6 Dicembre

<u>Teorema</u> (confronto asintotico) Siano $f, g \in L_{loc}([a,b))$

$f(x) \geq 0 \quad \forall x \qquad g(x) > 0 \quad \forall x.$

e supponiamo che esista $\displaystyle\lim_{x \to b^-} \frac{f(x)}{g(x)} = \ell$

Vale quanto segue

1) Se $0 < \ell < +\infty$

$$f \in L([a,b)) \iff g \in L([a,b))$$

2) Se $\ell = 0$

$$g \in L([a,b)) \implies f \in L([a,b))$$

$$\Updownarrow$$

$$\Big( \; f \notin L([a,b)) \implies g \notin L([a,b))$$

3) Se $\ell = +\infty$

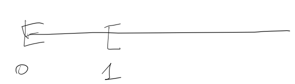
$$f \in L([a,b)) \implies g \in L([a,b))$$

$$\Updownarrow$$

$$\Big( \; g \notin L([a,b)) \implies f \notin L([a,b))$$

Esempi $\qquad f(x) = \dfrac{x^3}{1+x^4} \qquad f \overset{?}{\in} L([0,+\infty))$

$f \in L([0,+\infty)) \iff f \in L([1,+\infty))$



$$\lim_{x \to +\infty} \frac{f(x)}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{\frac{x^3}{1+x^4}}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{x^4}{1+x^4} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^4}}{\cancel{x^4}} = \lim_{x \to +\infty} 1 = 1 \neq 0$$

$$\boxed{\frac{ad}{bc} = \frac{\frac{a}{b}}{\frac{c}{d}}}$$

$f \in L([1,+\infty)) \iff x^{-1} \in L([1,+\infty))$

sappiamo che è falso

$\implies f \notin L[1,+\infty)$

Esempio $\displaystyle\int_0^{+\infty} (1-th(x))\, dx$ è sommabile?

$$1-th(x) = 1 - \frac{e^x - e^{-x}}{e^x+e^{-x}} = \frac{\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}}{e^x+e^{-x}} =$$

$$= \frac{2e^{-x}}{e^x+e^{-x}} = \frac{2e^{-x}}{e^x(1+e^{-2x})} =$$

$$= 2e^{-2x}\frac{1}{1+e^{-2x}} = 2e^{-2x}(1+o(1))$$

$$\lim_{x\to+\infty}\frac{1-th(x)}{e^{-2x}} = \lim_{x\to+\infty}\frac{2\cancel{e^{-2x}}}{\cancel{e^{-2x}}}\cancel{(1+o(1))} = 2$$

$$1-th(x) \in L([0,+\infty)) \iff e^{-2x} \in L([0,+\infty))$$

$$\int_0^{+\infty} e^{-2x}\,dx = \lim_{R\to+\infty}\int_0^R e^{-2x}\,dx = \lim_{R\to+\infty}\left.\frac{e^{-2x}}{-2}\right]_0^R$$

$$= \frac{1}{2}$$

$$\int_1^{+\infty} \lg\left(1+\frac{1}{x^p}\right) dx \qquad p > 0$$

Per quali $p > 0$ è sommabile?

$$\lim_{y \to 0} \frac{\lg(1+y)}{y} = 1$$

$$\lg(1+y) = y\,(1+o(1)) \qquad \text{per } y \text{ vicino a } 0$$

$$\lg\left(1+\frac{1}{x^p}\right) = \frac{1}{x^p}\,(1+o(1)) \qquad \text{per } x \to +\infty$$

$$\lim_{x \to +\infty} \frac{\lg\left(1+\frac{1}{x^p}\right)}{\frac{1}{x^p}} = \lim_{\substack{y \to 0^+ \\ y = \frac{1}{x^p}}} \frac{\lg(1+y)}{y} = 1 \neq 0$$

$$\lg\left(1+\frac{1}{x^p}\right) \in L([1,+\infty)) \iff \frac{1}{x^p} \in L([1,+\infty))$$

$$\iff p > 1$$

Esempio $[x] : \mathbb{R} \longrightarrow \mathbb{Z}$ definito da $[x] \le x < [x]+1$

(Esercizio $\displaystyle\lim_{x\to\infty} \frac{[x]}{x} = 1$)

Per quali $p$ risulta

$$\int_2^{+\infty} \frac{1}{[x]^p}\, dx \quad \text{sommabile ?}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

Abbiamo appena osservato che $[x] = x\,(1+o(1))$

$$\lim_{x\to+\infty} \frac{\frac{1}{[x]^p}}{\frac{1}{x^p}} = \lim_{x\to+\infty} \frac{x^p}{[x]^p} = \lim_{x\to+\infty} \left(\frac{x}{[x]}\right)^p = 1$$

