993SM - Laboratory of Computational Physics
Unit XII
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Deterministic chaos

- Determinism and predictability
- **Deterministic chaos and absolute chaos**
- Logistic map (several visualisation methods)
- Fractals
- **Measuring chaos**
- Chaos in classical billiards
- **Deterministic chaos and roundoff**

M. Peressi - UniTS - Laurea Magistrale in Physics
Laboratory of Computational Physics - Unit XII

Determinism and predictability

Deterministic chaos and absolute chaos

Determinism

Determinism indicates that every event is determined by a chain of prior occurrences.

Pierre Simon de Laplace (1749-1827) strongly believed in **causal determinism**:

" We ought to regard the present state of the universe as the effect of its antecedent state and as the cause of the state that is to follow. An intelligence knowing all the forces acting in nature at a given instant, as well as the momentary positions of all things in the universe, would be able to **comprehend in one single formula** the motions of the largest bodies as well as the lightest atoms in the world, provided that its intellect were sufficiently powerful to subject all data to analysis; to it **nothing would be uncertain**, the future as well as the past would be present to its eyes."

(from: "Essai philosophique sur les probabilites")

Predictability

Determinism ≠ predictability

The world could be highly predictable, in some senses, and yet not deterministic; and it could be deterministic yet highly unpredictable...

Determinism: related to the nature of the physical system

Predictability: related to what we can do (observe, analyze, calculate); to predict something we need:

- knowledge of initial conditions
- capability of solving exactly the equation of evolution

Chaos and determinism

a system is **chaotic** if its trajectory through the configuration space is sensitively dependent on the initial conditions, that is, if very small causes can produce large effects

(in meteorology: "butterfly effect")

Chaos and determinism

In the last few decades, physicists have become aware that even systems studied by classical mechanics can behave in an intrinsically unpredictable manner. Although such a system may be perfectly deterministic in principle, its behavior is completely unpredictable in practice. This phenomenon was called **deterministic chaos.**

Deterministic chaos is not randomness

Deterministic chaos is not the same as absolute chaos. Absolute chaos or randomness is when you don't know nothing at all of what will be the next value: it can be any value!

Another important difference is that for deterministic chaos we have a simple law that will produce all the values in the “attractor”. Instead for randomness there is no known recipe to produce past and future values.

Chaos and determinism:
logistic map;
Mandelbrot function and fractals

Chaos and determinism

Deterministic chaos described by
intrinsically NON LINEAR equations.
E.g., dynamics of population:

$$x_{n+1} = 4rx_n(1 - x_n)$$

x_n is the ratio of the population in the nth generation to a reference population.

WHICH DYNAMICAL BEHAVIOR?

The logistic map

realistic model in which the population is bounded

$$P_{n+1} = P_n(a - bP_n)$$

rescale the population by letting $P_n = (a/b)x_n$

$$x_{n+1} = ax_n(1 - x_n)$$

define the parameter $r = a/4$ and obtain

$$x_{n+1} = f(x_n) = 4rx_n(1 - x_n)$$

- f is called a *one-dimensional map*
- The sequence of values $x_0,\ x_1,\ x_2,\ \cdots$ is called the *trajectory* or the *orbit*.
- x^* is a **fixed point** if $x_{n+1} = x_n = x^*$, i.e., $f(x^*) = x^*$

The logistic map

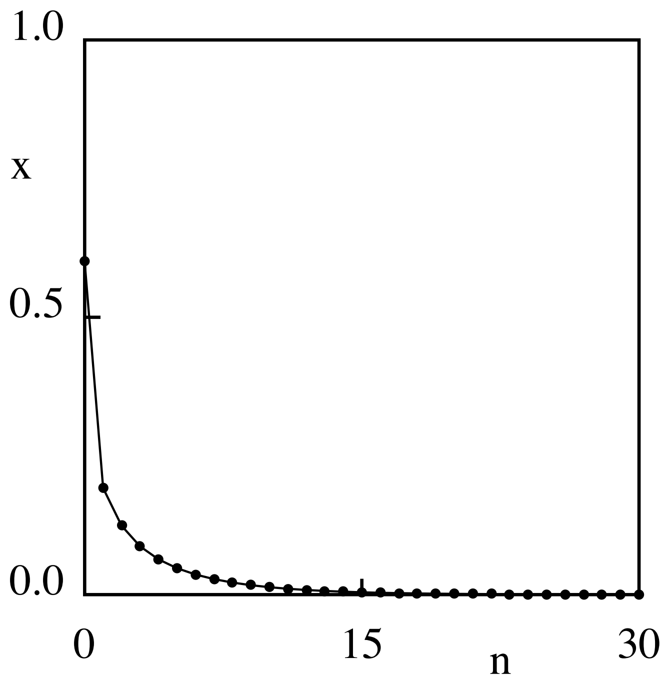
$$x_{n+1} = 4rx_n(1 - x_n)$$

$$0 \le x \le 1; \quad 0 < r \le 1 \quad (*)$$

(*): condition $(f(x))_{max} \le 1 \Rightarrow r \le 1$; x^*=fixed point $\le 1 \Rightarrow r > 0$

1D plots: x(n)

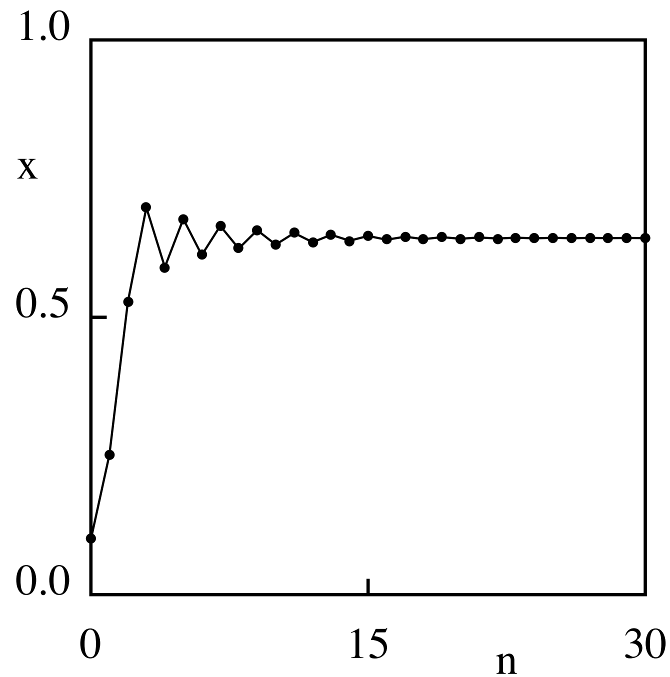
examples of **convergent** trajectories:



(a)

$r = 0.2$ and $x_0 = 0.6$

(stable fixed point is $x = 0$)



(b)

$r = 0.7$ and $x_0 = 0.1$.

initial transient behavior

The logistic map

fixed-point condition is given by $f(x^*) = x^*$

$$x_{n+1} = 4rx_n(1 - x_n)$$

$$\Rightarrow \quad x^*_1 = 0 \quad \text{and} \quad x^*_2 = 1 - \frac{1}{4r}$$

stable fixed point :
for sufficiently small r, the iterated values of x converge to $x = 0$ independently of the value of x_0

unstable if for almost all x_0 near the fixed point, the trajectories diverge from it

It can be demonstrated that:

$$x^*_1 = 0 \text{ is stable for } 0 < r < 1/4$$

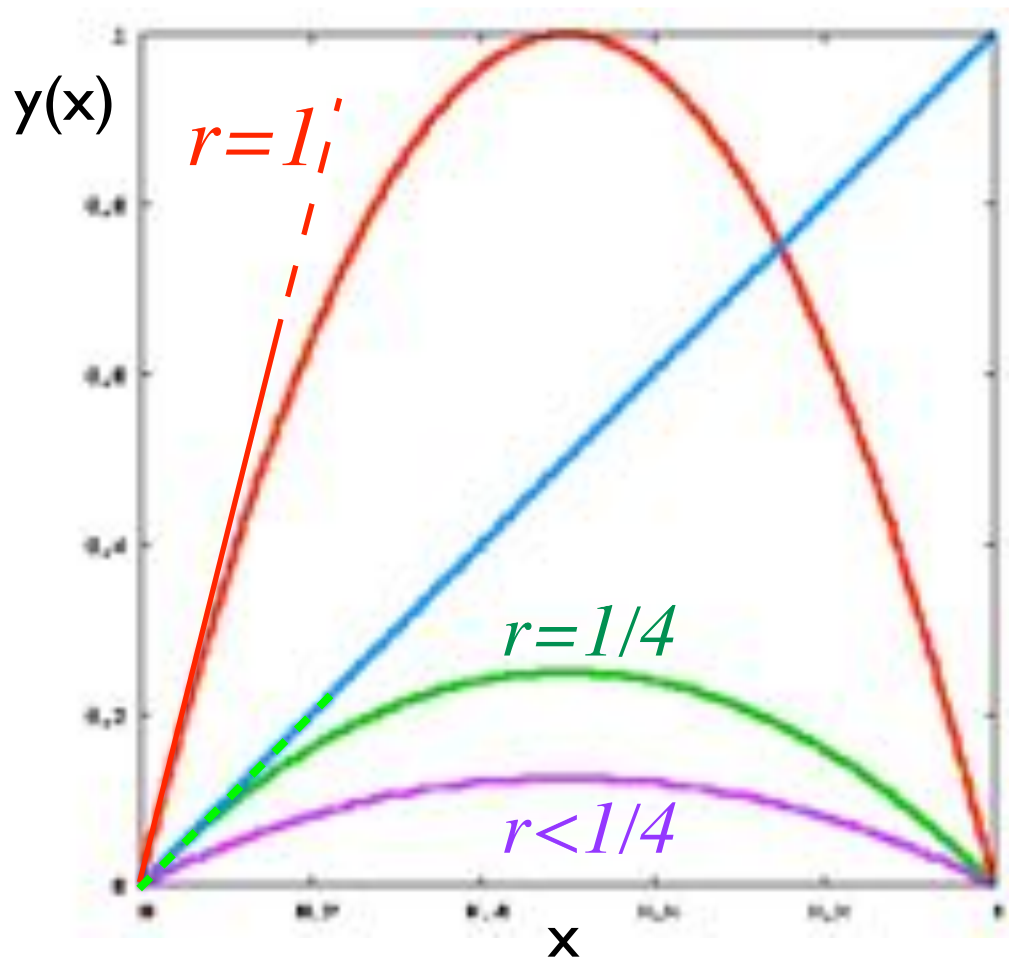
$$x^*_2 = 1 - \frac{1}{4r} \quad \text{is stable for} \quad \frac{1}{4} < r < \ldots ? \qquad (< 1)$$

(condition $x^*_2 > 0$)

The logistic map

$$x_{n+1} = 4rx_n(1-x_n) \quad \Rightarrow \quad y(x) = 4rx(1-x)$$

$$y'(x=0) = 4r$$

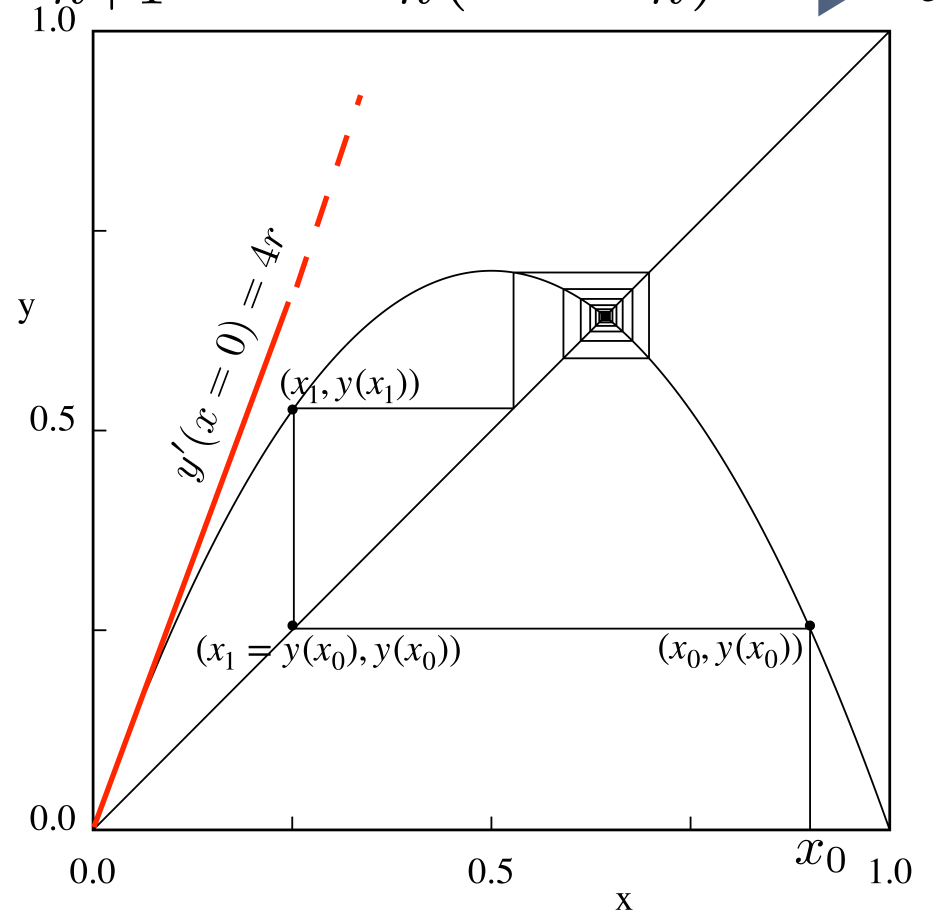


Graphical interpretation of the logistic map: intersection with the diagonal **(solution other than x=0) for** $1/4 \leq r \leq 1$

