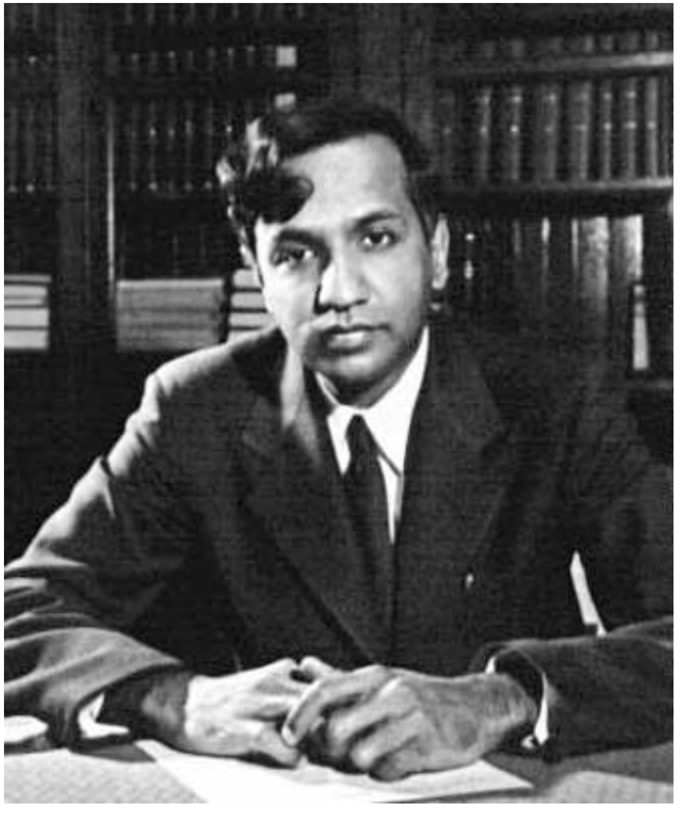
The structure and evolution of stars

Lecture 12:White dwarfs, neutron stars and black holes



Learning Outcomes

The student will learn

- How to derive the equation of state of a degenerate gas
- How polytropic models can be applied to degenerate stars - white dwarfs
- How to derive the stable upper mass limit for white dwarfs
- How the theoretical relations compare to observations
- What a neutron star is and what are their possible masses
- How to measure the masses of black-holes and what are the likely production mechanisms

Introduction and recap

So far have assumed that stars are composed of ideal gases

In lecture on low mass stars:

- Several times have mentioned degeneracy pressure - in the case of low-intermediate mass stars, they develop a degenerate He core.
- Degeneracy pressure can resist the gravitational collapse
- We will recap this idea in this lecture
- Will use our knowledge of polytropes and the Lane-Emden equation

In lecture on high mass stars:

- Found that that high mass stars develop Fe core at the end of their lives
- What will happen when core is composed of Fe ?

Equation of state of a degenerate gas

At high densities, gas particles may be so close, that that interactions between them cannot be neglected.

What basic physical principle will become important as we increase the density and pressure of a highly ionised ideal gas ?

The Pauli exclusion principle - the e^- in the gas must obey the law:

No more than two electrons (of opposite spin) can occupy the same quantum cell

The quantum cell of an e^- is defined in phase space, and given by 6 values:

x, y, z, p_x, p_y, p_z

The volume of allowed phase space is given by

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$$

The number of electrons in this cell must be at most 2

Consider the centre of a star, as the density increases

The e^- become crowded, eventually 2 e^- occupy almost same position

Volume of phase space “full” (from exclusion principle)

Not possible for another e^- to occupy space, unless δp significantly different

Consider a group of electrons occupying a volume V of position space which have momenta in the range $p+\delta p$. The volume of momentum space occupied by these electrons is given by the volume of a spherical shell of radius p, thickness δp:

$$4\pi p^2 \delta p$$

Volume of phase space occupied is ***volume occupied in position space multiplied by volume occupied in momentum space***

$$V_{ph} = 4\pi p^2 V \delta p$$

Number of quantum states in this volume is V_{ph} divided by volume of a quantum state (h^3)

$$\frac{4\pi p^2 V}{h^3} \delta p$$

Define $N_p\delta p$ = number of electrons with momenta in the range $p+\delta p$.

Pauli's exclusion principle tells us:

$$N_p \delta p \le \frac{8\pi p^2 V}{h^3}\delta p$$

Define a *completely degenerate gas* : one in which all of momentum states up to some critical value p_0 are filled, while the states with momenta greater than p_0 are empty.

$$p \le p_0 \qquad N_p = \frac{8\pi p^2 V}{h^3}$$

$$p > p_0 \qquad N_p = 0$$

$$\Rightarrow \qquad N = \frac{8\pi V}{h^3}\int_0^{p_0} p^2 dp = \frac{8\pi p_0^3 V}{3h^3}$$

The pressure *P* is mean rate of transport of momentum across unit area

$$P = \frac{1}{3}\int_0^{\infty} \frac{N_p}{V} p v_p dp$$

Where

v_p= *velocity of* e^- *with momentum p*

Use relation between *p and* v_p from theory of special relativity

$$p = \frac{m_e v_p}{(1 - v_p^2/c^2)^{\frac{1}{2}}} \qquad\qquad v_p = \frac{p/m_e}{(1 + p^2/m_e^2 c^2)^{\frac{1}{2}}}$$

Where m_e=*rest mass of* e^-

Combining the three expressions for *N, P,* and v_p , we obtain pressure of a completely degenerate gas

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} \frac{p^4}{(1 + p^2/m_e c^2)^{1/2}} dp$$

Non-relativistic degenerate gas ($p_0 << m_e c$)

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} p^4 dp = \frac{8\pi p_0^5}{15 h^3 m_e}$$

By defining $n_e=N/V$ and recalling

$$N = \frac{8\pi p_0^3 V}{3h^3}$$

The electron degeneracy pressure for a non-relativistic degenerate gas:

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2 n_e^{5/3}}{m_e}$$

Relativistic degenerate gas

(p_0 >> $m_e c$; when v approaches c and momentum → ∞)

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{4/3}$$

Aim is to obtain equation of state for a degenerate gas. We must convert n_e *to mass density* ρ (using similar arguments to derivation of mean molecular weight: lecture 7). For each mass of H (m_H) there is one e^- . For He and heavier elements there is approximately 1/2 e^- for each m_H. Thus:

$$n_e = \frac{\rho X}{m_H} + \frac{\rho(1-X)}{2m_H} = \frac{\rho(1+X)}{2m_H}$$

$$P_{gas} = K_1 \rho^{5/3} \qquad \text{Equation of state of non-relativistic degenerate gas}$$

$$P_{gas} = K_2 \rho^{4/3} \qquad \text{Equation of state of a relativistic degenerate gas}$$

where

$$K_1 = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1+X}{2m_H}\right)^{5/3}$$

$$K_2 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{1+X}{2m_H}\right)^{4/3}$$

In a completely degenerate gas the pressure depends ***only*** on the density and chemical composition. It is independent of temperature

Suggested further reading: See Prialnik (Chapter 3), Taylor (Appendix 3) for full discussions of derivation