Pertanto

$$\frac{1}{[x]^p} \in L([2,+\infty)) \iff \frac{1}{x^p} \in L([2,+\infty)) \iff p > 1.$$

Calcolare

$$\int_3^{+\infty} \underbrace{\frac{1}{x^3-3x^2+2x}}_{R(x)} \, dx = ?$$

$$= \int_3^{+\infty} \frac{1}{x(x-1)(x-2)} \, dx = \lim_{R \to +\infty} \int_3^{R} \frac{1}{x(x-1)(x-2)} \, dx$$

$$R \in L_{loc}([3,+\infty)) \quad e \quad \lim_{x \to +\infty} \frac{R(x)}{\frac{1}{x^3}} = 1$$

$$\Rightarrow R \in L([3,+\infty))$$

$$\frac{1}{x-3} \notin L((3,4]) \iff \frac{1}{x} \notin L((0,1])$$

$$\frac{1}{(x-3)^2} \qquad \frac{1}{x^p} \qquad \frac{1}{(x-1)^p}$$

$$\frac{1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} = \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}$$

$$\int_3^R \left[ \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2} \right] dx =$$

$$= \frac{1}{2} \int_3^R \frac{dx}{x} - \int_3^R \frac{1}{x-1} \, dx + \frac{1}{2} \int_3^R \frac{1}{x-2} \, dx$$

$$= \frac{1}{2} \lg x \Big]_3^R - \lg(x-1) \Big]_3^R + \frac{1}{2} \lg(x-2) \Big]_3^R =$$

$$= \frac{1}{2} \lg(R) - \lg(R-1) + \frac{1}{2} \lg(R-2)$$

$$- \frac{1}{2} \lg 3 + \lg 2 - \frac{1}{2} \underbrace{\lg 1}_{0}$$

$$= \underbrace{\lg \underbrace{\frac{\sqrt{R}\,\sqrt{R-2}}{R-1}}_{\downarrow\, 1}}_{\downarrow\, 0} + \underbrace{\lg \frac{2}{\sqrt{3}}}_{>0}$$

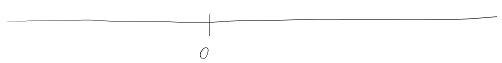
**Def** Sia $f \in L_{loc}([a,b))$ e sia $|f| \in L([a,b))$

Allora diciamo che $f$ è assolutamente integrabile in $[a,b)$

**Teorema** Se $f$ è assolutamente integrabile in $[a,b)$

allora $f \in L([a,b))$

**Es** $f = \sin(x) \dfrac{1}{1+x^2}$ è integrabile in $\mathbb{R}$ ?



$$f \in L(\mathbb{R}) \iff f \in L([0,+\infty)) \text{ e } f \in L((-\infty,0])$$

$$\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{+\infty} f(x)\,dx$$

$$f \in C^\infty(\mathbb{R}) \Rightarrow f \in L_{loc}(\mathbb{R})$$

$$|f(x)| = \frac{|\sin x|}{1+x^2} \leq \frac{1}{1+x^2} \Rightarrow f \text{ assolutamente integrabile}$$

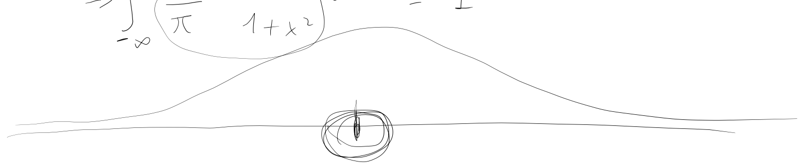
$$\Rightarrow f \in L(\mathbb{R})$$

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \arctan(R) = \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{R\to-\infty} \int_R^0 \frac{1}{1+x^2}\,dx = \lim_{R\to-\infty} -\arctan(R) = \frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} = \pi$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{1}{1+x^2}\,dx = 1$$



$\lambda > 0$

$$\frac{1}{\lambda}\,\frac{1}{\pi}\,\frac{1}{1+\lambda^2(x-x_0)^2}$$

Esistono funzioni integrabili ma non assolutamente integrabili

$$\int_1^{+\infty} \frac{\sin x}{x^p}\, dx \quad \text{è sommabile}$$

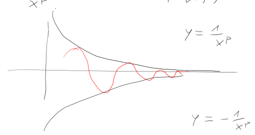
se $p > 1$ $\quad \left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} \leq \frac{1}{x^p} \in L([1,+\infty))$

$\Rightarrow \frac{\sin x}{x^p}$ è assolut int in $[1,+\infty)$ se $p > 1$

$\Rightarrow \frac{\sin x}{x^p} \in L([1,+\infty))$

Ora dimostriamo che $\frac{\sin x}{x^p} \in L([1,+\infty))$ $\forall\, p > 0$.