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \rightarrow y(x) = 4rx(1-x)$$

$$y'(x=0) = 4r$$

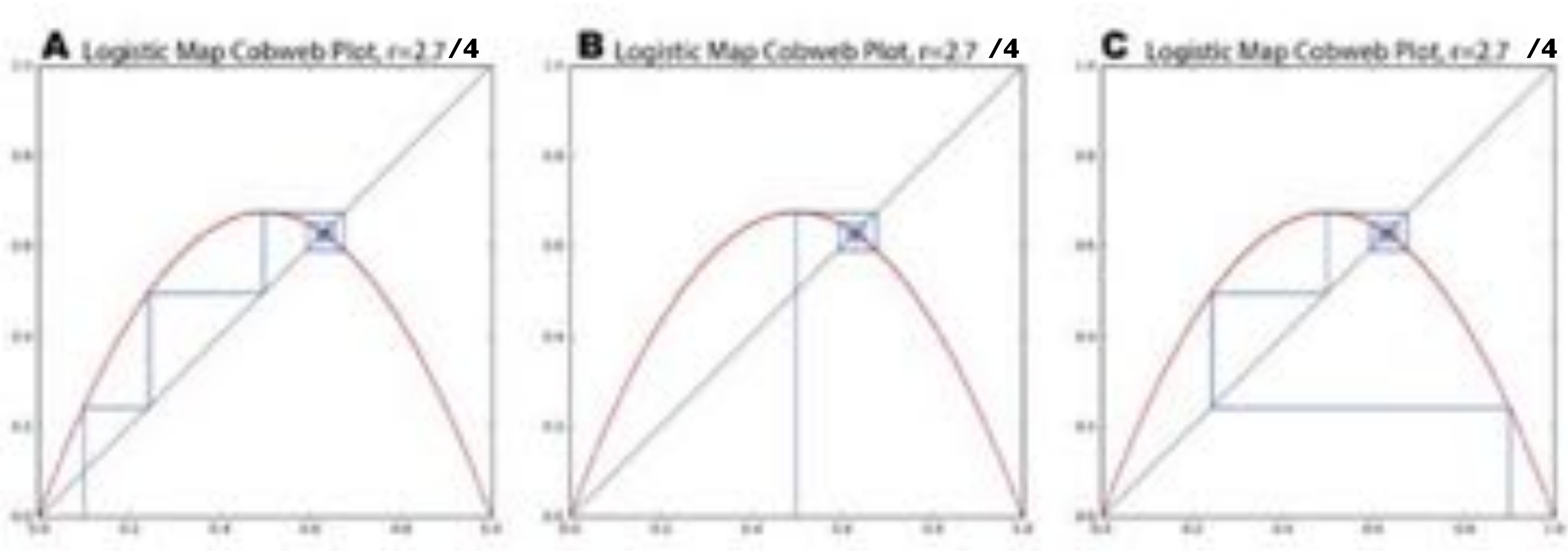


$r = 0.7$
$x_0 = 0.9$

Graphical representation of the iteration of the logistic map (cobweb plot) : the graphical solution converges to the fixed point $x^* \approx 0.643$

Note: the graphical intersection between y(x) and the diagonal gives the **fixed point**, but it is not sufficient to determine whether it is **stable or unstable**

The logistic map



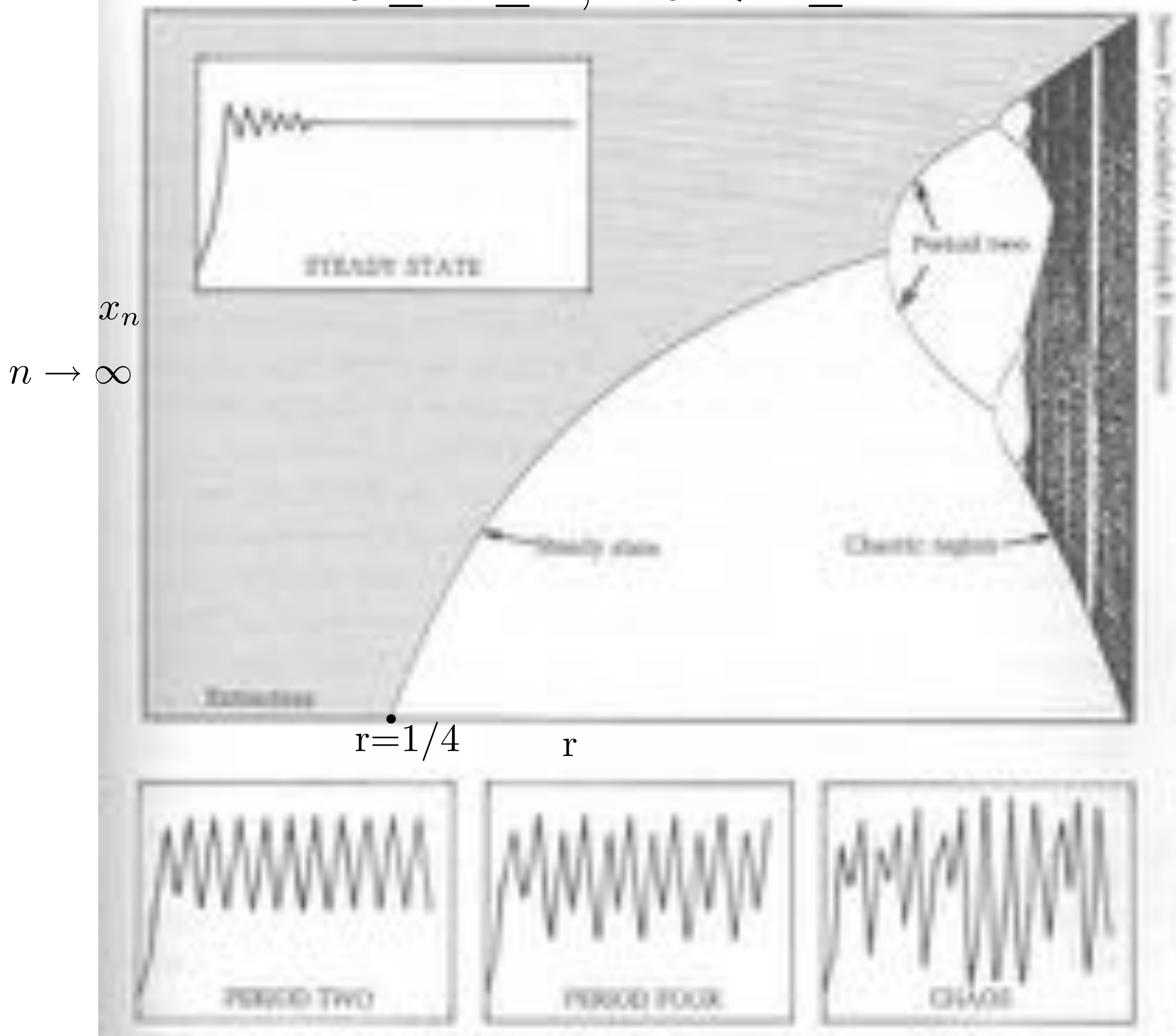
Cobweb plots of the logistic map pulling initial population values of 0.1 (A), 0.5 (B) and 0.9 (C) into the same fixed-point attractor over time.

From: G. Boeing, DOI: 10.3390/systems4040037

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n)$$

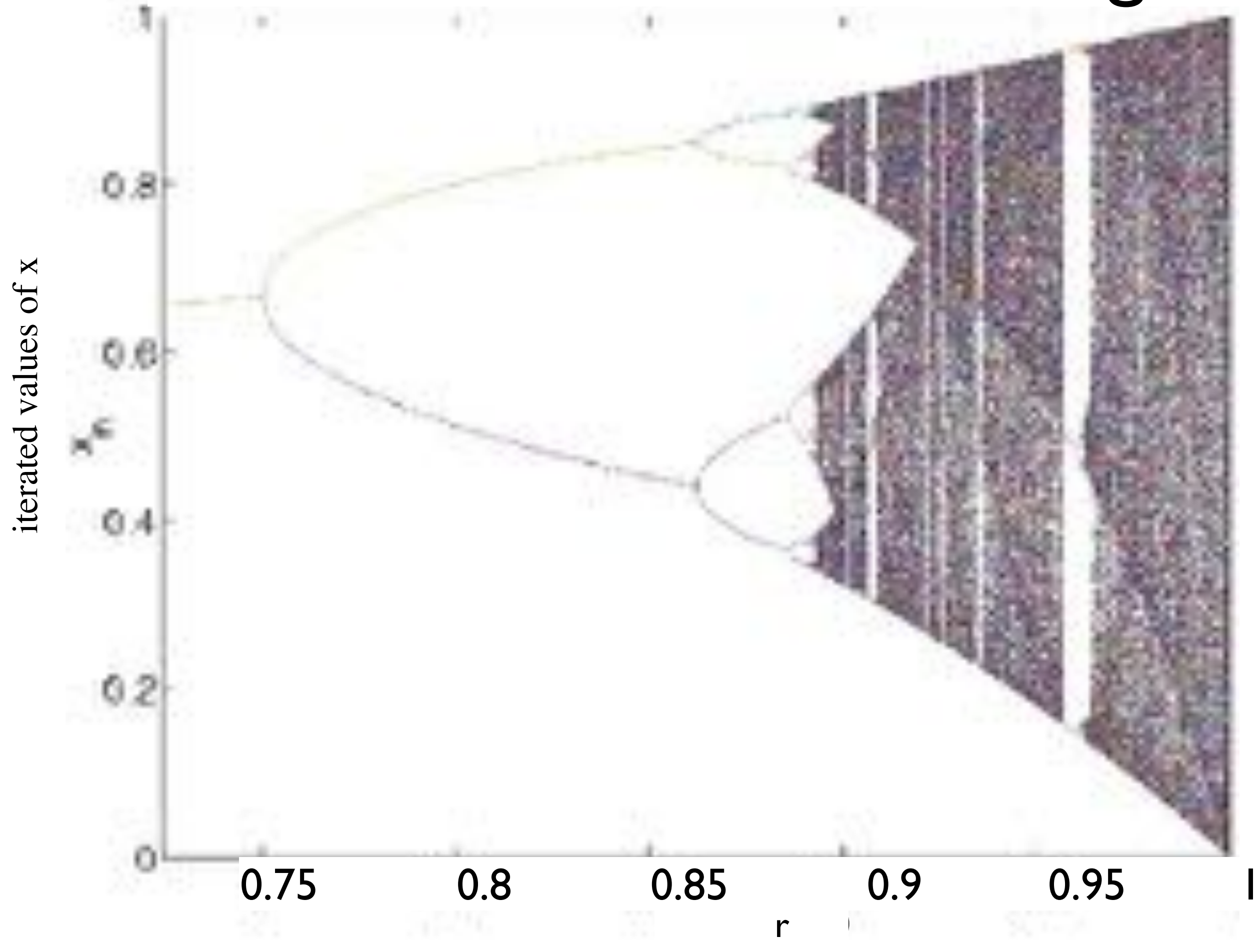
$$0 \leq x \leq 1; \quad 0 < r \leq 1$$



From Chaos by James Gleick (1987)

