13 dicembre

$$L[y] = y'' - 5y' + 6y = 2e^{x} + 3e^{2x} + e^{-x}$$

$$y_g = y_h + y_p$$

$$y_h = C_1 e^{2x} + C_2 e^{3x}$$

Dobbiamo trovare y_p $\quad L[y_p] = 2e^{x} + 3e^{2x} + e^{-x}$

Per trovare una y_p si sfrutta il fatto che $L[y]$ è un operatore lineare.

Cerchiamo soluzioni particolari dei seguenti tre problemi

$$L[y_1] = 2e^{x}$$

$$L[y_2] = 3e^{2x}$$

$$L[y_3] = e^{-x}$$

Allora avremo

$$L[\underbrace{y_1 + y_2 + y_3}_{y_p}] = L[y_1] + L[y_2] + L[y_3] = 2e^{x} + 3e^{2x} + e^{-x}$$

$$L[y] = y'' - 5y' + 6y \qquad p(r) = r^2 - 5r + 6$$

$$p'(r) = 2r - 5$$

$$L[y_1] = 2e^{x} \qquad y_1 = C e^{x} \qquad p(1) = 2$$

$$L[Ce^{x}] = C\,L[e^{x}] = C\,p(1)\,e^{x} = 2Ce^{x} = 2e^{x}$$

$$\Rightarrow C = 1$$

$$y_1 = e^{x}$$

$$L[y_3] = e^{-x} \qquad y_3 = C e^{-x}$$

$$L[Ce^{-x}] = C\,L[e^{-x}] = C\,p(-1)\,e^{-x} = 12Ce^{-x}$$

$$= e^{-x}$$

$$p(-1) = (-1)^2 - 5(-1) + 6 = 12 \qquad C = \frac{1}{12}$$

$$y_3 = \frac{1}{12} e^{-x}$$

$$L[y_2] = 3e^{2x} \qquad y_2 = C x e^{2x}$$

$$L[y_2] = C\,L[xe^{2x}] =$$

$$= C\left\{ (xe^{2x})'' - 5(xe^{2x})' + 6xe^{2x} \right\}$$

$$L[ue^{rx}] = (ue^{rx})'' - 5(ue^{rx})' + 6ue^{rx} =$$

$$(ue^{rx})' = u'e^{rx} + rue^{rx}$$

$$(ue^{rx})'' = u''e^{rx} + 2ru'e^{rx} + u(e^{rx})''$$

$$= u''e^{rx} + 2ru'e^{rx} + u\underbrace{(e^{rx})''}_{r^2e^{rx}} - 5(u'e^{rx} + rue^{rx}) + 6ue^{rx} =$$

$$= \Big[u'' + u'\underbrace{(2r-5)}_{p'(r)} + u\underbrace{(r^2 - 5r + 6)}_{p(r)}\Big] e^{rx}$$

$$y_2 = Cxe^{2x} = -3xe^{2x}$$

$$C\,L[xe^{2x}] = C\Big[\underbrace{(x)''}_{0} + \underbrace{(x)'}_{1}\,p'(2) + x\,\underbrace{p(2)}_{0}\Big] e^{2x}$$

$$= C\,p'(2)\,e^{2x} = 3e^{2x} \qquad -Ce^{2x} = 3e^{2x}$$

$$p'(r) = 2r - 5 \qquad p'(2) = -1 \qquad C = -3$$

$$y'' - 5y' + 6y = \sin(2x)$$

Qui applichiamo il metodo della somiglianza per trovare

y_p.