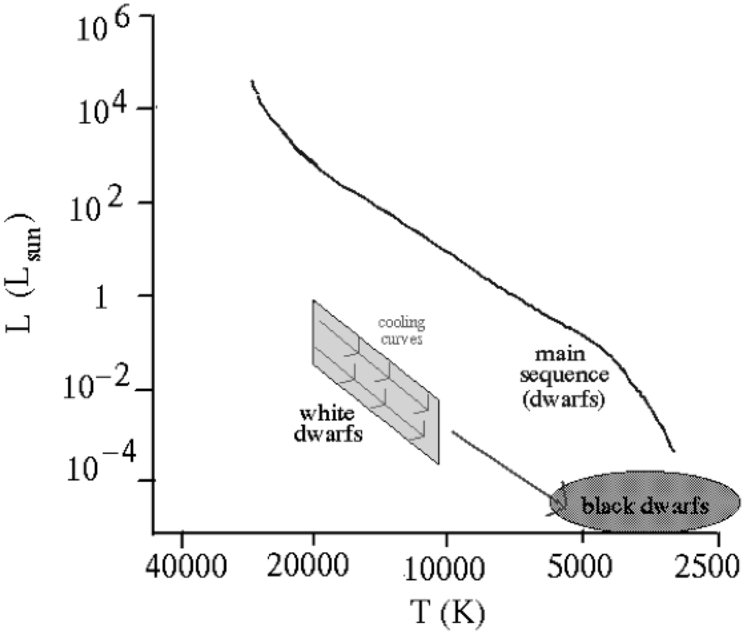
Degenerate stars

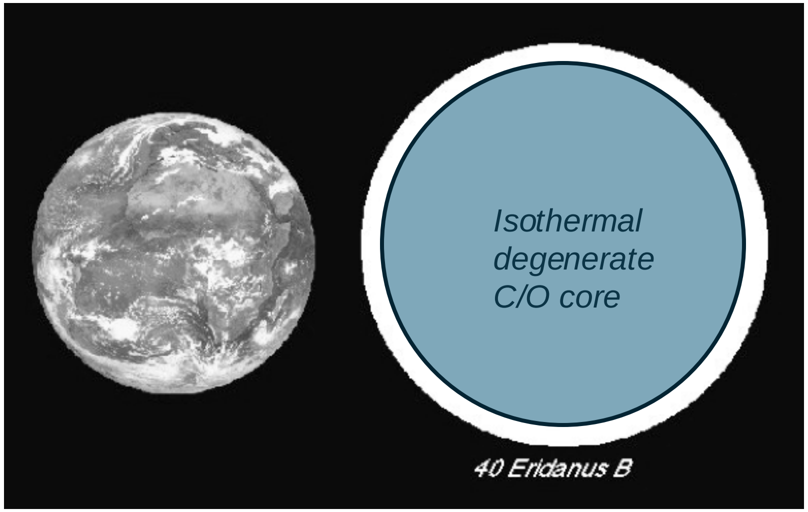
There is not a sharp transition between relativistically degenerate and non-relativistically degenerate gas. Similarly, there is no sharp transition between an ideal gas and a completely degenerate one. ***Partial degeneracy*** situation requires much more complex solution.

White dwarfs

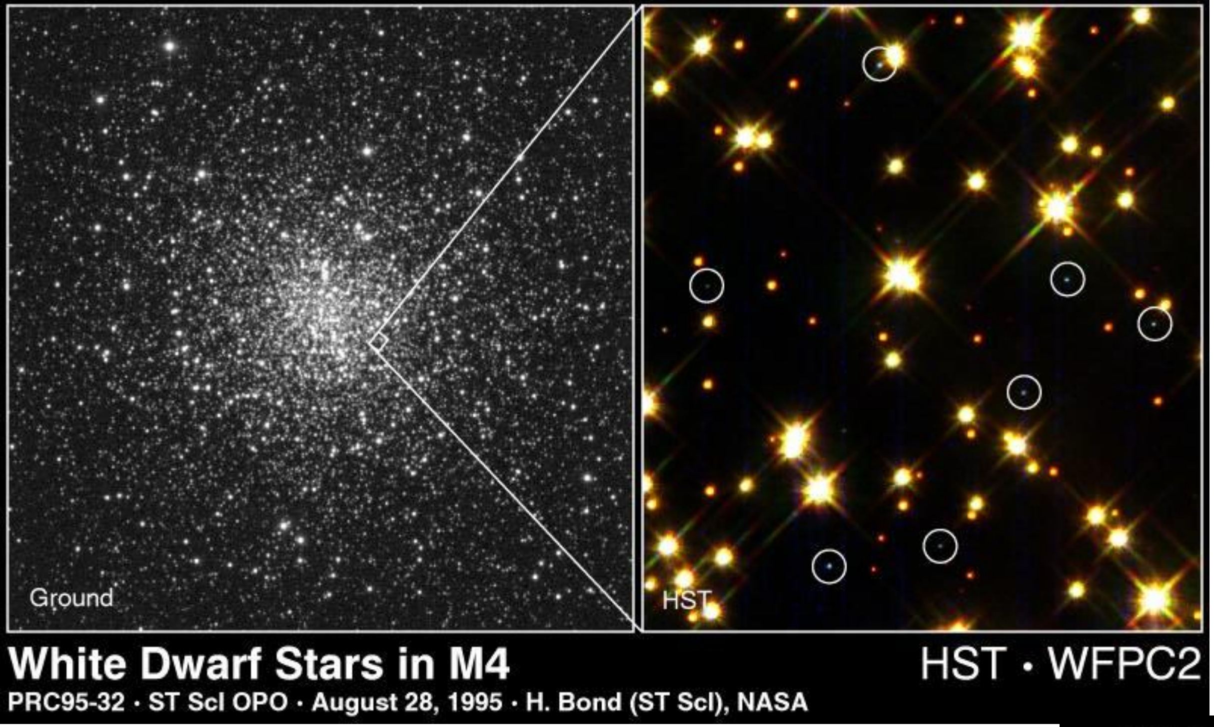
Intrinsically faint, hot stars. Typical observed masses 0.1-1.4$M_\odot$

Calculate typical radius and density of a white dwarf (σ= 5.67x10^{-8} $Wm^{-2}K^{-4}$)



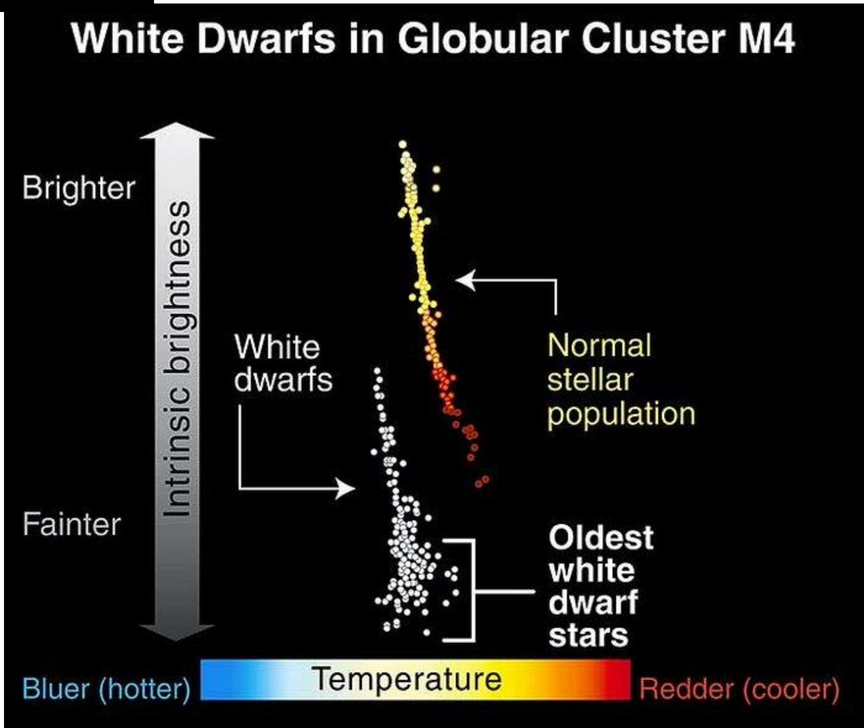


Thin nondegenerate surface layer of H or He



Example of WD discovered in Globular cluster M4

- Cluster age ~ 13Myrs
- WDs represent cooling sequence
- Similar intrinsic brightness as MS members, but much hotter (hence bluer)



The Chandrasekhar mass

Recall the equations of state for a degenerate gas - what could these be used for ?