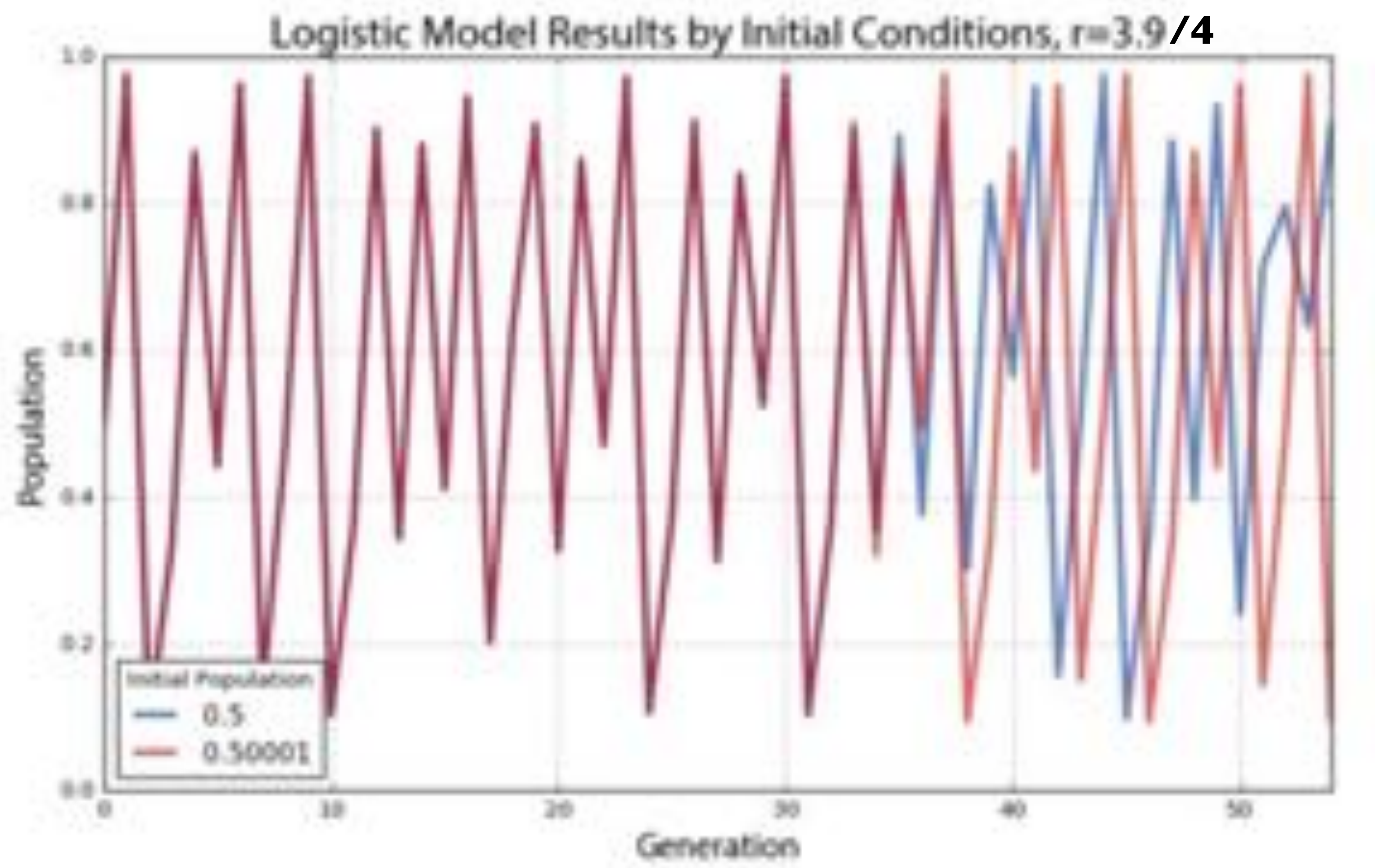
The logistic map

zoom on the bifurcation diagram



Bifurcation diagram of the logistic map. For each value of r, the iterated values of x_n are plotted after the first 1000 iterations are discarded. Note the transition from periodic to chaotic behavior and the narrow windows of periodic behavior within the region of chaos.

The logistic map



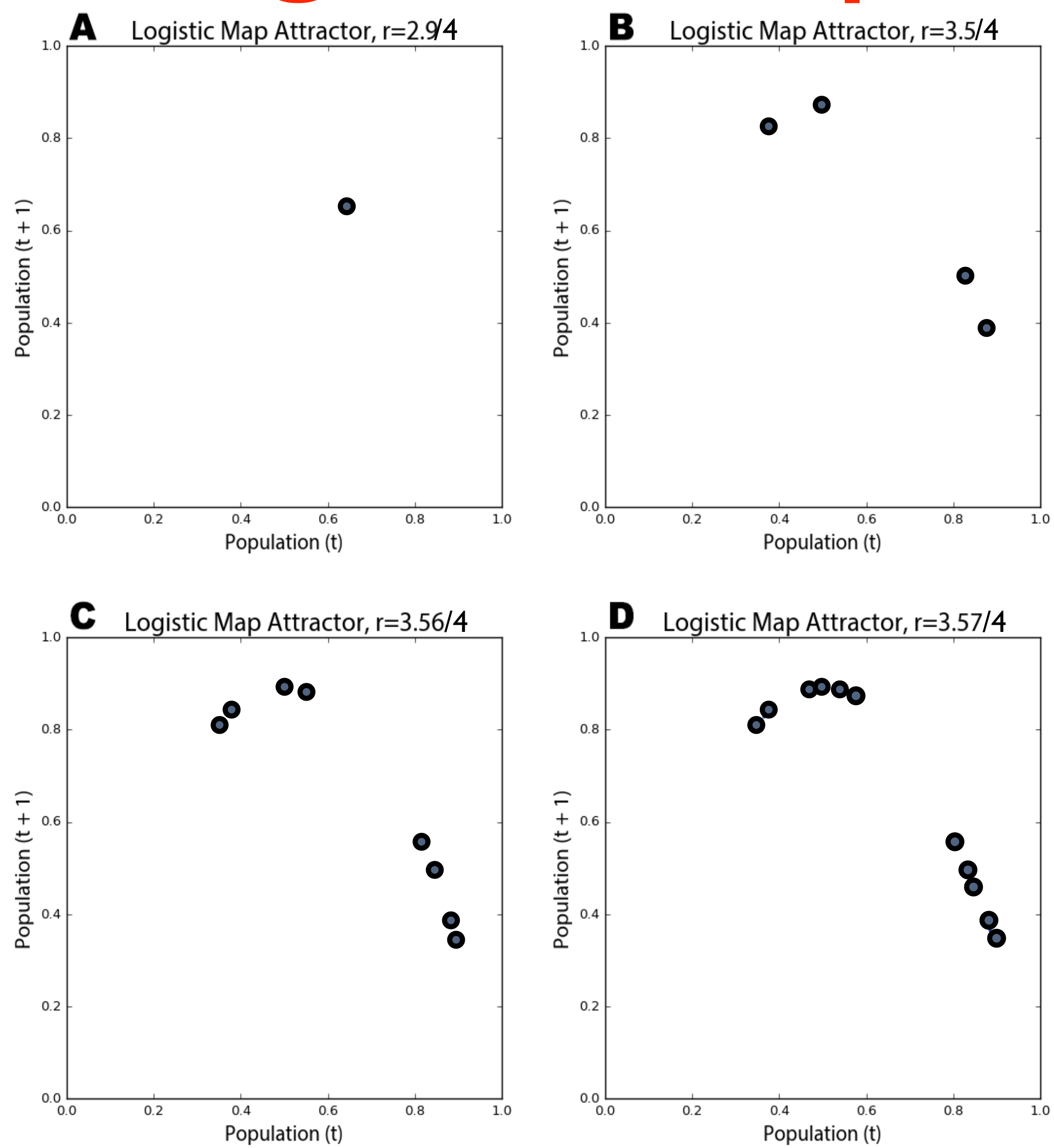
In the chaotic region, the trajectory is extremely sensitive to the initial conditions

From: G. Boeing, DOI: 10.3390/systems4040037

The logistic map

state-space reconstruction: plot (x_{i+1}, x_i)

(each plot: fix r, consider different x_0)



The logistic map

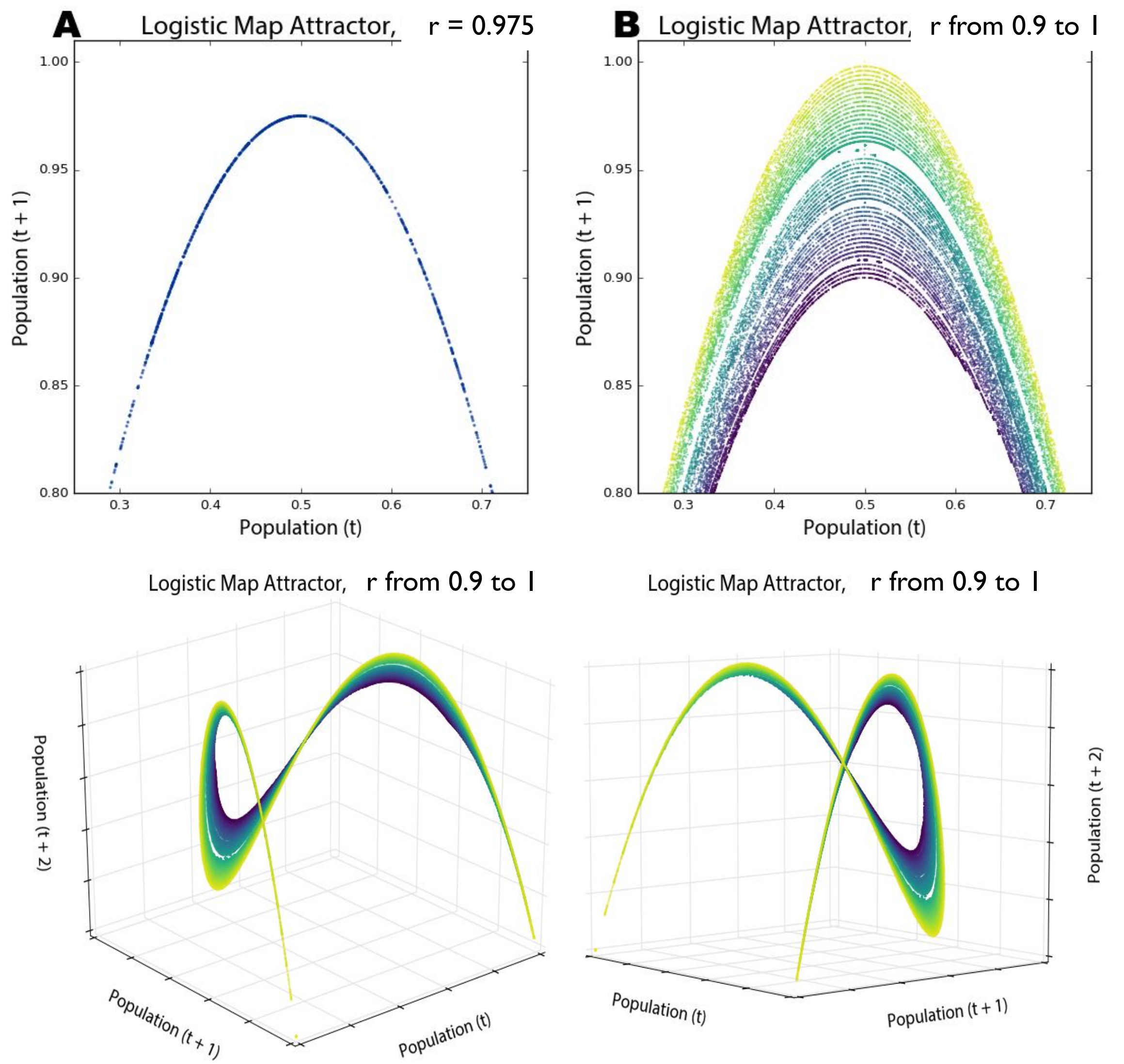
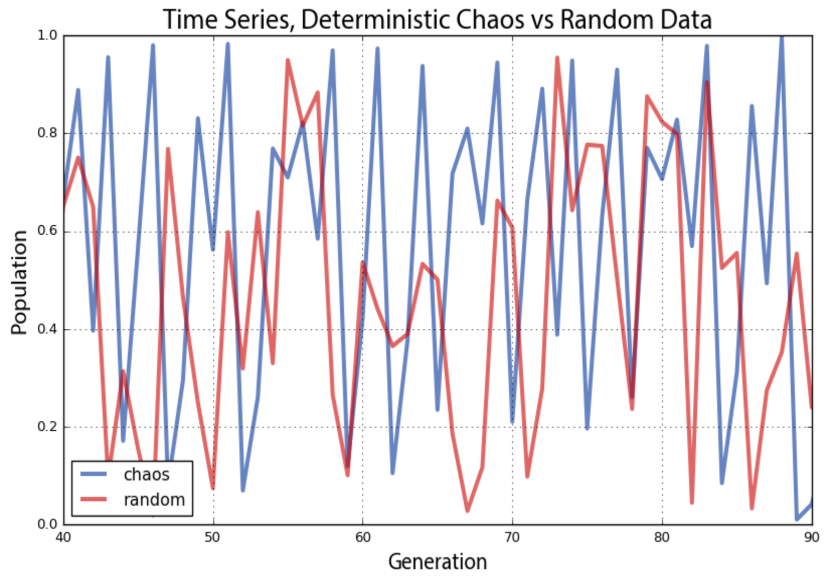
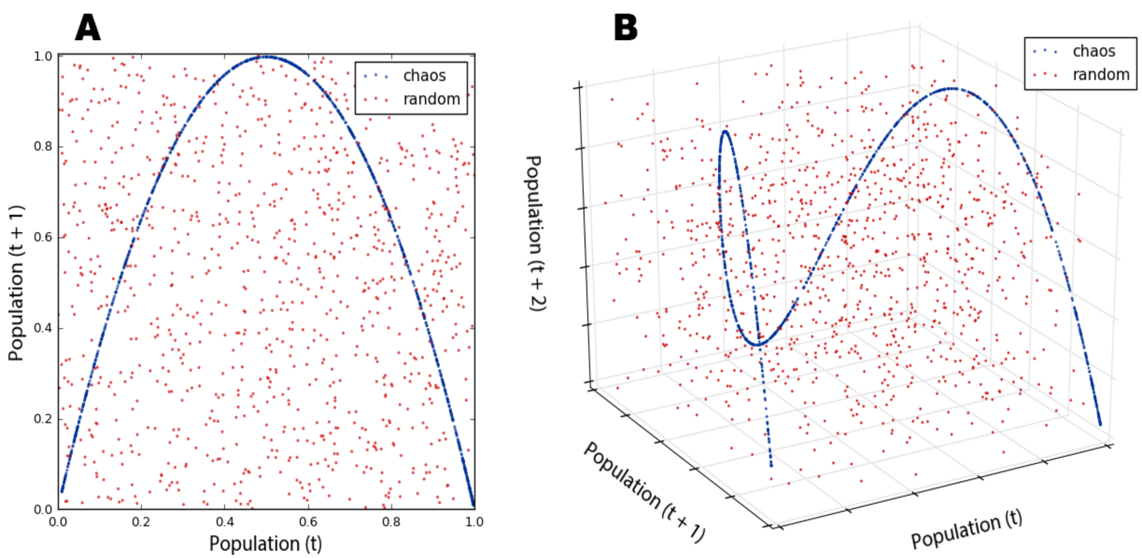