ma per $0 < p \leq 1$ $\quad \left|\frac{\sin x}{x^p}\right| \notin L([1,+\infty))$

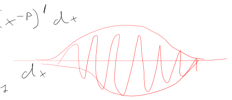



$$\int_1^R \frac{\sin x}{x^p}\, dx =$$

$$= \int_1^R (-\cos x)'\, x^{-p}\, dx =$$

$$= -\cos x\, x^{-p}\Big]_1^R - \int_1^R (-\cos x)\,(x^{-p})'\, dx$$

$$= \cos 1 - \frac{\cos R}{R^p} - \int_1^R \frac{\cos x}{x^{p+1}}\, dx$$



$\frac{\cos x}{x^{p+1}}$ è assolutamente integrabile in $[1,+\infty$

$$\frac{|\cos x|}{x^{p+1}} \leq \frac{1}{x^{p+1}} \in L([1,+\infty)) \quad \text{dove } p > 0 \Rightarrow p+1 > 1$$

$$\Rightarrow \frac{\cos x}{x^{p+1}} \in L([1,+\infty)$$

$$\Rightarrow \lim_{R\to+\infty} \int_1^R \frac{\cos x}{x^{p+1}}\, dx = \int_1^{+\infty} \frac{\cos x}{x^{p+1}}\, dx \in \mathbb{R}$$

$$\lim_{R\to+\infty} \int_1^R \frac{\sin x}{x^p}\, dx = \lim_{R\to+\infty}\left[\cos 1 - \frac{\cos R}{R^p} - \int_1^R \frac{\cos x}{x^{p+1}}\, dx\right] =$$

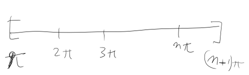
$$= \cos(1) - \int_1^{+\infty} \frac{\cos x}{x^{p+1}}\, dx \in \mathbb{R}$$

ora dimostriamo che

$$\frac{|\sin x|}{x^p} \notin L([1,+\infty) \quad \text{se} \quad 0 < p \leq 1$$

$$\Updownarrow$$

$$\frac{|\sin x|}{x^p} \notin L([\pi,+\infty)$$




$$\int_\pi^{(n+1)\pi} \frac{|\sin x|}{x^p}\, dx = \sum_{j=1}^{n} \int_{j\pi}^{(j+1)\pi} \frac{|\sin x|}{x^p}\, dx \geq \sum_{j=1}^{n} \int_{j\pi}^{(j+1)\pi} \frac{|\sin x|}{(j+1)^p \pi^p} =$$

$j\pi \leq x \leq (j+1)\pi$ $\qquad \frac{1}{x} \geq \frac{1}{(j+1)\pi}$

$$\frac{1}{x^p} \geq \frac{1}{(j+1)^p\, \pi^p}$$

$$= \frac{1}{\pi^p} \sum_{j=1}^{n} \frac{1}{(j+1)^p} \underbrace{\int_{j\pi}^{(j+1)\pi} |\sin x|\, dx}_{2} = \frac{2}{\pi^p} \sum_{j=1}^{n} \frac{1}{(j+1)^p}$$

$|\sin(x+k\pi)| = |\sin(x)|$ $\quad \forall\, k \in \mathbb{Z}$

$$\int_{j\pi}^{(j+1)\pi} |\sin x|\, dx = \qquad x = y + j\pi, \quad dx = dy$$

$$= \int_0^\pi |\sin(y+j\pi)|\, dy = \int_0^\pi |\sin y|\, dy = \int_0^\pi \sin y = -\cos y\Big]_0^\pi$$

$$= 2$$

$$\int_{\pi}^{(n+1)\pi} \frac{|\operatorname{sen} x|}{x^p}\, dx \geq \frac{2}{\pi^p} \sum_{j=1}^{n+1} \frac{1}{(j+1)^p} =$$

$$= \frac{2}{\pi^p} \int_{1}^{n+1} \frac{1}{([x]+1)^p}\, dx \xrightarrow{n \to +\infty} +\infty$$

$$= \frac{2}{\pi^p} \sum_{j=1}^{n} \int_{j}^{j+1} \frac{1}{([x]+1)^p}\, dx$$

$$= \frac{2}{\pi^p} \sum_{j=1} \frac{1}{(j+1)^p}$$

perchè

$$\frac{1}{([x]+1)^p} \notin L([1,+\infty)) \quad \text{se} \quad p \leq 1$$

$$\Rightarrow \lim_{n \to +\infty} \int_{\pi}^{(n+1)\pi} \frac{|\operatorname{sen} x|}{x^p}\, dx = +\infty \quad \text{per } 0 < p \leq 1$$