$$P_{gas} = K_1 \rho^{5/3} \qquad \text{(non - relativistic degenerate gas)}$$

$$P_{gas} = K_2 \rho^{4/3} \qquad \text{(relativistic degenerate gas)}$$

A polytrope of index *n=1.5 with* $K=K_1$ would describe non-relativistic case, and *n=3,* $K=K_2$ would describe relativistic case.

$$P = K\rho^{\gamma} = K\rho^{\frac{n+1}{n}}$$

Now recall from Lecture 7, the mass of a polytropic star is given by

$$M = -4\pi\alpha^3 \rho_c \xi_R^2 \left[\frac{d\theta}{d\xi}\right]_{\xi=\xi_R}$$

Using this, and eliminating ρc and substituting in for α (as from previous lecture). We obtain a relation between stellar mass and radius:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{\left[(n+1)K\right]^n}{4\pi G}$$

$$\frac{(n+1)K}{4\pi G\rho^{\frac{n-1}{n}}} = \alpha^2$$

M_n and R_n are constants that vary with polytropic index n (from solution of Lane-Emden equation shown in Lecture 7).

where $$M_n = -\xi_R^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_R} \qquad R_n = \xi_R$$

For *n=1.5,* the relation between mass - radius, and mass density become

$$R = AM^{-1/3}$$

$$\bar{\rho} \propto MR^{-3} \propto M^2$$

Imagine degenerate gaseous spheres with higher and higher masses, what will happen ?

Density becomes so high that the degenerate gas becomes relativistic, hence the degenerate gaseous sphere is still a polytrope but with index *n=3*

$$\left(\frac{GM}{M_3}\right)^2 = \frac{[4K]^3}{4\pi G} \qquad \Rightarrow \qquad M = 4\pi M_3\left(\frac{K}{\pi G}\right)^{3/2}$$

Substituting in for K_2, gives us this limiting mass. First found by Chandrasekhar in 1931, it is the ***Chandrasekhar mass***

$$M_{Ch} = \frac{M_3\sqrt{1.5}}{4\pi}\left(\frac{hc}{Gm_H^{4/3}}\right)^{3/2}\left(\frac{1+X}{2}\right)^2$$

Inserting the values for the constants we get

$$M_{Ch} = 5.86\left(\frac{1+X}{2}\right)^2 M_{Sol}$$

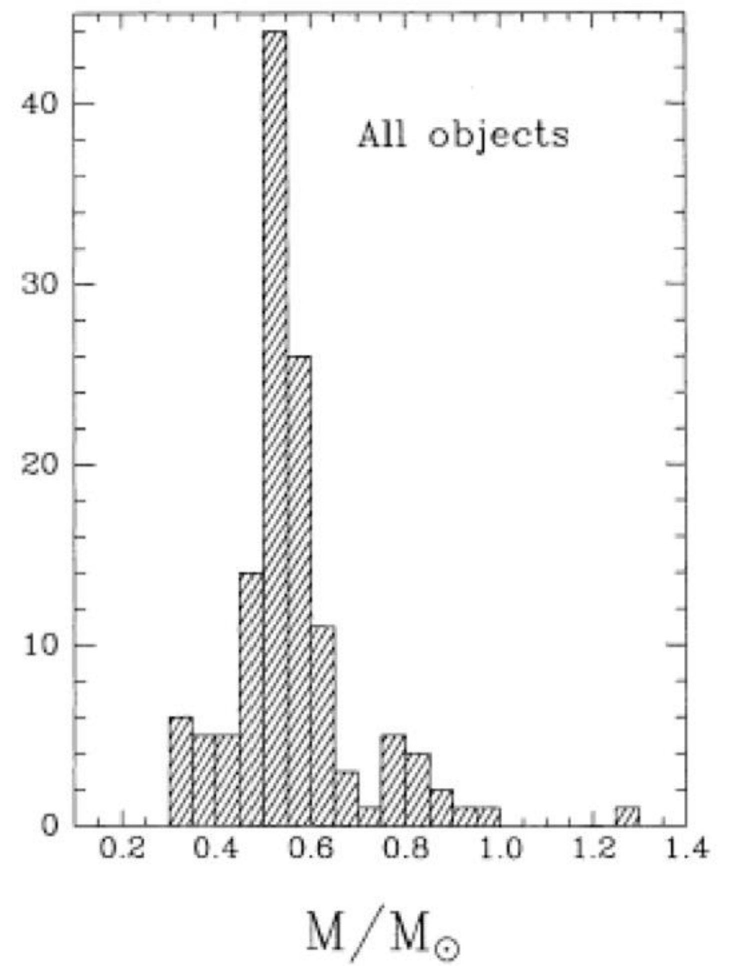
For $X \sim 0$; $M_{Ch} = 1.46 M_\odot$ (He, C, O…. composition)

Measured WD masses

Mass estimates for 129 white dwarfs

From Bergeron et al. 1992, ApJ

Mean $M = 0.56 \pm\ 0.14\, M_{\odot}$



How is mass determined ?

Note sharp peak, and lack of high mass objects.

Observed mass-radius relation

Mass/radius relation and initial mass vs. final mass estimate for WD in stellar clusters. How would you estimate the initial mass of the progenitor star of a WD ?

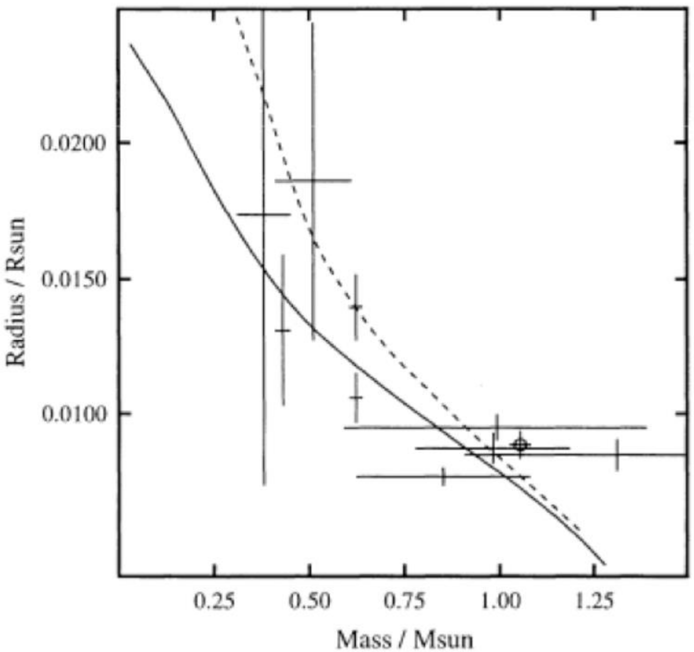


Fig. 3. Mass-radius relation for four white dwarfs in NGC 2516 together with five DA in astrometric binaries. See text for further explanations

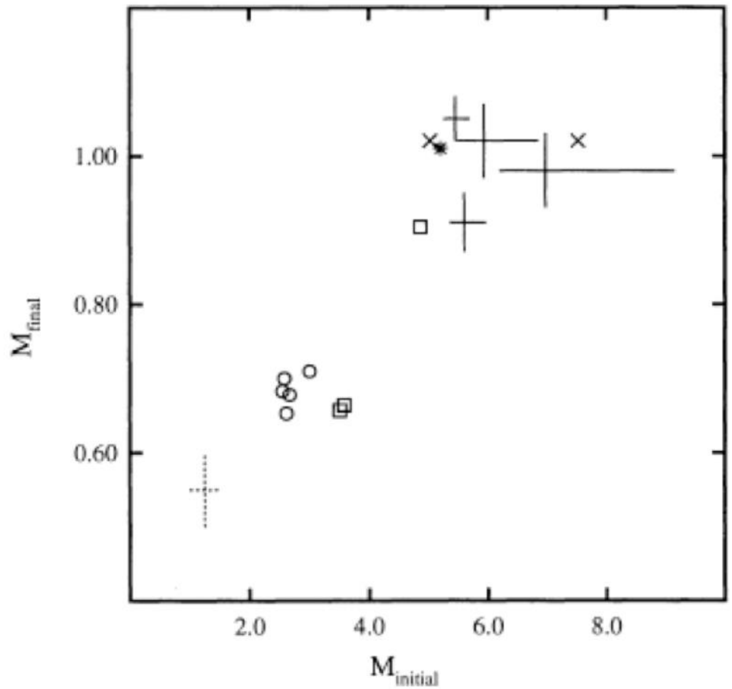


Fig. 4. Remnant (final) masses as a function of main sequence (initial) mass for the objects in NGC 2516. Also shown are the best observed objects in NGC 3532, the Pleiades white dwarf, and five white dwarfs in the Hyades. See text for references and further explanations

Koester & Reimers 1996, A&A, 313, 810 *White dwarfs in open clusters (NGC2516)*

White Dwarf Cooling I

Without nuclear reactions, a white dwarf will slowly cool over time, radiating away its thermal energy.