Figure 11. Two different viewing perspectives of a single three-dimensional phase diagram of the logistic map over 200 generations for 50 growth rate parameter values between 0.9 to 1, each with its own colored line.

Chaos or randomness?





The logistic map

Numerics:

for a given parameter r:

- for a given x_0, iterate the map and plot the trajectory (n, x_n);

- verify whether it converges and, in case, to which value(s)

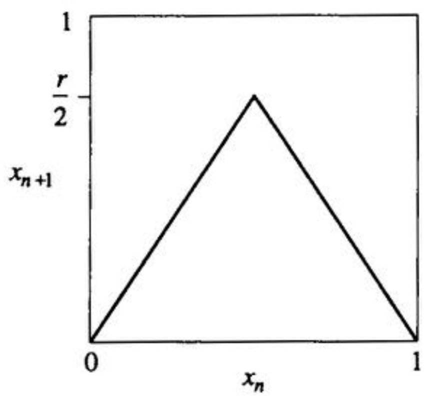
- verify numerically if the analytically predicted fixed points x_1^*, x_2^* are stable or unstable fixed points

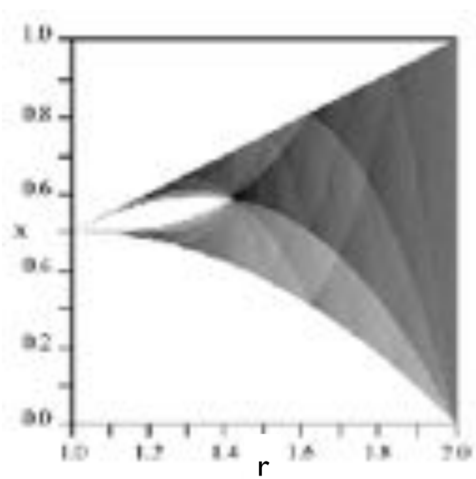
Other unimodal maps

Consider the so-called *tent map*

$$f(x) = \begin{cases} rx & 0 \le x \le 1/2 \\ r - rx & 1/2 \le x \le 1 \end{cases}$$

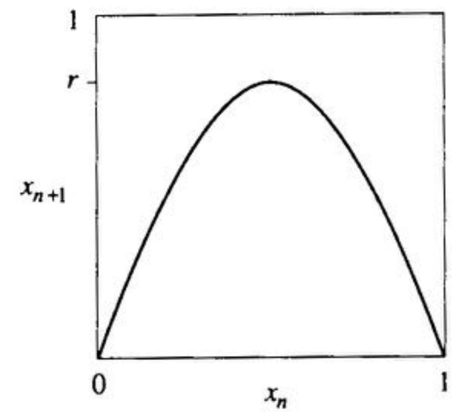
(for $0 \le r \le 2$ and $0 \le x \le 1$).

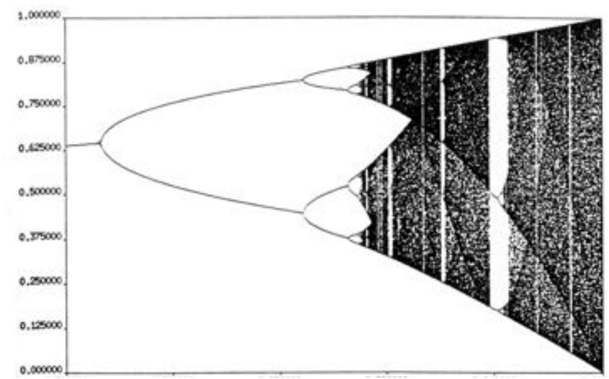




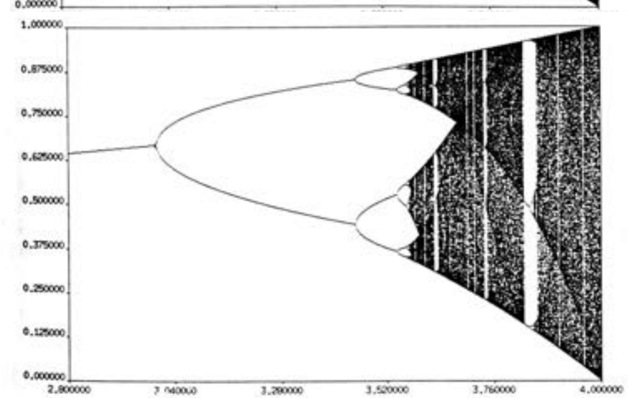
https://en.wikipedia.org/wiki/Tent_map#/media/File:TentMap_BifurcationDiagram.png

Consider the *sine map* $x_{n+1} = r \sin \pi x_n$ for $0 \le r \le 1$ and $0 \le x \le 1$.





sine map



logistic map

Chaos and fractals

Another famous example

other equations intrinsically NON LINEAR can show a chaotic behavior for certain values of the parameters.

E.g.

quadratic recurrence equation

Mandelbrot function (in general in the complex field):

$$Z(n+1) = Z(n)^2 + C$$

with C constant (also negative) and $n = 0, 1, 2, \ldots$

Start with an initial value Z(0), then calculate:

$Z(1) = Z(0)^2 + C$

then:

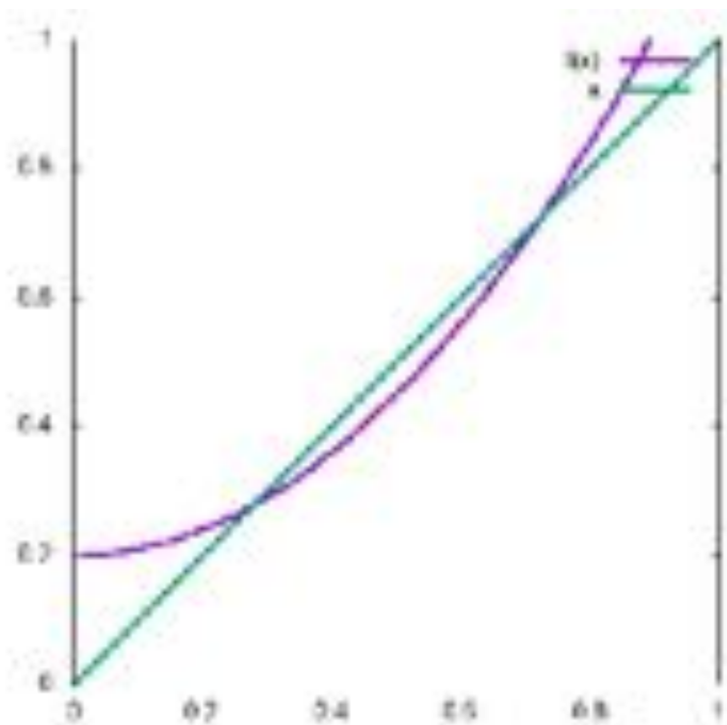
$Z(2) = Z(1)^2 + C$

etc etc ...

Some examples in the real field

$$Z(n+1) = Z(n)^2 + C$$

$C = 0.2$ and $Z(0) = 0$
Convergence to $Z^* = 0.2764$



n	Z1(n)
0	0.0000
1	0.2000
2	0.2400
3	0.2576
4	0.2664
5	0.2709
6	0.2734
7	0.2748
8	0.2755
9	0.2759
10	0.2761
11	0.2762
12	0.2763
13	0.2763
14	0.2764
15	0.2764
16	0.2764

Some examples in the real field

$$Z(n+1) = Z(n)^2 + C$$

Previous example: $C = 0.2$ and $Z(0) = 0$ => Convergence to $Z^* = 0.2764$

In general:

Starting from Z(0) = 0:

For $0 < C <= 0.25$: convergence to a fixed point, solution of $Z = Z^2 + C$ (**attractor)**

For $C <\sim -0.75$: convergence with damped oscillation

For $C \sim -0.76$: bifurcation (two-values attractor)

Decreasing C: further bifurcations

Further decreasing, at $C \sim -1.42$: chaotic behavior
(infinite points of attraction; and very small change of Z(0)=> very different behavior of the sequence - "butterfly effect")

Some examples in the real field

$$Z(n+1) = Z(n)^2 + C$$

Chaotic sequence at C = -1.7:

The values of the sequence do not repeat
However they are within a certain range

Range including all points of the series:
chaotic attractor or strange attractor

n	Z1(n)
0	0.0000
1	-1.7000
2	1.1900
3	-0.2839
4	-1.6194
5	0.9225
6	-0.8491
7	-0.9791
8	-0.7414
9	-1.1503
10	-0.3768
11	-1.5581
12	0.7275
13	-1.1707
14	-0.3295
15	-1.5914
16	0.8326

Some examples in the complex field - fractal sets

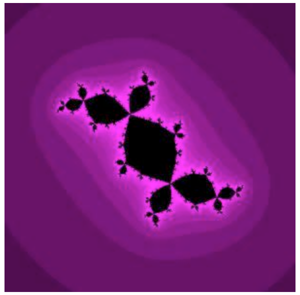
Remainder: **$Z(n+1) = Z(n)^2 + C$**; in general, C and Z(n) are complex numbers.

Repeat the iteration either until |z| > 2 or until a maximum number of iterations is reached.

For **fixed C** complex, **the set of the values Z(0) whose "evolution" Z(n→∞) tends to a finite value**: such set produces a fractal figure (Z(0) is represented in black if Z(n→∞) is finite). In general, if Z(n→∞)→∞, color the corresponding pixel; better, use a color derived from the number of iterations keeping Z(n) within a certain value.