In linea di principio basta usare la formula di Eulero

$$\sin(2x) = \frac{e^{2ix} - e^{-2ix}}{2i}$$

$$y'' - 5y' + 6y = \frac{e^{2ix}}{2i} - \frac{e^{-2ix}}{2i}$$

$$y_p = A\sin(2x) + B\cos(2x)$$

$$L[A\sin(2x) + B\cos(2x)] =$$

$$= A\,L[\sin(2x)] + B\,L[\cos(2x)]$$

$$= A\left[(\sin(2x))'' - 5(\sin(2x))' + 6\sin(2x)\right] +$$

$$+ B\left[(\cos(2x))'' - 5(\cos(2x))' + 6\cos(2x)\right]$$

$$= A[-4\sin(2x) - 10\cos(2x) + 6\sin(2x)]$$

$$+ B[-4\cos(2x) + 10\sin(2x) + 6\cos(2x)] = \sin(2x)$$

$$= \sin(2x)[-4A + 6A + 10B] +$$

$$+ \cos(2x)[-10A - 4B + 6B] =$$

$$= \sin(2x)[2A + 10B] + \cos(2x)[-10A + 2B] =$$

$$= \sin(2x) = \sin(2x) + 0\cos(2x)$$

$$\begin{cases} 2A + 10B = 1 \\ -10A + 2B = 0 \end{cases} \qquad -5A + B = 0 \qquad B = 5A$$

$$2A + 50A = 52A = 1 \qquad A = \frac{1}{52}$$

$$B = \frac{5}{52}$$

$$y_p = \frac{1}{52}\sin(2x) + \frac{5}{52}\cos(2x)$$

$$y'' + y = 2\cos x = e^{ix} + e^{-ix}$$

$$P(r) = r^2 + 1$$

$$P(\pm i) = 0$$

$$y_p = A\,x\,\mathrm{sen}(x) + B\,x\cos x$$

$$L[y_p] = L[A\,x\sin(x) + B\,x\cos x] =$$

$$= A\,L[x\,\mathrm{sen}(x)] + B\,L[x\cos x]$$

$$= A\left[(x\,\mathrm{sen}\,x)'' + x\sin x\right] + B\left[(x\cos x)'' + x\cos x\right]$$

$$(x\sin x)' = \mathrm{sen}\,x + x\cos x \qquad (x\sin x)'' = \cos x + \cos x - x\sin x$$

$$(x\cos x)' = \cos x - x\sin x, \qquad (x\cos x)'' = -\mathrm{sen}\,x - \mathrm{sen}\,x - x\cos x$$

$$= A\left[2\cos x - \cancel{x\,\mathrm{sen}\,x} + \cancel{x\,\mathrm{sen}\,x}\right] + B\left[-2\sin x - \cancel{x\cos x} + \cancel{x\cos x}\right]$$

$$= 2A\cos x - 2B\,\mathrm{sen}\,x = 2\cos x + 0\sin x$$

$$\begin{cases} 2A = 2 \\ -2B = 0 \end{cases} \qquad \begin{array}{l} A = 1 \\ B = 0 \end{array}$$

$$y_p = x\,\mathrm{sen}\,x$$

$$y_g = C_1\,\mathrm{sen}\,x + C_2\cos x + x\,\mathrm{sen}\,x$$

$$y'' + y = \operatorname{sen}(2x) \qquad p(2i) \neq 0$$

$$p(-2i) \neq 0$$

$$y_p = A \sin(2x) + B\cos(2x)$$

$$L[A \sin(2x) + B \cos(2x)] =$$

$$= A \cdot [(\sin(2x))'' + \sin(2x)] + B\,[(\cos 2x)'' + \cos(2x)]$$

$$= A\,[-4 \operatorname{sen}(2x) + \sin(2x)] + B[-4\cos(2x) + \cos(2x)]$$

$$= -3A \sin(2x) - 3B\cos(2x) = \sin(2x)$$

$$-3A = 1 \qquad -3B = 0$$

$$A = -\frac{1}{3} \qquad y_p = -\frac{1}{3}\operatorname{sen}(2x)$$

$$y = C_1 \sin(x) + C_2 \cos x - \frac{1}{3}\sin(2x)$$

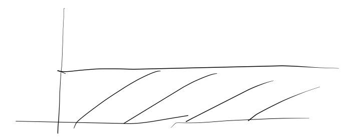
Dall'esame 16/1/2023 Esercizio 3

$$\lim_{x\to+\infty} \int_0^x e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}}\ dt$$

Per prima cosa esiste per il teorema dell'Aut aut.