• In normal stars the mean free path for photons is much greater than that of electrons or heavier particles; consequently, energy transport is mainly by radiative diffusion.

• In a white dwarf, **degenerate electrons** can travel **long distances** before losing energy in a collision with a nucleus, since the vast majority of lower-energy electron states are already occupied.

• Thus, in a white dwarf energy is carried by electron conduction (similar to conduction in metals) rather than by radiation.

• Electron conduction is so efficient that the interior of a white dwarf is nearly isothermal, with the temperature dropping significantly only in the non-degenerate surface layers.

• The thin (1% of the white dwarf radius) non-degenerate envelope transfers heat less efficiently and acts as an insulating “blanket” allowing energy to leak out slowly.

• A steep temperature gradient near the surface results in the outer non-degenerate envelope being convective.

• The initial temperature of a white dwarf may be estimated by recalling that it forms from the contraction of a thermally unsupported stellar core, a process which is eventually stopped by degeneracy pressure.

White Dwarf Cooling II

- By the Virial Theorem, just before reaching the point of equilibrium, the thermal energy ($E_{\rm th}$) will equal half of the potential energy:

$$E_{\rm th} \sim \frac{1}{2}\frac{G\,M^2}{R}.$$

- For a pure He composition, the number of nuclei in the core is $M/4m_{\rm H}$ and the number of electrons is is $M/2m_{\rm H}$. The total thermal energy is therefore

$$E_{\rm th} = \frac{3}{2}\mathcal{N}\,k\,T = \frac{3}{2}\frac{M}{m_{\rm H}}\left(\frac{1}{2}+\frac{1}{4}\right)\,k\,T = \frac{9}{8}\frac{M}{m_{\rm H}}\,k\,T,$$

so that

$$k\,T \sim \frac{4}{9}\frac{G\,M\,m_{\rm H}}{R}.$$

- It was previously shown that a degenerate non-relativistic star is well represented by a $n = 1.5$ polytrope for which $R \propto M^{-1/3}$ and therefore $k\,T \propto M^{4/3}$.

White Dwarf Cooling III

- Specifically, the solution of the Lane Emden Equation for a $n = 1.5$ polytrope yields $\xi_1 = 3.6538$ and $|d\theta/d\xi|_{\xi=\xi_1} = -0.20325$ which with the previously derived result

$$\left(\frac{G\,M}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)\,K]^n}{4\,\pi\,G}$$

gives

$$R = 2.679 \times 10^{17} M^{-1/3}$$

with $K = K_1$, $R_{1.5} = \xi_1$ and $M_{1.5} = {\xi_1}^2\,|d\theta/d\xi|_{\xi=\xi_1}$.

- Substituting for R in the above expression for kT gives

$$k\,T \sim \frac{4}{9}\,G\,m_{\rm H}\,(2.679 \times 10^{17})^{-1}\,M^{4/3} = 1.848 \times 10^{-55}\,M^{4/3}$$

- For a white dwarf having $M = 0.5\,M_\odot$

$$k\,T \sim 1.848 \times 10^{-15}\,\text{Joules}$$

$$T \sim 1.34 \times 10^{8}\,\text{K}.$$

- Clearly a just-formed degenerate core is a very hot object with thermal emission that peaks at X-ray wavelengths; this radiation ionises the layers of gas that were blown off during the AGB phase, giving rise to planetary nebulae.

White Dwarf Cooling IV

Energy radiated away from the surface of a white dwarfs is the thermal energy stored in the still classical gas of nuclei within the star's volume.

- Degeneracy of the electron gas limits almost completely the ability of electrons to lose their kinetic energies.
- An upper limit to the cooling rate is obtained by neglecting the envelope and assuming a uniform temperature throughout.
- The rate at which $E_{\rm th}$ decreases is then determined by L from

$$L = 4\,\pi\,R^2\,\sigma\,T^4 \sim -\frac{dE_{\rm th}}{dt} = -\frac{3}{8}\,\frac{M\,k}{m_{\rm H}}\frac{dT}{dt}$$

  where only the thermal contribution of nuclei has been included.

- On integrating and assuming a fixed radius, the time required for a white dwarf to cool from $T = T_1$ to $T = T_2$ is

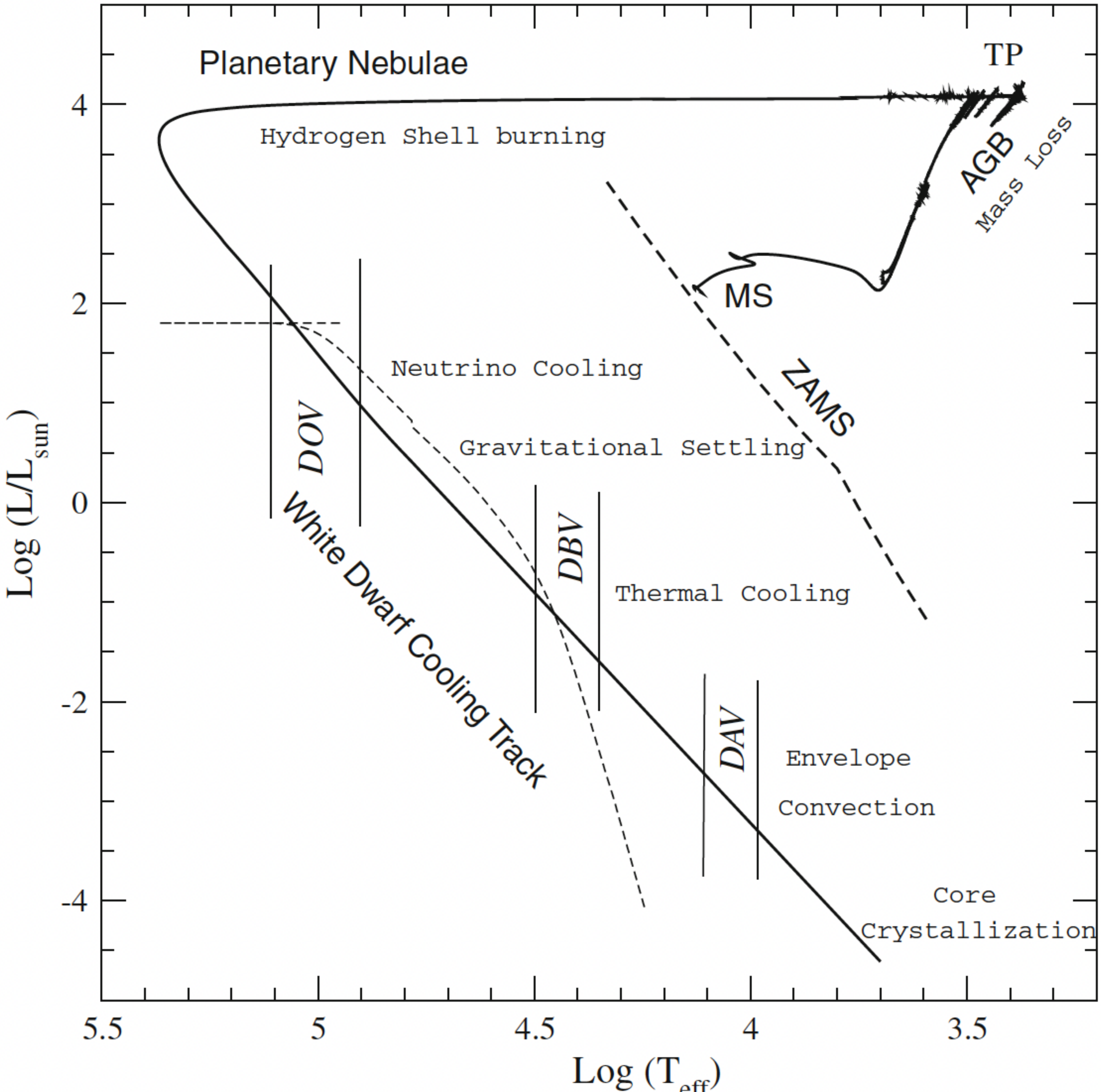
$$\tau_{12} = \frac{3}{8}\,\frac{M\,k}{m_{\rm H}}\,\frac{1}{4\,\pi\,R^2\,\sigma}\,\frac{1}{3}\,\left[\frac{1}{{T_2}^3} - \frac{1}{{T_1}^3}\right]$$