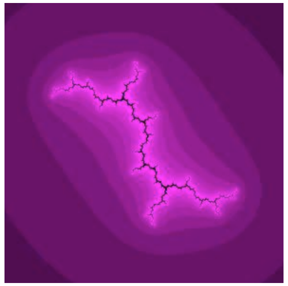
Maps of Z(0) in the complex plane for three different values of C:

$c = -0.123 + 0.745i$



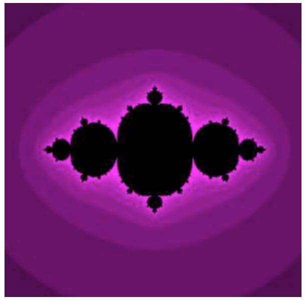
Douady rabbit

$c = i$



Dendrite

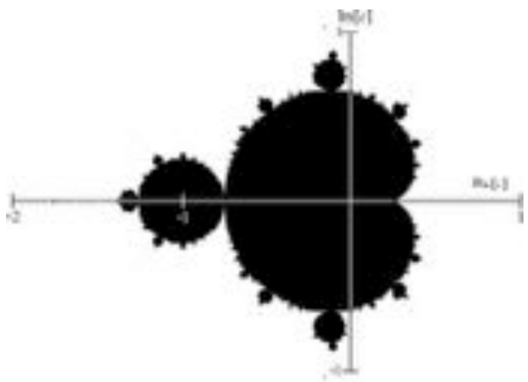
$c = -0.75$



San Marco fractal

extreme points on x axis: Z(0)=0,1

“The” Mandelbrot set



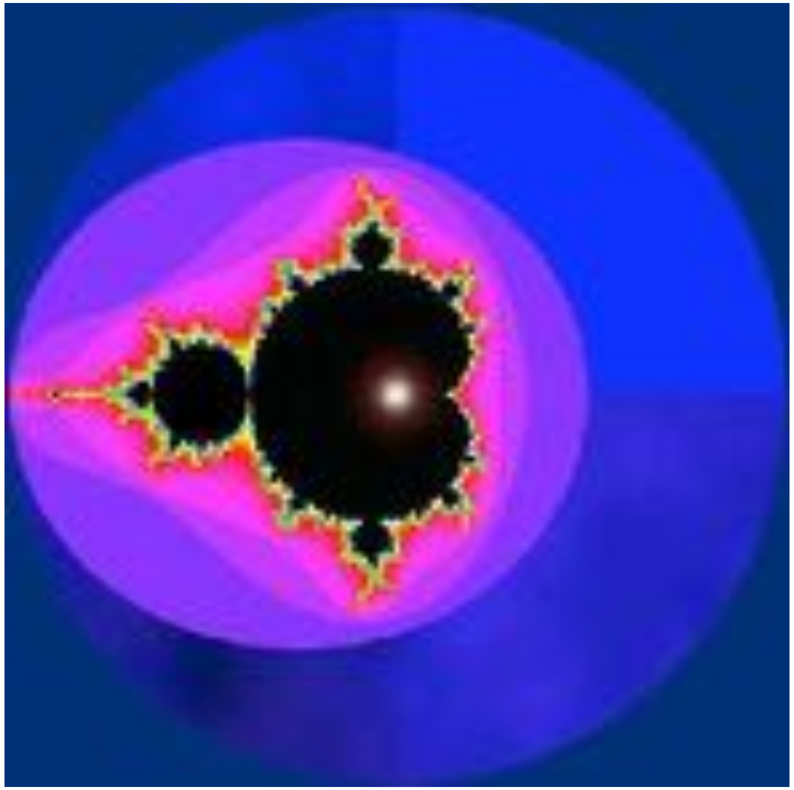
the set of those points C in the **complex plane** for which the “evolution” of Z(0)=0 under iteration of Z(n) remains “bounded”, i.e., |Z(n)| never diverges as n grows.

The Mandelbrot set can be plotted: in practice, a maximum number of iterations n_{max} and a maximum value of $|Z|=r_{max}=2$ is considered (it can be demonstrated that if there is a $|Z_n|>2$, then the sequence diverges)

one-color plots: black pixel: C is in the Mandelbrot set (|Z| remains limited)/ white: C is NOT

=> FRACTAL CHARACTERISTICS

http://mathworld.wolfram.com/MandelbrotSet.html



"The" Mandelbrot set

the set of those points C in the **complex plane** for which the "evolution" of Z(0)=0 under iteration of Z(n) remains "bounded", i.e., |Z(n)| never diverges as n grows.

The Mandelbrot set can be plotted: in practice, a maximum number of iterations n_{max} and a maximum value of $|Z|=r_{max}=2$ is considered (it can be demonstrated that if there is a $|Z_n|>2$, then the sequence diverges)

one-color plots: black pixel: C is in the Mandelbrot set (|Z| remains limited)/ white: C is NOT

multicolor plots: C points are colored according to the number of iterations $n<n_{max}$ required to have $|Z_n|>r_{max}$

=> FRACTAL CHARACTERISTICS

http://mathworld.wolfram.com/MandelbrotSet.html

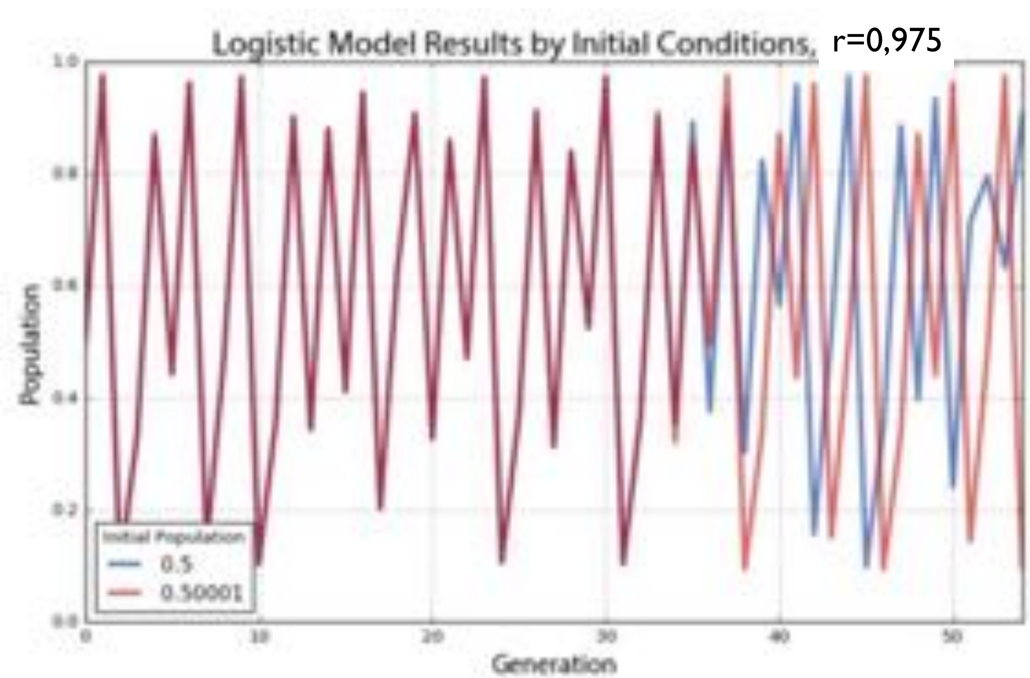
Measuring chaos

Measuring chaos

important characteristic of chaos

sensitivity to initial conditions

$$\Delta x_0 = 0.00001, \quad \Delta x_{n>40} = ???$$



Measuring chaos

important characteristic of chaos

sensitivity to initial conditions

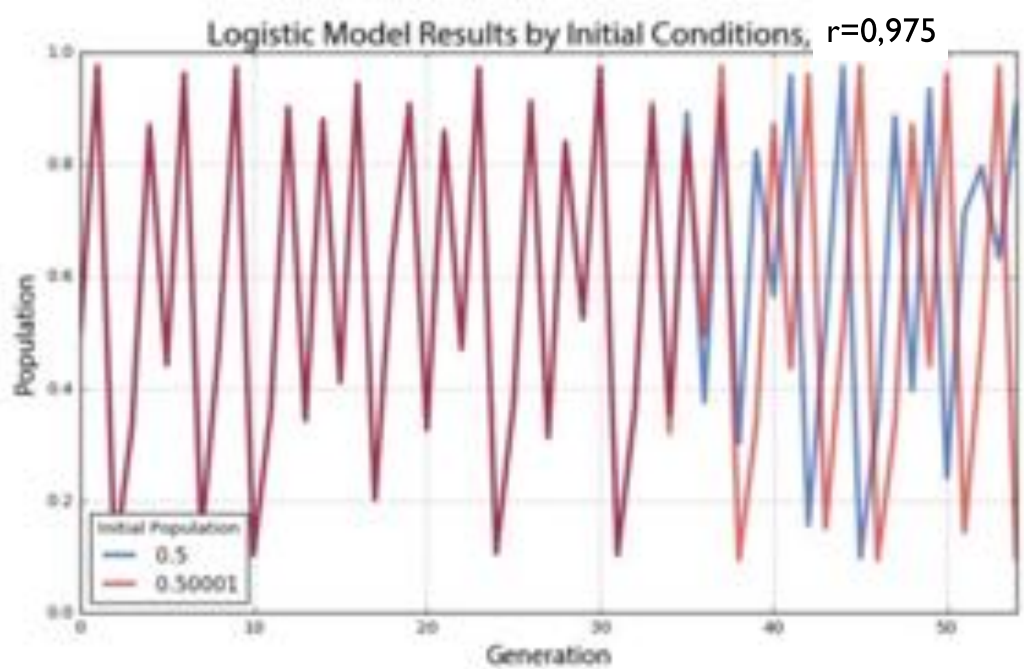
The difference between two trajectories may diverge exponentially :

$$\Delta x_0 = 0.00001, \qquad \Delta x_{n>40} = ???$$

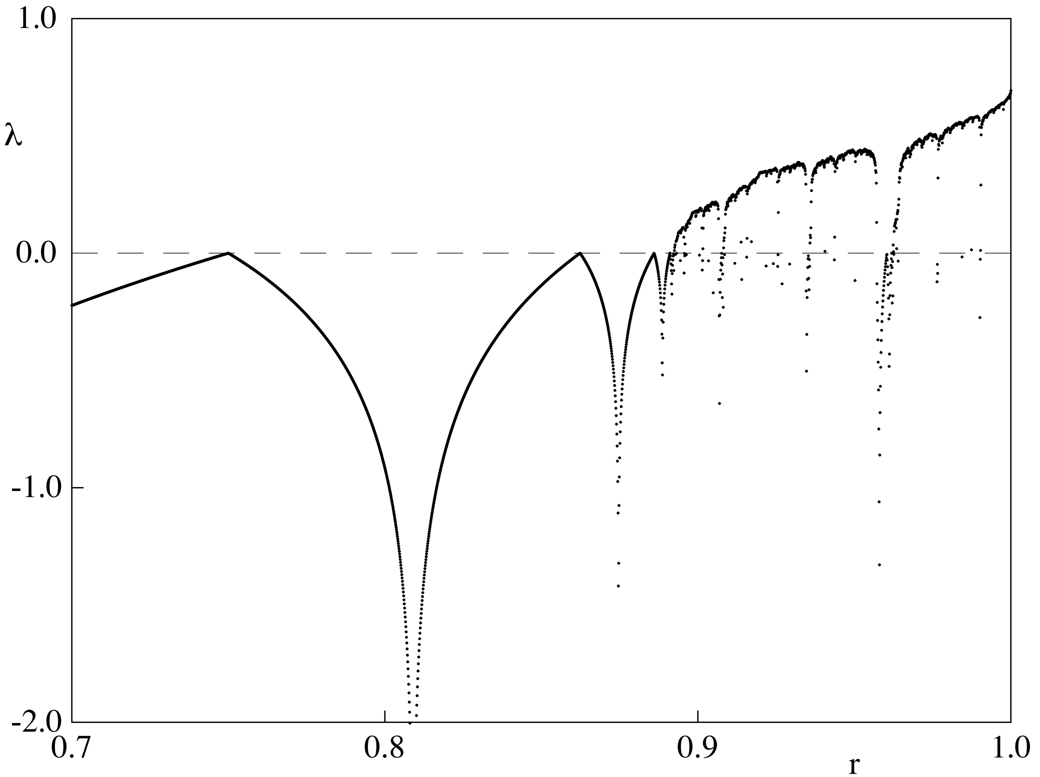
$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}$$

Lyapunov exponent

(LE)

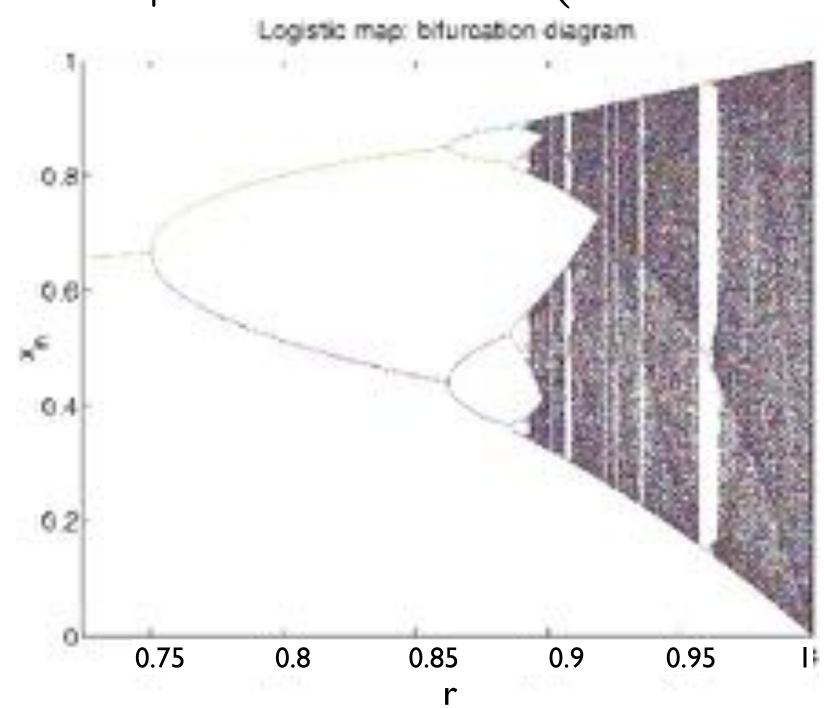


Measuring chaos



The Lyapunov exponent as a function of the control parameter r for the logistic map

$$x_{n+1} = 4rx_n(1 - x_n)$$



$$|\Delta x_n| = |\Delta x_0| \, e^{\lambda n}$$

Lyapunov exponent

Measuring chaos

A PROBLEM in a numerical approach:

ROUNDOFF:
small initial errors are exponentially amplified in time;
after some (?) iterations the trajectories can diverge!