$$e^{-\frac{1}{t}} = e^0 + o(1) = 1 + o(1) \quad \text{per } \cancel{x}\, t \to +\infty$$

$$\sqrt{1+\frac{1}{t}} = \left(1+\frac{1}{t}\right)^{\frac{1}{2}} = 1 + o(1) \quad \text{per } t \to +\infty$$



$$\lim_{t\to+\infty} \frac{e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}}}{1} = 1 .$$ Ma $1 \notin L[1,+\infty)$

$\Rightarrow$ per il confronto asintotico $e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}} \notin L[1,+\infty)$

$$\Rightarrow \lim_{x\to+\infty} \int_1^{+\infty} e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}}\ dt = +\infty$$

Per quanto riguarda $\displaystyle\int_0^1 e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}}\ dt$

qui $g(t) := e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}} \in C^0((0,1])$

ma inoltre $\displaystyle\lim_{t\to0^+} e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}} = \lim_{t\to0^+} \frac{\sqrt{1+\frac{1}{t}}}{e^{\frac{1}{t}}}$

$y = \frac{1}{t}$ $\displaystyle = \lim_{y\to+\infty} \frac{\sqrt{1+y}}{e^y} = 0$

Ponendo $g(t) = \begin{cases} e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}} & \text{per } t \neq 0 \\ 0 & t = 0 \end{cases}$

$g \in C^0([0,1]) \Rightarrow g \in L[0,1] \Rightarrow e^{-\frac{1}{t}} \sqrt{1+\frac{1}{t}} \in L[0,1]$

$$p(r) = r^2 + br + c$$

__Teorema__ Dato $y'' + by' + cy = P(x)\, e^{\alpha x} \sin(\beta x)$

vale quanto segue $(\beta \neq 0)$

1) Se $p(\alpha + i\beta) \neq 0$ esiste

$$y_p = P_1(x)\, e^{\alpha x} \sin(\beta x) + P_2(x)\, e^{\alpha x} \cos(\beta x)$$

dove $\deg P_j(x) \leq \deg P(x)$ $*$

2) Se $p(\alpha + i\beta) = 0$ esiste

$$y_p = x\, P_1(x)\, e^{\alpha x} \sin(\beta x) + x\, P_2(x)\, e^{\alpha x} \cos(\beta x)$$

con $P_1(x)$ e $P_2(x)$ che soddisfano $*$

$$y'' + \varepsilon y' + y = \sin(x)$$

$r = \alpha \pm i\beta$

$$y'' + \varepsilon y' + y = 0 \qquad r^2 + \varepsilon r + 1 = 0 \qquad e^{\alpha x} \cos(\beta x) \quad e^{\alpha x} \ldots(\beta x)$$

$$C_1 e^{-\frac{\varepsilon}{2}x} \sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\, x\right) + C_2 e^{-\frac{\varepsilon}{2}x} \cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\, x\right)$$

$$y_h = C_1 e^{-\frac{\varepsilon}{2}x} \sin\left(\frac{\sqrt{4-\varepsilon^2}}{2}\, x\right) + C_2 e^{-\frac{\varepsilon}{2}x} \cos\left(\frac{\sqrt{4-\varepsilon^2}}{2}\, x\right)$$

$$= C_1 e^{-\frac{\varepsilon}{2}x} \sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\, x\right) + C_2 e^{-\frac{\varepsilon}{2}x} \cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\, x\right)$$

$$y'' + \varepsilon y' + y = \sin(x) \qquad ix$$

$$y_p = A \sin(x) + B \cos(x)$$

$$L[y_p] = \sin x$$

$$L[y_p] = A\, L[\sin x] + B\, L[\cos x]$$

$$= A\left(\underbrace{\sin'' x + \sin x}_{=0} + \varepsilon \sin' x\right) + B\left(\underbrace{\cos'' x + \cos x}_{=0} + \varepsilon \cos' x\right)$$

$$= \varepsilon A \cos x - \varepsilon B \sin x = \sin x$$

$$A = 0 \qquad B = -\frac{1}{\varepsilon}$$

$$y_p = -\frac{1}{\varepsilon} \cos x \qquad y'' + \varepsilon y' + y = \sin x$$

$$y_g = y_h - \frac{1}{\varepsilon} \cos x$$