$S_\lambda : L^p(\mathbb{R}^d) \circlearrowleft$ $\qquad S_\lambda$ is an isometry $\forall \lambda$

$S_\lambda f(x) = \lambda^{\frac{d}{p}} f(\lambda x)$ $\qquad p < \infty \qquad \lambda_0 > 0$

$S_\lambda \xrightarrow{\lambda \to \lambda_0} S_{\lambda_0}$ strong convergence

$S_\lambda f \xrightarrow{\lambda \to \lambda_0} S_{\lambda_0} f \qquad \forall f \in L^p(\mathbb{R}^d)$

$S_\lambda \xrightarrow{\lambda \to \lambda_0} S_{\lambda_0}$ in $\mathcal{L}(L^p(\mathbb{R}^d))$

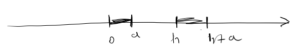
In other words $\|S_\lambda - S_{\lambda_0}\|_{\mathcal{L}(L^p(\mathbb{R}))} \xrightarrow{\lambda \to \lambda_0} 0$?

This is not true. $\lambda > 1$

$\|S_\lambda - 1\|_{\mathcal{L}(L^p(\mathbb{R}))} \geq \sqrt[p]{2}$

$\tau_h f(x) = f(x-h) \qquad \tau_h : L^p \circlearrowleft$

$|\tau_h - 1|_{\mathcal{L}(L^p)} \xrightarrow{h \to 0} 0$ false. Because $\forall h$



$L^p(\mathbb{R}) \qquad h > 0$

$\chi_a = \dfrac{1_{[0,a]}}{a^{\frac{1}{p}}}$ $\qquad \|\tau_h \chi_a - \chi_a\|^p_{L^p(\mathbb{R})} = \qquad |\chi_a|_{L^p(\mathbb{R})} = 1$

$= \| \underbrace{1_{[0,a]}(x-h)}_{1_{[h,h+a]}} - 1_{[0,a]} \|^p_{L^p(\mathbb{R})} \frac{1}{a}$

for $0 < a < h$

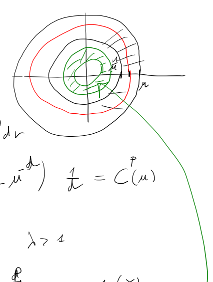
$= \frac{1}{a} \| 1_{[0,a]} \|^p_{L^p} + \frac{1}{a} \| 1_{[h,h+a]} \|^p_{L^p} = 2$

$\|\tau_h - 1\|_{\mathcal{L}(L^p)} \geq \|(\tau_h - 1)\chi_a\|_{L^p} = \sqrt[p]{2}$

$S_\lambda f(x) = \lambda^{\frac{d}{p}} f(\lambda x) \qquad \lambda > 1$

$\|S_\lambda - 1\|_{\mathcal{L}(L^p)} \geq \sqrt[p]{2}$

$\mu > 1 \qquad 1_{\frac{1}{\mu} < |x| < \mu}(x)$



$| 1_{\frac{1}{\mu} < |x| < \mu} |^p_{L^p} = \int_{S^{d-1}} dS \int_{\frac{1}{\mu}}^{\mu} r^{d-1} dr$

$= |S^{d-1}| \left(\mu^d - \mu^{-d}\right) \frac{1}{d} = C^p(\mu)$

$\chi_\mu(x) = \frac{1}{C(\mu)} 1_{\frac{1}{\mu} < |x| < \mu}(x) \qquad \lambda > 1$

$S_\lambda \chi_\mu(x) = \lambda^{\frac{p}{d}} \frac{1}{C(\mu)} 1_{\frac{1}{\mu} < \lambda|x| < \mu}(\lambda x) = \lambda^{\frac{p}{d}} \frac{1}{C(\mu)} 1_{\frac{1}{\lambda\mu} < |x| < \frac{\mu}{\lambda}}(x)$

For fixed $\lambda > 1$ I want $\mu > 1$ st

$\{x : \frac{1}{\lambda\mu} < |x| < \frac{\mu}{\lambda}\} \cap \{x : \frac{1}{\mu} < |x| < \mu\} = \emptyset$

This will happen $\frac{\mu}{\lambda} < \frac{1}{\mu} \qquad 1 < \mu < \sqrt{\lambda}$

$|S_\lambda \chi_\mu - \chi_\mu|^p_{L^p} = |S_\lambda \chi_\mu|^p_{L^p} + |\chi_\mu|^p_{L^p} = 2$

$\|S_\lambda - 1\|_{\mathcal{L}(L^p)} \geq |(S_\lambda - 1)\chi_\mu|_{L^p} \geq \sqrt[p]{2}$