How to calculate λ?
FIT over several trajectories

Measuring chaos

According to the previous definition, $|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}$
the Lyapunov parameter λ is given by:

$$\lambda = \frac{1}{n}\ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \frac{1}{n}\ln\left|\frac{\Delta x_n}{\Delta x_{n-1}}\cdot\frac{\Delta x_{n-1}}{\Delta x_{n-2}}\cdot\frac{\Delta x_{n-2}}{\Delta x_{n-3}}\cdots\frac{\Delta x_1}{\Delta x_0}\right| = \frac{1}{n}\sum_{i=0}^{n-1}\ln\left|\frac{\Delta x_{i+1}}{\Delta x_i}\right|$$

If we consider the map as a function, we have:

$$x_{i+1} = f(x_i) \;\Rightarrow\; \Delta x_{i+1} = \Delta f(x_i) \;\Rightarrow\; \frac{\Delta x_{i+1}}{\Delta x_i} = \frac{\Delta f(x_i)}{\Delta x_i} = f'(x_i)$$

if the Δx_i are sufficiently small, which is true in case of convergence towards fixed points

hence:

$$\lambda = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right)$$

Measuring chaos

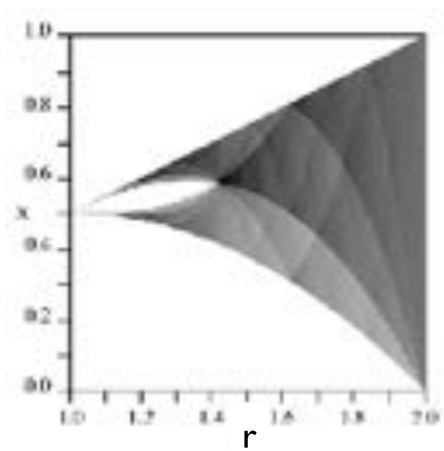
For the so-called *tent map*

$$f(x) = \begin{cases} rx & 0 \le x \le 1/2 \\ r - rx & 1/2 \le x \le 1 \end{cases}$$

(for $0 \le r \le 2$ and $0 \le x \le 1$)

Since $f'(x) = \pm r$ for all x, we find

$$\begin{aligned} \lambda &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1} \ln|f'(x_i)|\right) \\ &= \lim_{n\to\infty}\left(\frac{\ln r}{n}\sum_{i=0}^{n-1} 1\right) \\ &= \ln r \end{aligned}$$

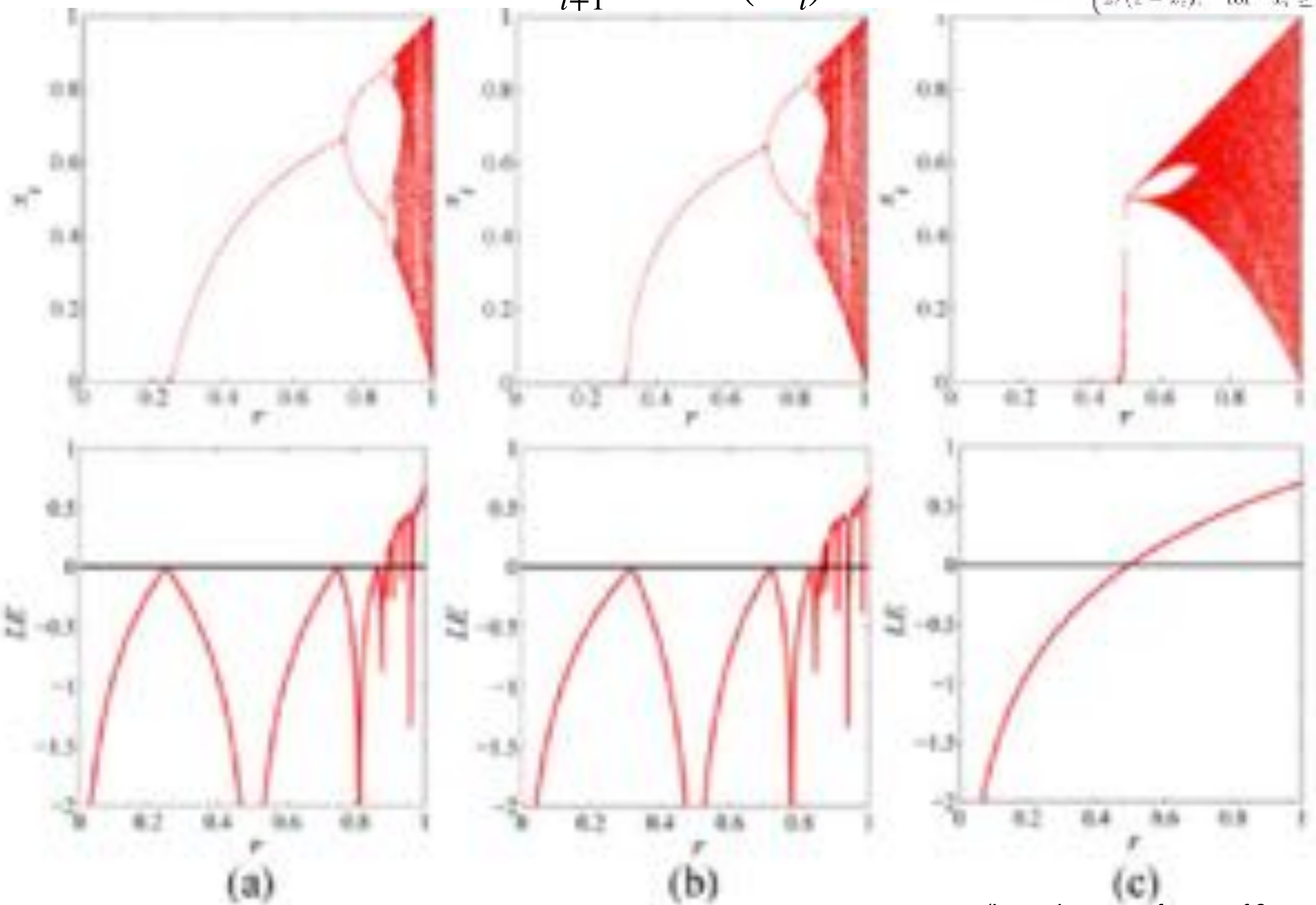


This suggests that the tent map has chaotic solutions for all $r > 1$, since $\lambda = \ln r > 0$.

Other 1D chaotic maps

$$x_{i+1} = r\sin(\pi x_i)$$

$$x_{i+1} = T(x_i) = \begin{cases} 2rx_i, & \text{for } x_i < 0.5 \\ 2r(1-x_i), & \text{for } x_i \geq 0.5 \end{cases}$$



(here there is a factor of 2 in front of r)

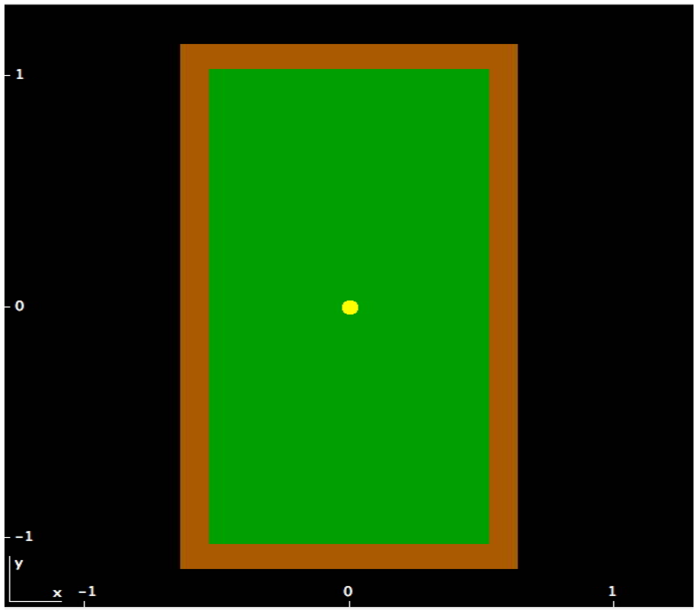
Fig. 1. Bifurcation diagrams (top row) and LEs (bottom row) of (a) the logistic map, (b) sine map, and (c) tent map.

As can be observed, the logistic, sine, and tent maps have chaotic behaviors when $r \in [0.89, 1]$, $r \in [0.87, 1]$, and $r \in (1, 2)$, respectively. Even the logistic and sine maps are two different maps with totally different definitions, they have similar behaviors, which can be seen from their bifurcation diagrams and LEs. Moreover, the logistic and sine maps do not have robust chaos as periodic windows exist in their chaotic ranges, but the tent map has robust chaos when its control parameter $r \in (1, 2)$.

(Finally, some physics…!)

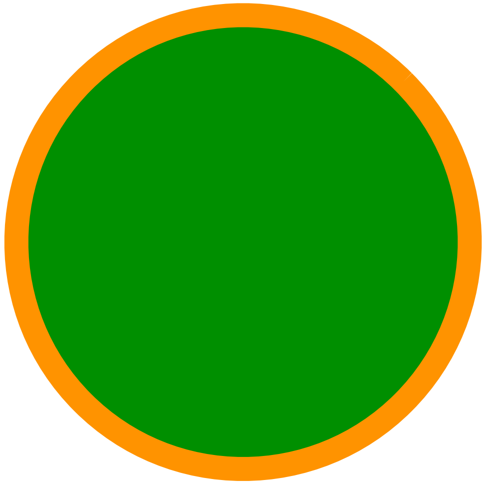
Chaos in classical billiards

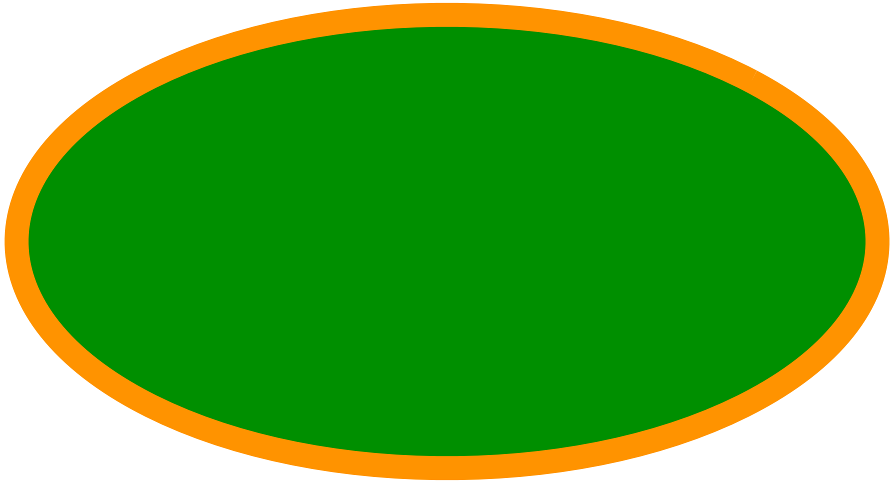
Billiards



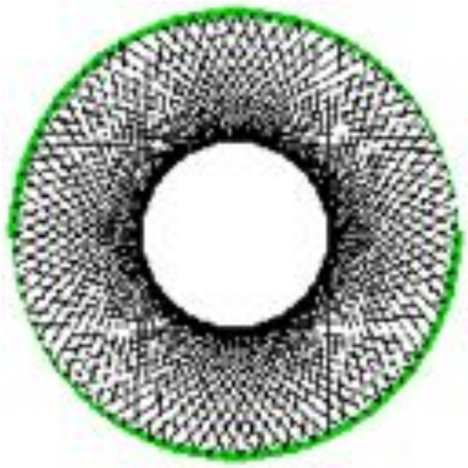
MODEL BILLIARDS
(conservation of energy law,
reflection law of geometric optics)

calculate trajectories
(which depend on:
shape of the billiard;
initial position and velocity)