- Substituting numerical values gives

$$\tau_{12} \sim 6.41E+24\,\left(\frac{M}{M_\odot}\right)^{5/3}\,\left[\frac{1}{{T_2}^3} - \frac{1}{{T_1}^3}\right] \quad \text{seconds.}$$

Thus, even with an unrealistically efficient cooling it would take a $1\,M_\odot$ white dwarf $\sim 1Gyr$ to cool to $\sim 10^3\,\mathrm{K}$.

- In reality, the insulation provided by the non-degenerate envelope results in a $T_{\rm eff}$ that is significantly lower than the interior temperature and this lowers the cooing rate.
- Furthermore, as a white dwarf cools, it crystallises in a gradual process that starts at the centre and moves outwards.
- The regular crystal structure is maintained by the mutual electrostatic repulsion of the nuclei; it minimises their energy as they vibrate about their average position in the lattice.
- As the nuclei undergo this phase change, the latent heat that is released is added to the thermal balance, further slowing down the decline in temperature.


Planetary Nebulae
TP
Hydrogen Shell burning
AGB
Mass Loss
MS
ZAMS
Neutrino Cooling
DOV
Gravitational Settling
White Dwarf Cooling Track
DBV
Thermal Cooling
DAV
Envelope
Convection
Core
Crystallization
4
2
0
-2
-4
Log (L/L$_{sun}$)
5.5
5
4.5
4
3.5
Log (T$_{eff}$)

Neutron stars

Will see in next lecture that the collapse of the Fe core of a massive star results in neutron star formation.

Landau (1932) - postulated formation of “one gigantic nucleus” from stars more compact than critical value. Walter Baade and Fritz Zwicky (1934) suggested they come from supernovae

Neutrons are fermions - neutron stars supported from gravitational collapse by neutron degeneracy.

NS structure can be approximated by a polytrope of *n=1.5 (ignoring relativistic effects)* which leads to similar mass/radius relation. But constant of proportionality for neutron star calculations implies much smaller radii.

$$R = CM^{-1/3} \qquad C = \text{constant}$$

$1.4M_\odot$ NS has R~10-15 km

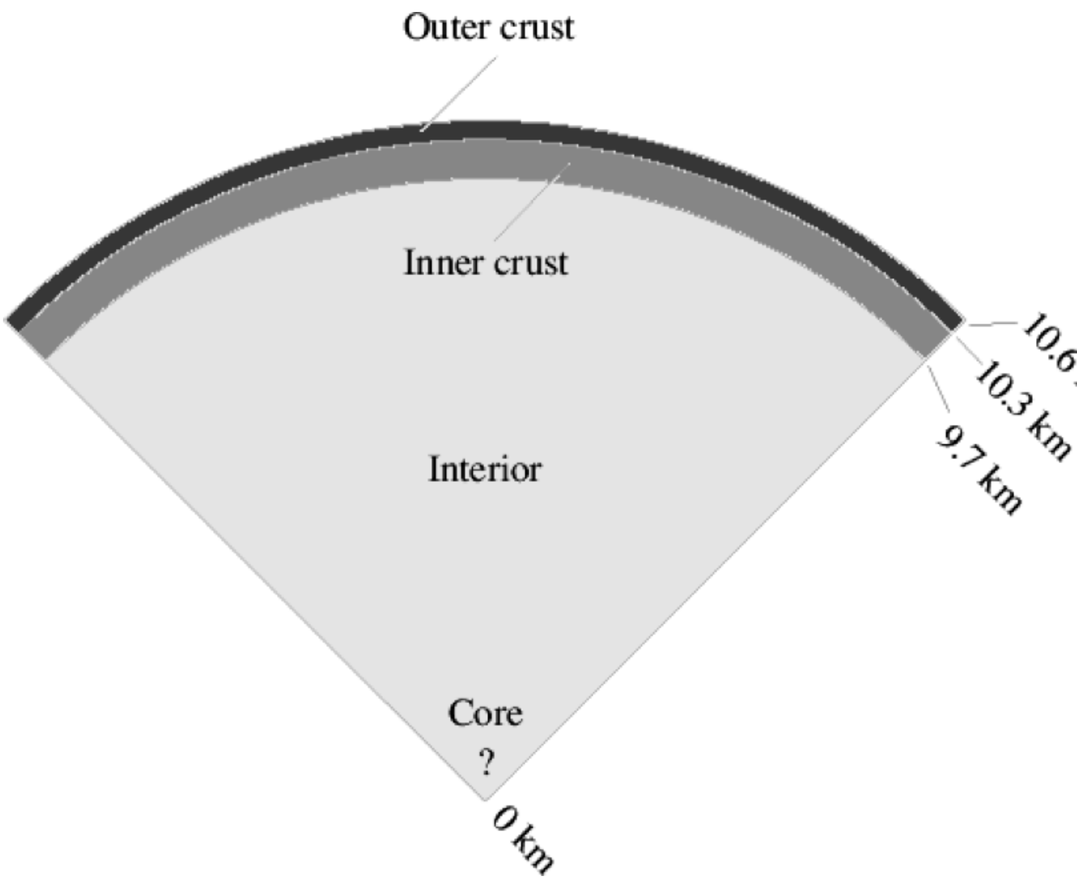
$\rho \sim 6 \times 10^{14}$ gm cm^{-3} (nuclear density)

Relativistic treatment of the equation of state imposes upper limit on NS mass. Above this mass, degeneracy pressure unable to balance self-gravity.

Complications:

- General Theory of Relativity required
- Interactions between neutrons (strong force) important
- Structure and maximum mass equations too complex for this course

Various calculations predict M_{max}=1.5 – 3$M_{\odot}$ solar



Outer Crust: Fe and n-rich nuclei, relativistic degenerate e–
Inner Crust: n-rich nuclei, relativistic degenerate e–
Interior: superfluid neutrons
Core: unknown, pions ?quarks ?