$\begin{cases} \partial_t u = \Delta u \\ u(0) = u_0 \in L^p(\mathbb{R}^d) \end{cases} \qquad p < +\infty$

$u(t) = e^{t\Delta} u_0 = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy$

$= K_{\sqrt{t}} * u_0 \xrightarrow{t \to 0^+} u_0$

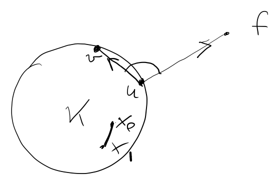
$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right) \qquad K(x) = \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4}}$

$e^{t\Delta} \xrightarrow{t \to 0^+} 1$ is not true in the norm of $\mathcal{L}(L^p(\mathbb{R}^d))$!

Thm: Let H be Hilbert space $K \subseteq H$ K is convex and closed. Then $\forall f \in H \; \exists! \; u \in K$

s.t. $\|f-u\| \le \|f-v\| \quad \forall v \in H$

u is characterized by



$\mathrm{Re}(f-u, v-u) \le 0$

$\forall v \in K$

Pf $\phi(x) = \underset{x\in K}{\|x-f\|}$ $\quad \lim_{x\to\infty} \phi(x) = +\infty$

$\phi \in C^0(K, \mathbb{R})$ and since H is reflexive

ϕ has a minimizer. $\quad \cdot \phi(x_0) \quad \phi^{-1}([0, \phi(x_0)]) = \tilde{K}$

$\tilde{K}$ $\quad \exists$ a point abs minimum

$\phi((1-t)x_0+tx_1) = \|(1-t)x_0 + tx_1 - f\| = \|(1-t)(x_0-f) + t(x_1-f)\| \le$

$\le (1-t)\|x_0-f\| + t\|x_1-f\|$

$\le (1-t)\phi(x_0) + t\phi(x_1)$

u_1 a minim $\quad \mathrm{Re}(f-u_1, v-u_1) \le 0 \quad \forall v \in K$

$u_2 \quad \mathrm{Re}(f-u_2, v-u_2) \le 0$

$\mathrm{Re}(f-u_1, u_2-u_1) \le 0$

$\mathrm{Re}(f-u_2, u_1-u_2) \le 0$

$0 \ge \mathrm{Re}(\cancel{f}-u_1, u_2-u_1) + \mathrm{Re}(\cancel{f}-u_2, u_1-u_2)$

$= \mathrm{Re}(u_1, u_1-u_2) + \mathrm{Re}(-u_2, u_1-u_2)$

$= \mathrm{Re}(u_1-u_2, u_1-u_2) = \|u_1-u_2\|^2 \Rightarrow u_1 = u_2$

$\|u-f\| \le \|v-f\| \quad \forall v \in K \Rightarrow \mathrm{Re}(u-f, v-u) \le 0 \quad \forall v \in K$

$t \in [0,1]$

$\frac{d}{dt}\|f-u-t(v-u)\|^2\Big|_{t=0} = \frac{d}{dt}\left[\|f-u\|^2 - 2\,\mathrm{Re}(f-u, v-u)\,t + t^2\|v-u\|^2\right]\Big|_{t=0}$

$u + t(v-u) = (1-t)u + tv$

$= -2\,\mathrm{Re}(f-u, v-u) \ge 0$

$\Rightarrow \mathrm{Re}(f-u, v-u) \le 0 \quad \forall v \in K$

Let now $u \in K$ satisfy

$\mathrm{Re}(f-u, v-u) \le 0 \quad \forall v \in K \Rightarrow u$ is the minimizer

If u is not the minimizer then $\bar{u}$ is the minimizer

Prop $K \subseteq H$ closed and convex and for any $f \in H$ let $P_K f \in K$ be the u of the previous theorem. Then

$$\| P_K f - P_K g \| \leq \| f - g \| .$$

Pf $u = P_K f$ $v = P_K g$

$Re\,(f-u, w-u) \leq 0$ $\forall w \in K$

$Re\,(g-v, w-v) \leq 0$ $\forall w \in K$

$Re\,(f-u, v-u) \leq 0$
$Re\,(g-v, u-v) \leq 0$

$0 \geq Re\,(f-u, v-u) - Re\,(g-v, v-u) =$

$= Re\,(f-g, v-u) + \underbrace{Re\,(v-u, v-u)}_{\|v-u\|^2}$

$\|u-v\|^2 \leq Re\,(f-g, u-v) \leq \|f-g\| \, \|u-v\|$

$\|u-v\| \leq \|f-g\|$

Theorem Let $f \in H'$. Then $\exists\, y \in H$ st

$$f(x) = (x, y)_H \qquad \forall x \in H.$$

Pf $T : H \longrightarrow H'$

$$\underset{\underset{H}{\uparrow}}{z} \longrightarrow Tz = (\ , z)_H$$

$$\left| \langle Tz, w \rangle_{H' \times H} \right| = \left| (w, z)_H \right| \le \|w\|_H \|z\|_H$$