Billiards

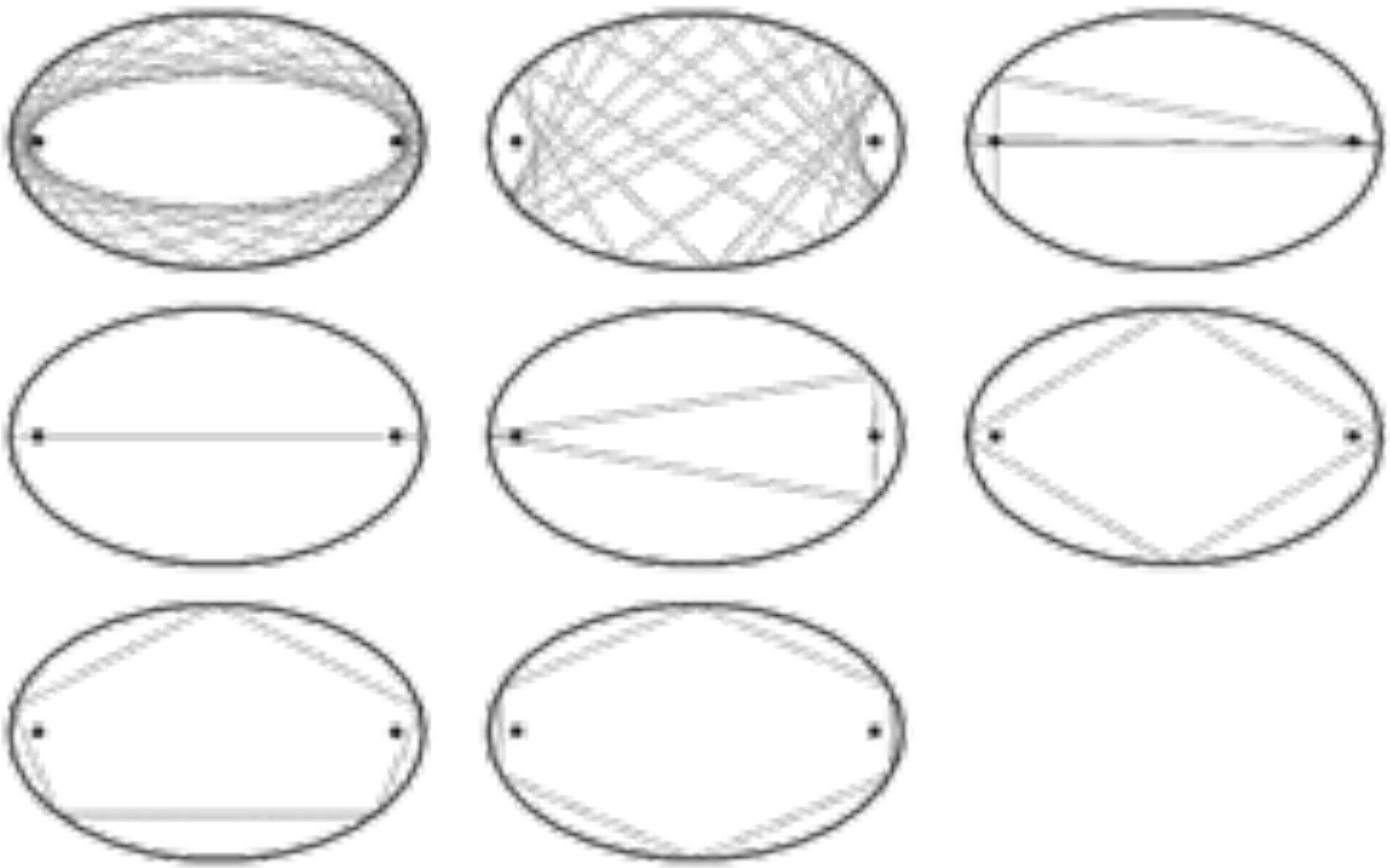


Circular billiards support regular (periodic or non - periodic) trajectories, but in any case **non - ergodic**.
(note also:
conservation of angular momentum, incidence angle constant)

In phase space (q(t),p(t)):
limited region (a line: q(t) varies, p(t) constant)

Billiards

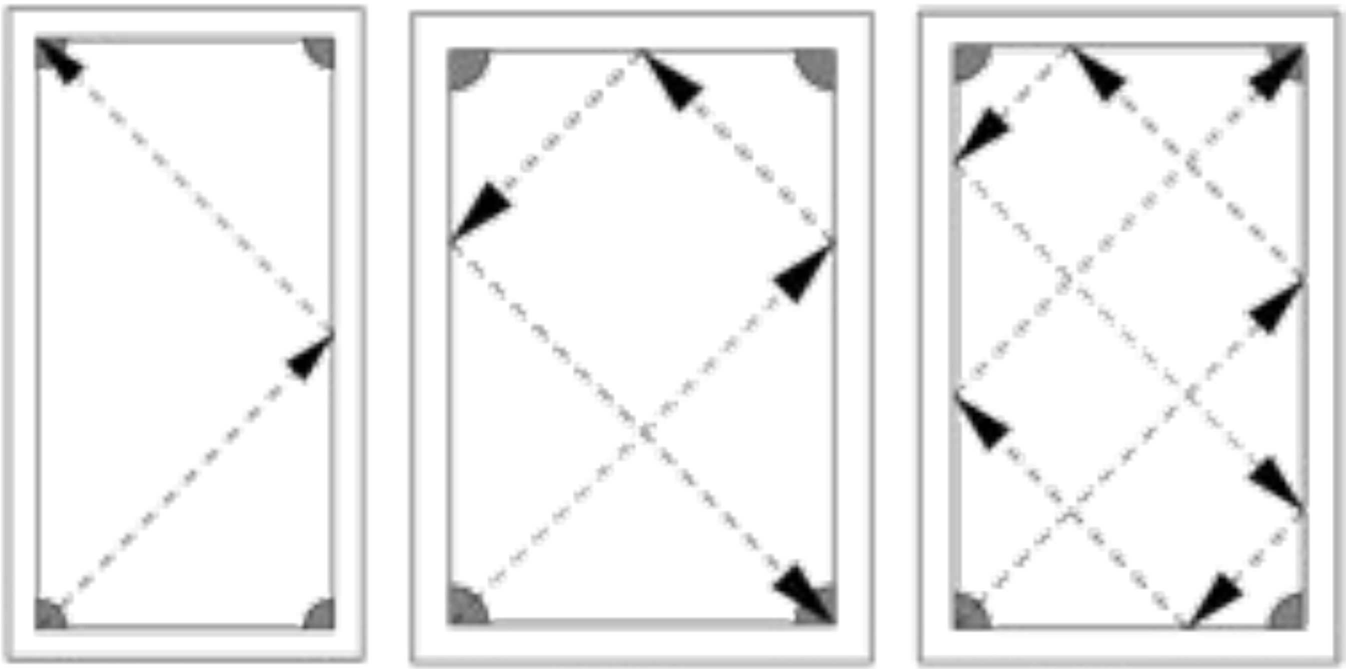
Also **elliptical billiards** support regular trajectories:



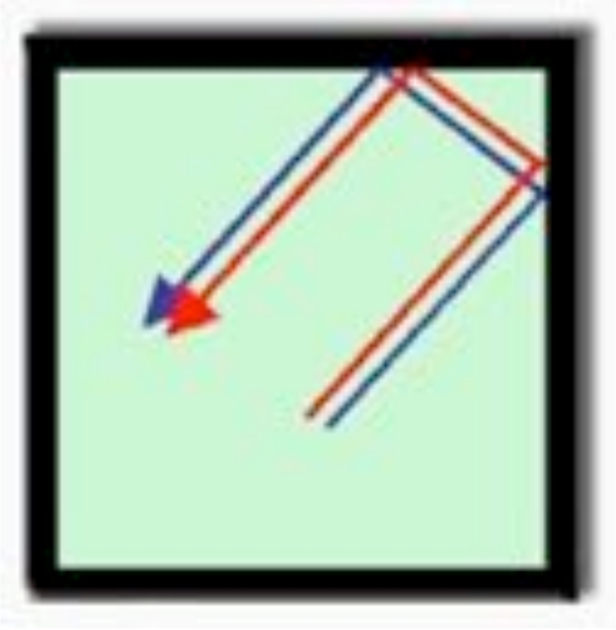
The convolution of a trajectory can be: ellipse, hyperbole, regular polygon

Billiards

Rectangular billiards also support regular (periodic or non - periodic) trajectories, which in this case can be also ergodic

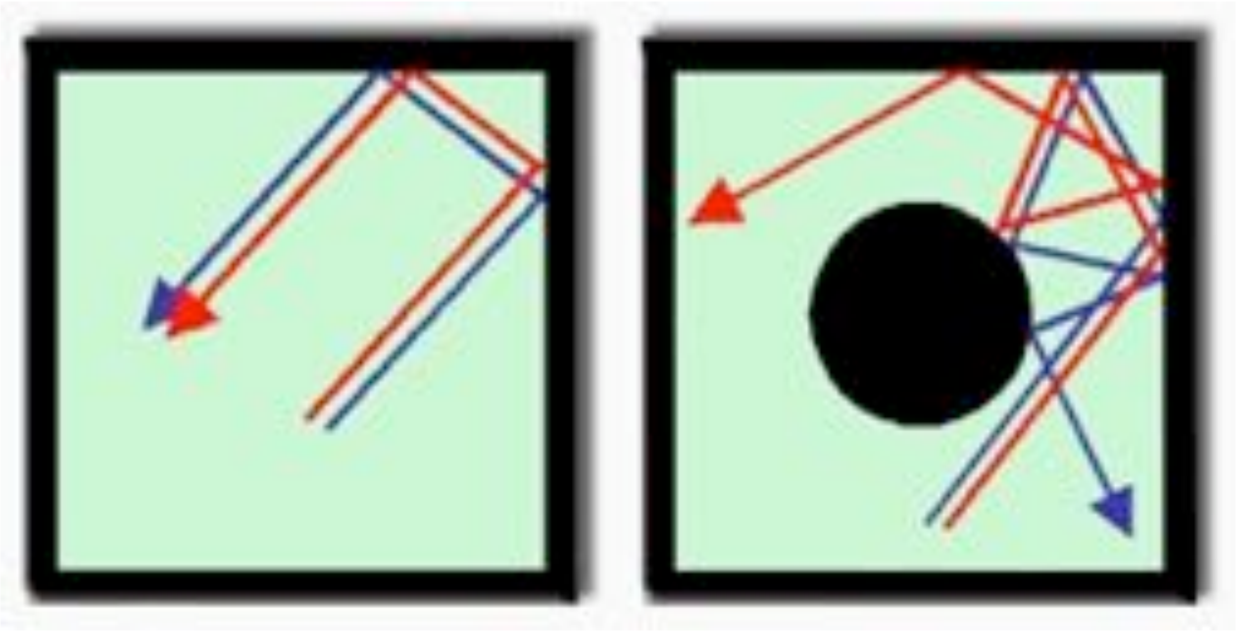


Billiards



In perfectly rectangular/square/elliptic billiards the trajectories are **regular** but also **stable**, i.e. changing the initial conditions, they remain close each other