Neutron star properties

Neutron stars are predicted to rotate fast and have large magnetic fields. Simple arguments:

Angular momentum

$$I_i\omega_i = I_f\omega_f$$

$$M_iR_i^2\omega_i = M_fR_f^2\omega_f$$

$$\omega_f = \omega_i\left(\frac{R_i}{R_f}\right)^2$$

$$P_f = P_i\left(\frac{R_f}{R_i}\right)^2$$

Magnetic field

$$B_i 4\pi R_i^2 = B_f 4\pi R_f^2$$

$$B_f = B_i\left(\frac{R_i}{R_f}\right)^2$$

Luminosity ($T_s \sim 10^6$)

$$\mathrm{L} \sim 4\pi \mathrm{R}^2\sigma T_s^4 \sim 10^{26} W$$

$$\lambda_{BB-peak} = \frac{2.9\times10^7}{T}\,\text{Angs.} \quad \sim 29\ \text{Angs.}$$

Initial rotation period is uncertain but if it is similar to typical WDs (e.g. 40Eri B has P_{WD}=1350s), $P_{NS} \sim 4$ ms

Magnetic field strengths in WDs typically measured at B=5x10^8 Gauss, hence B_{NS}~10^{14} Gauss (compare with $B_{\odot}$ ~2 Gauss!)

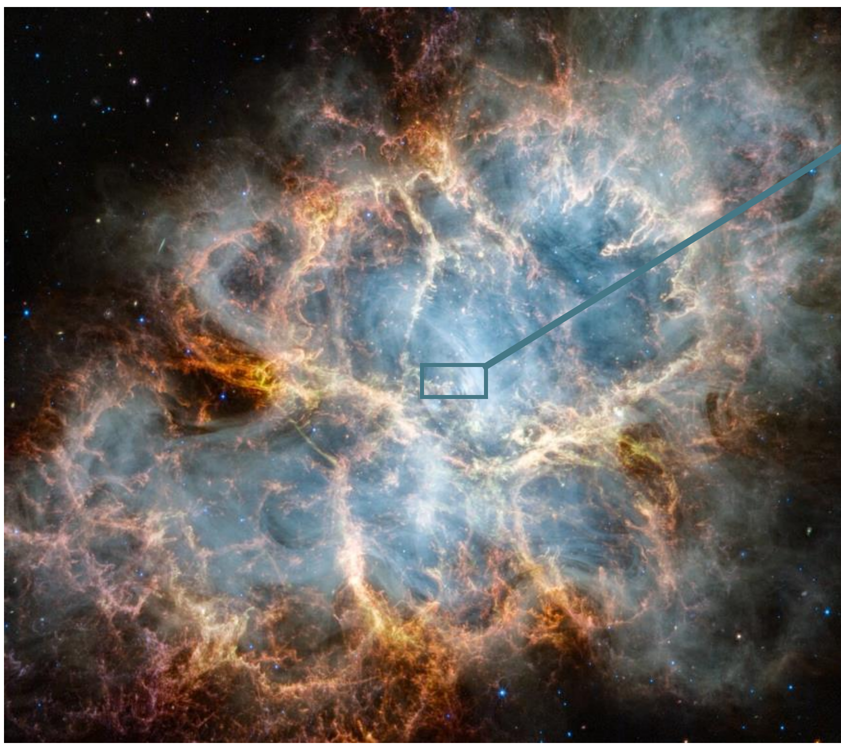
Similar luminosity to Sun, but mostly in X-rays (optically very faint)

Discovery of neutron stars

1967: Hewish and Bell discovered regularly spaced radio pulses P=1.337s, repeating from same point in sky.

Approx. 1500 ***pulsars*** now known, with periods on range $0.002 < P < 4.3$ s

Crab pulsar - embedded in Crab nebula, which is remnant of supernova historically recorded in 1054AD



Crab pulsar emits X-ray, optical, radio pulses P=0.033s

Spectrum is power law from hard X-rays to the IR

Synchrotron radiation: relativistic electrons spiralling around magnetic field lines.

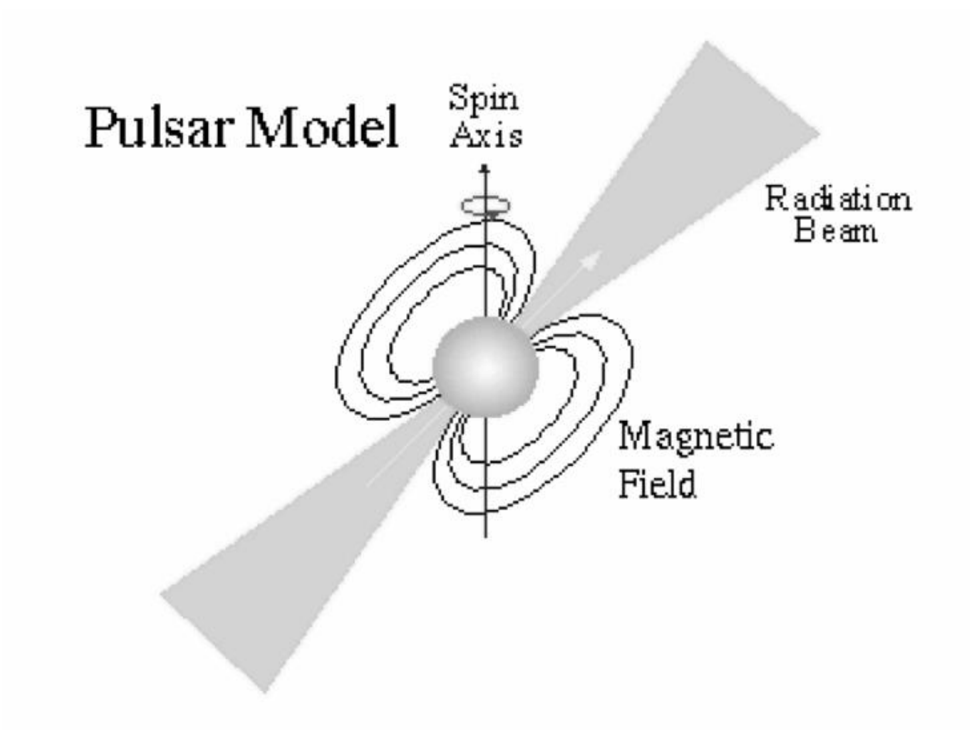
Pulsar mechanism

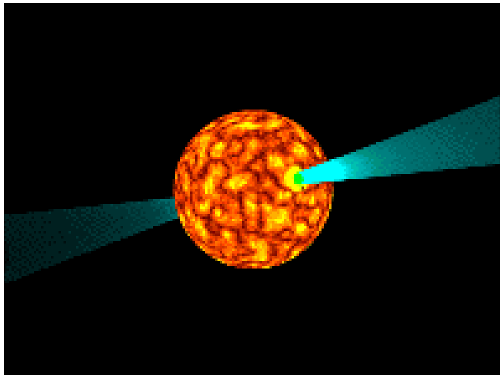
Rapidly rotating NS with strong dipole magnetic field.

Magnetic field axis is not aligned with rotational axis.

Spectrum of Crab pulsar is non-thermal. Suggestive of synchrotron radiation - relativistic charged particles emit radiation dependent on particle energy.