$$\Rightarrow \quad \|Tz\|_{H'} \le \|z\|_H$$

$$\|Tz\|_{H'} \ge \left| \langle Tz, \frac{z}{\|z\|_H} \rangle_{H' \times H} \right| = \left| \left(\frac{z}{\|z\|_H}, z \right)_H \right| = \frac{1}{\|z\|_H} \|z\|_H^2 = \|z\|_H$$

$$\|Tz\|_{H'} \ge \|z\|_H$$

$$\|Tz\|_{H'} = \|z\|_H \qquad R(T) = H'$$

Suppose $\boxed{R(T) \subsetneq H'}$. $\exists\, h \overset{\ne 0}{\in} H''$ st.

$$\langle Tx, h \rangle_{H' \times H''} = 0 \qquad \forall x \in H$$

H is reflexive $J : H \longrightarrow H''$ is an isomorphism

$$\Rightarrow h = Jy \quad \text{for an } y \in H \qquad y \ne 0$$

$$0 = \langle Tx, Jy \rangle_{H' \times H''} = \langle Tx, y \rangle_{H' \times H} = (y, x)_H \quad \forall x \in H$$

false because $y \ne 0 \Rightarrow (y, y)_H > 0$

$$\Rightarrow R(T) = H'$$

Def a subset $S \subset H$ is orthonormal if

$\|x\| = 1 \quad \forall x \in S$ and

$(x, y) = 0 \quad \forall \; x \neq y$ in S

Thm Let $S \subset H$ orthonormal. Then

1) $\forall u \in H$

$$\sum_{s \in S} |(u,s)|^2 \le \|u\|^2 \quad \text{(Bessel inequality)}$$

2) Let $V_S = \overline{\mathrm{Span}\{s : s \in S\}}$. The following are equivalent

a) $u \in V_S$

b) $\sum_{s \in S} |(u,s)|^2 = \|u\|^2$

c) $\sum_{s \in S} (u,s)_H \, s = u$ in H

Pf Let card S = card $\mathbb{N}$

$S = \{ s_j \}_{j \in \mathbb{N}} \qquad s_1, \dots, s_n$

$$S_n u = \sum_{j=1}^{n} (u, s_j) s_j$$

$$\| S_n u \|^2 = \sum_{j=1}^{n} |(u, s_j)|^2 \qquad \checkmark$$

$$\| u - S_n u \|^2 = \| u \|_H^2 - \| S_n u \|_H^2$$

$$(u - S_n u, u - S_n u) = (u, u - S_n u) - \underbrace{(S_n u, u - S_n u)}_{0}$$

$$(S_n u, u - S_n u) = \sum_{j=1}^{n} (u, s_j) \left(s_j, u - \sum_{k=1}^{n} (u, s_k) s_k \right) =$$

$$= \sum_{j=1}^{n} (u, s_j) \left((s_j, u) - \sum_{k=1}^{n} \overline{(u, s_k)} \underbrace{(s_j, s_k)}_{\delta_k} \right)$$

$$= \sum_{j=1}^{n} (u, s_j) \left((s_j, u) - \overline{(u, s_j)} \right) = 0$$

$$= \|u\|^2 - (u, S_n u) =$$

$$= \|u\|^2 - \sum_{j=1}^{n} (u, (u, s_j) s_j)$$

$$= \|u\|^2 - \sum_{j=1}^{n} \underbrace{\overline{(u, s_j)} \, (u, s_j)}_{|(u, s_j)|^2} = \|u\|^2 - \|S_n u\|^2$$

$$\| u - S_n u \|^2 = \|u\|^2 - \|S_n u\|^2 \le 0 \qquad \forall n$$

$$\| S_n u \|^2 \le \|u\|^2$$

$$\sum_{j=1}^{n} |(u, s_j)|^2 \le \|u\|^2$$

$$\sum_{s \in S} |(u, s)|^2 \le \|u\|^2$$

If card $S >$ card $\mathbb{N}$

$$\hat{S} = \{ s \in S : (u, s)_H \neq 0 \}$$

I claim card $\hat{S} \le$ card $\mathbb{N}$

otherwise $\exists \quad \alpha > 0$ s.t.

$\{ s \in \hat{S} : |(u,s)| \ge \alpha \}$ has infinite cardinality