Billiards



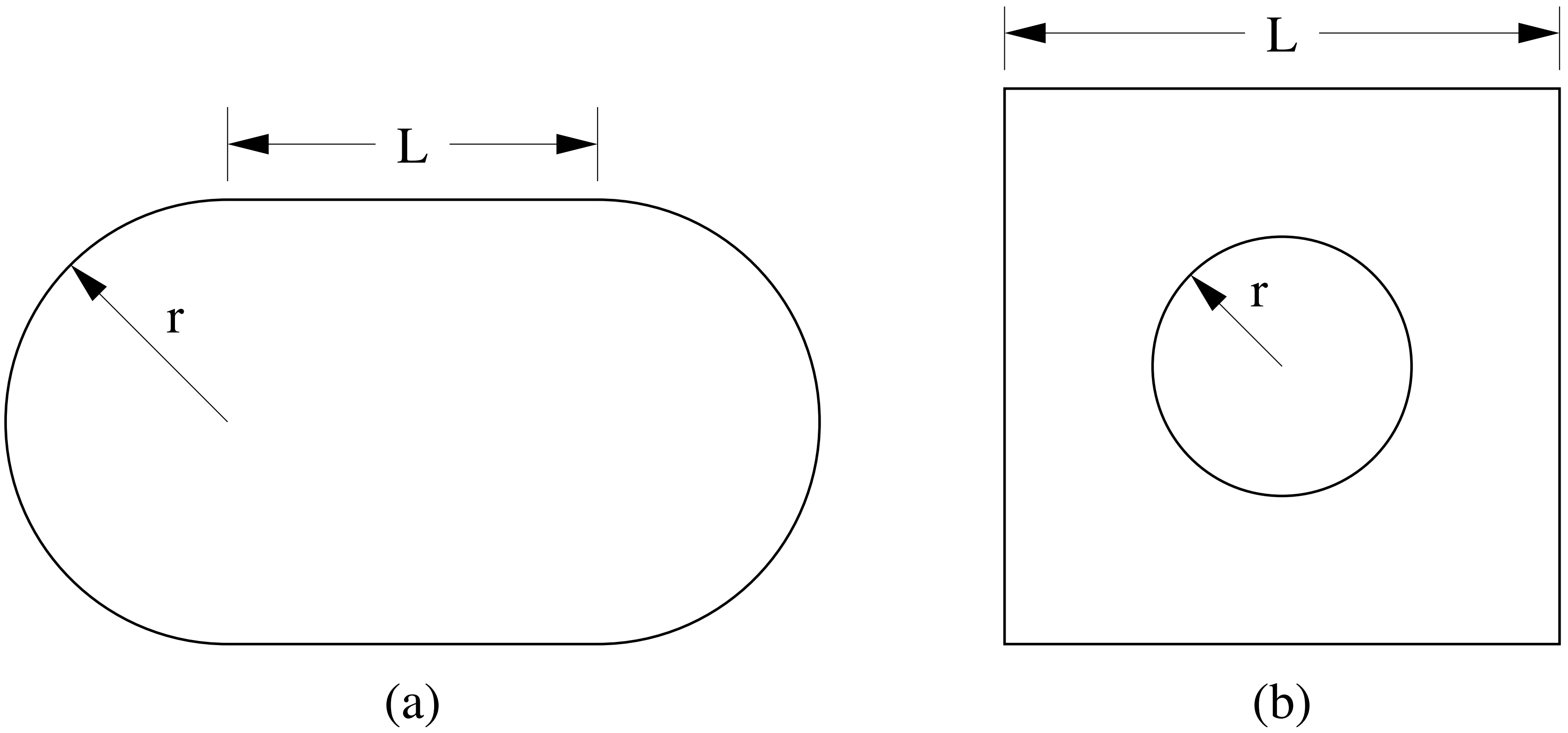
In perfectly rectangular/square/elliptic billiards the trajectories are **regular** but also **stable**, i.e. changing the initial conditions, they remain close each other

By inserting a circle in a rectangular or square billiard, chaotic trajectories, strongly dependent on the initial conditions, are generated
("dynamical billiard" or "Sinai billiard", 1963)

https://www.abelprize.no

Billiards

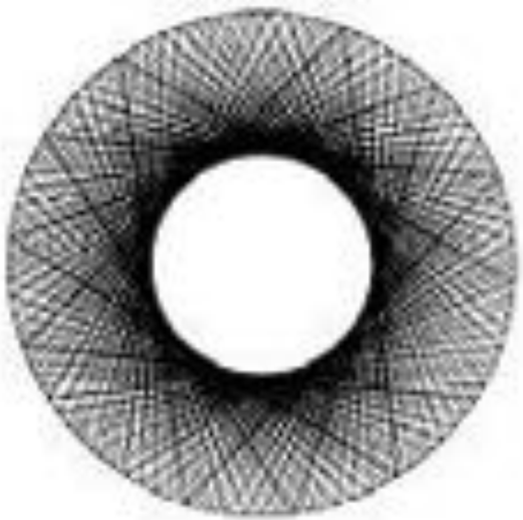
Stadium (Bunimovich) billiard has a geometry simpler than Sinai billiard, also resulting in chaotic trajectories



: (a) Geometry of the stadium billiard model. (b) Geometry of the Sinai billiard model.

Billiards

NON Ergodicity
of circular
billiards



Ergodicity
of chaotic
billiards

Conservation of the energy,

but in some cases (stable trajectories):
- another physical constant
(e.g. angular momentum in case of circular billiards;
x and y "components" of the kinetic energy
in rectangular billiards)

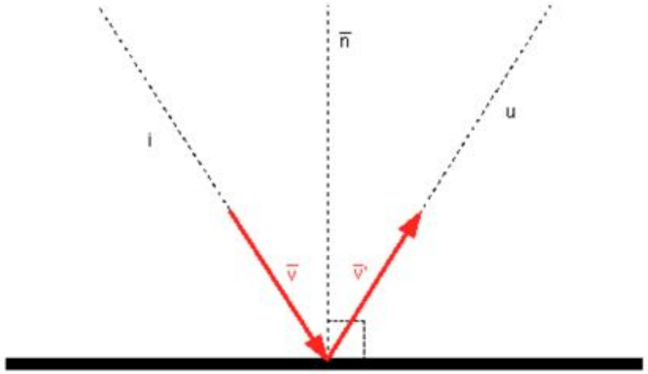
- no physical constant for stadium billiards

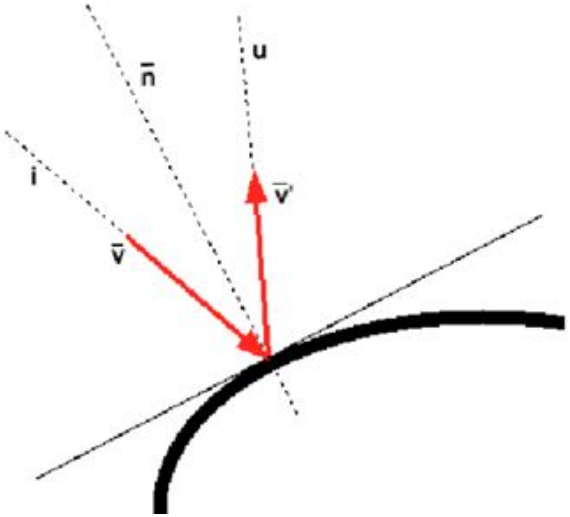
our model

point-like spheres

no friction:
forces normal to the boundaries
$\Rightarrow v'_{//} = v_{//} \Rightarrow v'_{\perp} = -v_{\perp}$

perfectly elastic collisions:
energy conservation: $|v'| = |v|$





the algorithm

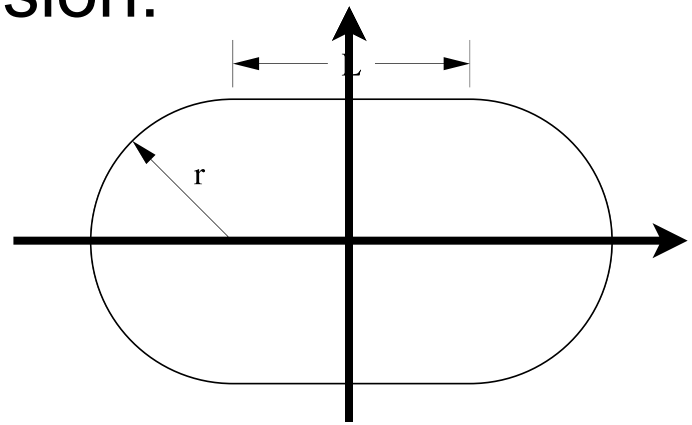
given x, y, v_x, v_y at time t

calculate :
time to the next collision
the position of collision
velocity after the collision (reflection)

Iterate N times (N collisions)

collision time

Calculation of time to the next collision:



$x(t) = x_0 + v_x t$
$y(t) = y_0 + v_y t$

boundaries: $f(x,y)=0$: (e.g. : $y_0 + v_y t_c = 0$)

at the collision time t_c:

$$f(x(t_c),y(t_c))=f(x_0 + v_x t_c,\ y_0 + v_y t_c)=0$$

collision point

Specify f: here (half) circular boundary, with equation:

$[x(t_c) - x_c]^2 + [y(t_c) - y_c]^2 = 1$

i.e.:

$(x_0 + v_x t_c - x_c)^2 + (y_0 + v_y t_c - y_c)^2 = 1$

=> 0, 1 o 2 solutions:

(0 sol.) no collision

(1 sol.) collision (tangent line)

(2 sol.) collision (consider only the larger t_c)

velocity after collision

For reflection off of a circular boundary:

$(x - x_c)^2 + y^2 = 1$

$v'_x = (y^2 - (x-x_c)^2)\, v_x - 2\,(x - x_c)y\, v_y$

$v'_y = -\,2\,(x-x_c)\, y\, v_x + ((x - x_c)^2 - y^2)\, v_y$

(valid if $v_x^2 + v_y^2 = 1$)

Lyapunov exponent

Dynamics is chaotic:
start with two particles with almost identical positions and/or momenta (varying by say 10^{-5}); compute the difference Δs of the two phase space trajectories as a function of the number of reflections n, where:

$$\Delta s_n = \sqrt{|\mathbf{r}_{1,n} - \mathbf{r}_{2,n}|^2 + |\mathbf{p}_{1,n} - \mathbf{p}_{2,n}|^2}$$

Lyapunov exponent can be calculated by a semilog plot of Δs versus n *(of course, consider only the initial part, since Δs is limited!)*

- L dependence?

- role of single/double precision?
- Time inversion symmetry?

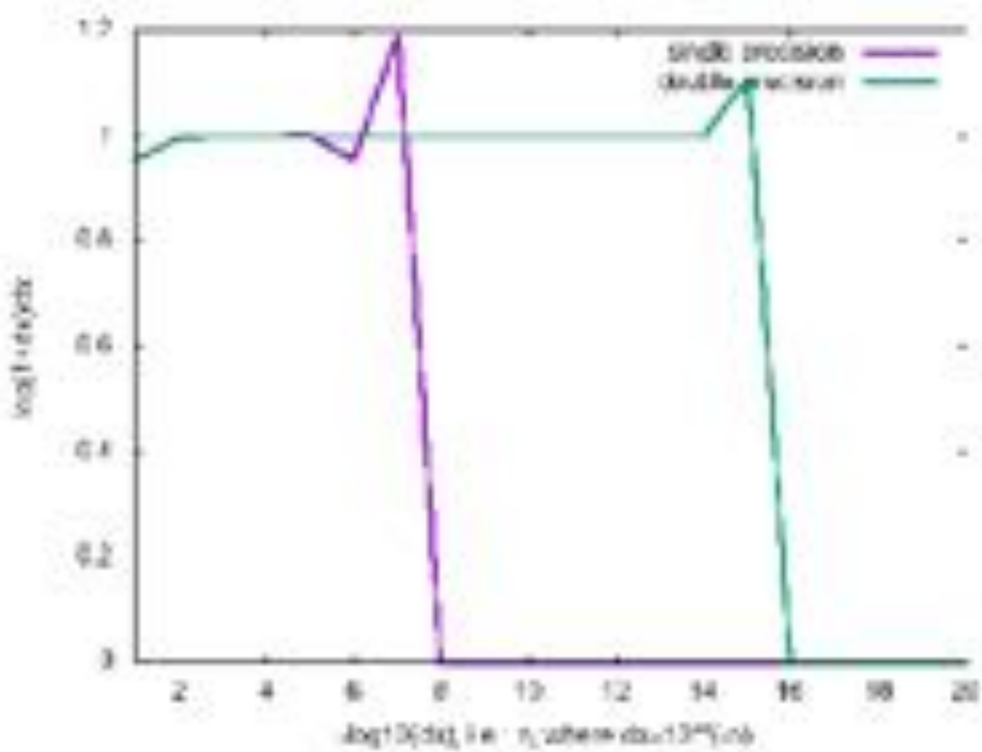
recommendation:

don’t forget roundoff errors…

Analytically:

$$\lim_{x \to 0} \frac{log(1+x)}{x} = 1$$

… but… numerically?



Some programs and materials:

on moodle2:

map.f90

billiard.f90

and

biliardi2.zip (material in java, from the Lab activity with High School students, with G. Pastore)

And also:
julia.f90
Mandelbrot.f90
(taken somewhere from the web, Author unknown)

From ICTP web site:
https://www.ictp.it/about-ictp/media-centre/news/2018/6/yorke-interview.aspx