Charged particles (e-) accelerated along magnetic field lines, radiation is beamed in the the acceleration direction. If axes are not aligned, leads to the “lighthouse effect”

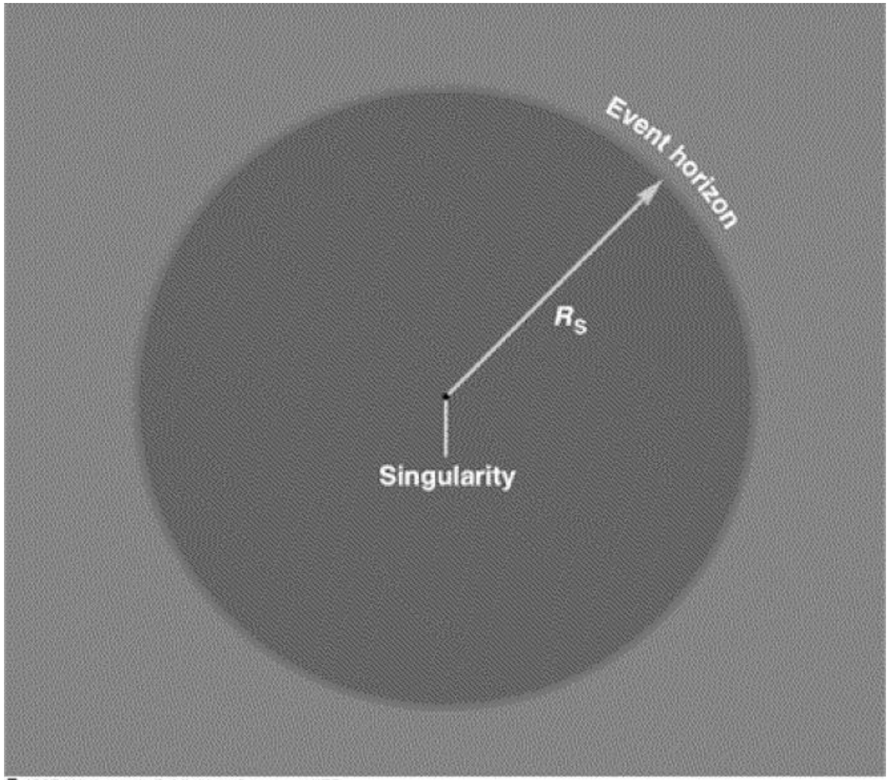




Black Holes

Description of a black hole is entirely based on theory of General Relativity - beyond scope of this course. But simple arguments can be illustrative:

Black holes are completely collapsed objects - radius of the "star" becomes so small that the escape velocity approaches the speed of light:



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Escape velocity for particle from an object of mass *M* and radius *R*

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

If photons cannot escape, then $v_{esc} > c$. ***Schwarzschild*** radius is

$$R < R_S \equiv \frac{2GM}{c^2} = 3\ \text{km} \frac{M}{M_{Sol}}$$

Size of black holes determined by mass. Example Schwarzschild radius for various masses given by:

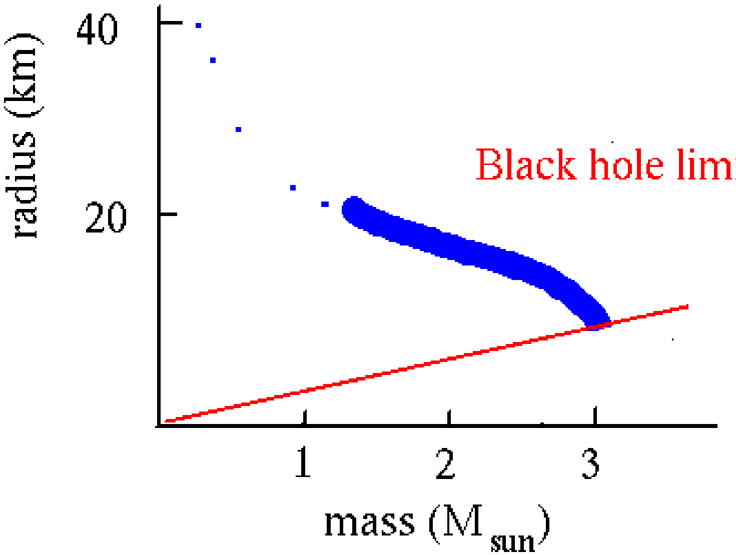
Object	M ($M_\odot$)	R_s
Star	10	30 km
Star	3	9 km
Sun	1	3 km
Earth	$3x10^{-6}$	9 mm

The event horizon is located at R_s - everything within the event horizon is lost. The event horizon hides the singularity from the outside Universe.

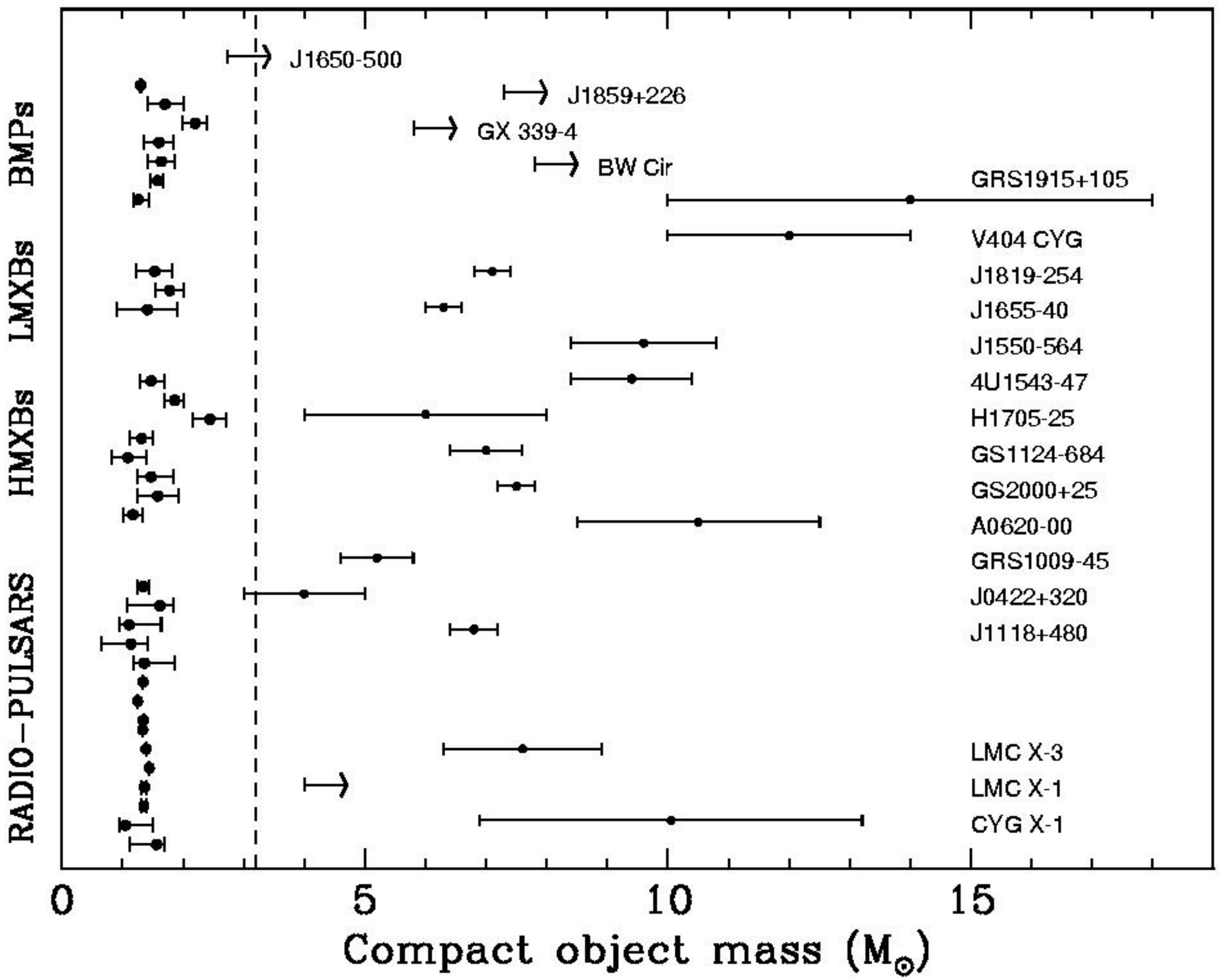
Two more practical questions:

What could collapse to form a black hole ?

How can we detect them and measure their masses ?

