) + \mu B(y,z)$$
$$B(x, \lambda y + \mu z) = \bar{\lambda} B(x,y) + \bar{\mu} B(x,z)$$

is Hermitian if
$$B(x,y) = \overline{B(y,x)}$$

positive if $B(x,x) \geq 0 \quad \forall x \in H$

nondegenerate if $B(x,x) = 0 \Rightarrow x = 0$

if $(x,y)_H$ has all the above properties then H is a Pre Hilbert space on $\mathbb{C}$.

$$\|x\| = \sqrt{(x,x)} \quad \text{defines a norm}$$

and if H is complete it is called a Hilbert space

$$(f,g)_{L^2(X,d\mu)} = \int_X f(x)\,\overline{g(x)}\,d\mu$$

Properties

$$\left\| \frac{a+b}{2} \right\|^2 + \left\| \frac{a-b}{2} \right\|^2 = \frac{1}{2} \left(\|a\|^2 + \|b\|^2 \right)$$

$$\left(\frac{a+b}{2}, \frac{a+b}{2} \right) + \left(\frac{a-b}{2}, \frac{a-b}{2} \right) = \frac{2}{4} (a, a) + \frac{2}{4} (b, b)$$

$$= \frac{1}{2} (a, a) + \frac{1}{2} (b, b)$$

$$|(a, b)| \leq \|a\| \, \|b\|$$

$$2(a, b) + 2(b, a) = \|a+b\|^2 - \|a-b\|^2 \leq \|a+b\|^2 + \|a-b\|^2 =$$

$$= (a+b, a+b) - (a-b, a-b) \qquad = 2\|a\|^2 + 2\|b\|^2$$

$$= 2(a, b) + 2(b, a)$$

$$2 \operatorname{Re}(a, b) \leq \|a\|^2 + \|b\|^2$$

$$2 \operatorname{Re}\left(\lambda a, \frac{b}{\lambda}\right) \leq \lambda^2 \|a\|^2 + \frac{1}{\lambda^2} \|b\|^2 \qquad \forall \lambda > 0$$

$$\lambda \|a\| = \frac{1}{\lambda} \|b\| \qquad \lambda = \left(\frac{\|b\|}{\|a\|} \right)^{\frac{1}{2}}$$

$$2 \operatorname{Re}(a, b) \leq \lambda^2 \|a\|^2 + \frac{1}{\lambda^2} \|b\|^2 \qquad \forall \lambda > 0$$

$$= 2 \frac{\|b\|}{\|a\|} \|a\|^2 = 2 \|a\| \, \|b\|$$

$$\Rightarrow \quad \operatorname{Re}(a, b) \leq \|a\| \, \|b\|$$

$$\|a + b\| \leq \|a\| + \|b\| \qquad \checkmark$$

$$\|\lambda a\| = |\lambda| \, \|a\|$$

$$\|a + b\|^2 = (a+b, a+b) = \|a\|^2 + \|b\|^2 + 2 \underbrace{\operatorname{Re}(a, b)}_{\leq \|a\| \|b\|}$$

$$\leq \|a\|^2 + \|b\|^2 + 2 \|a\| \, \|b\| = (\|a\| + \|b\|)^2$$

$$\Rightarrow \quad \checkmark$$

Then H hilbert is uniformly convex.

$$\left\| \frac{a+b}{2} \right\|^2 = \frac{1}{2}\|a\|^2 + \frac{1}{2}\|b\|^2 - \left\| \frac{a-b}{2} \right\|^2$$

$\|a\| \le 1$

$\|b\| \le 1$

$\|a-b\| \ge \varepsilon$

$$\le 1 - \frac{\varepsilon^2}{4}$$

$$\left\| \frac{a+b}{2} \right\| \le \sqrt{1 - \frac{\varepsilon^2}{4}} = 1 - \overbrace{\left(1 - \sqrt{1 - \frac{\varepsilon^2}{4}}\right)}^{\delta(\varepsilon)}$$