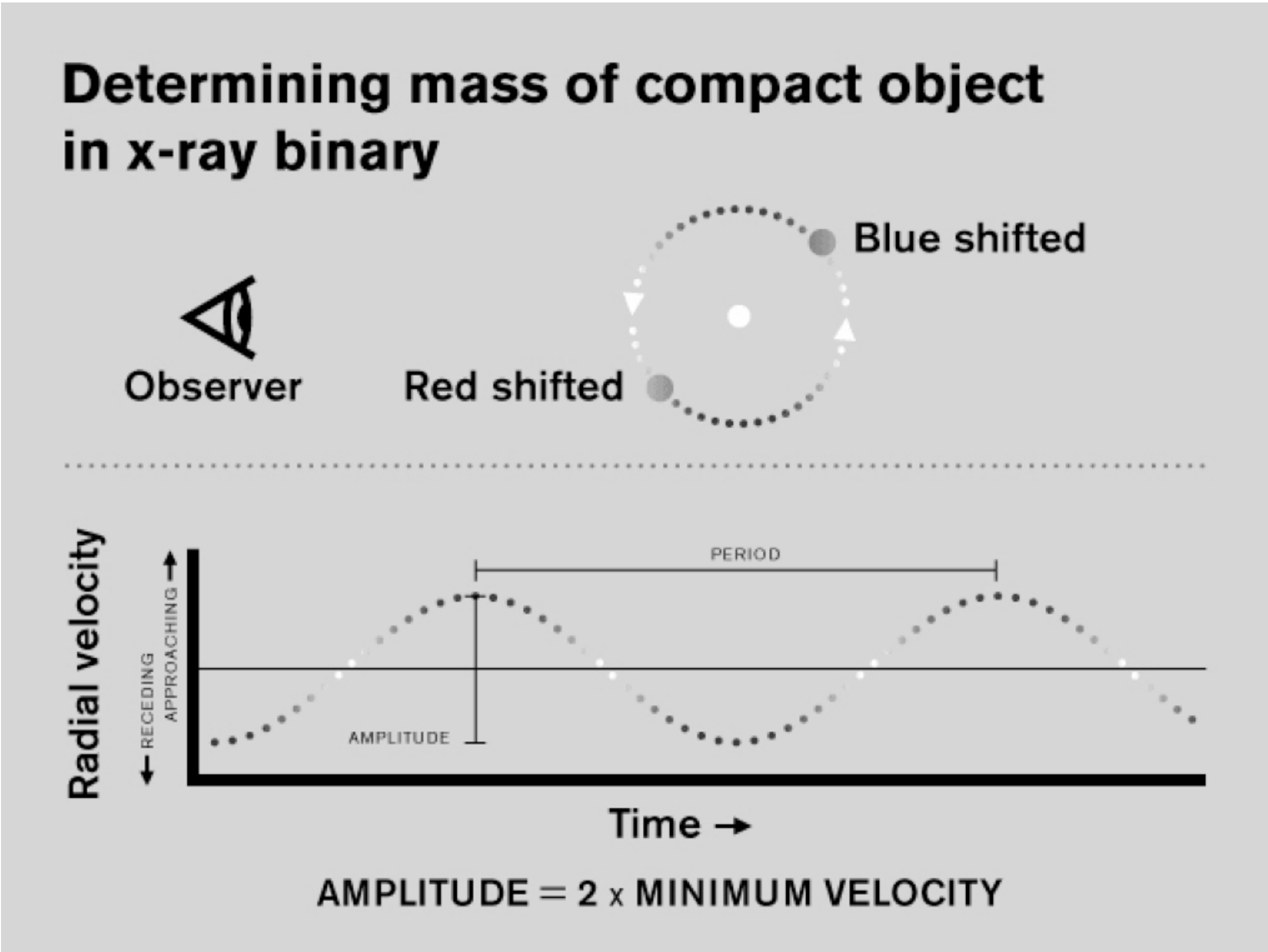


Black hole and neutron star masses from binary systems



From J. Caseres, 2005, astro-ph/0503071

How to determine compact object masses



P = orbital period

K_c = semiamplitude of companion star

i = inclination of the orbit to the line of sight (90º for orbit seen edge on)

M_{BH} and M_c = masses of invisible object and companion star

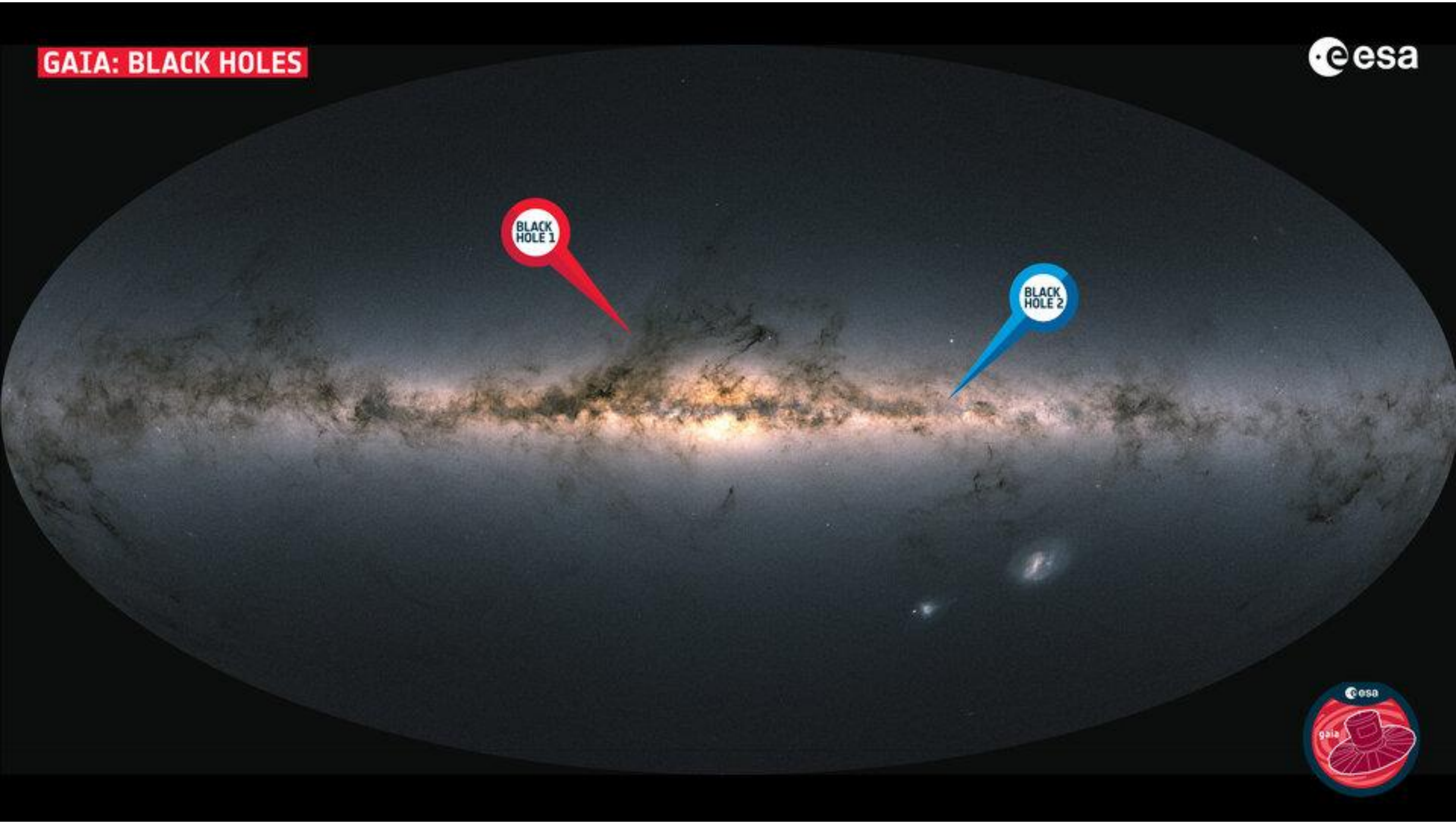
Keplers Laws give:

$$\frac{PK_c^3}{2\pi G} = \frac{M_{BH}^3 \sin^3 i}{\left(M_{BH} + M_c\right)^2}$$

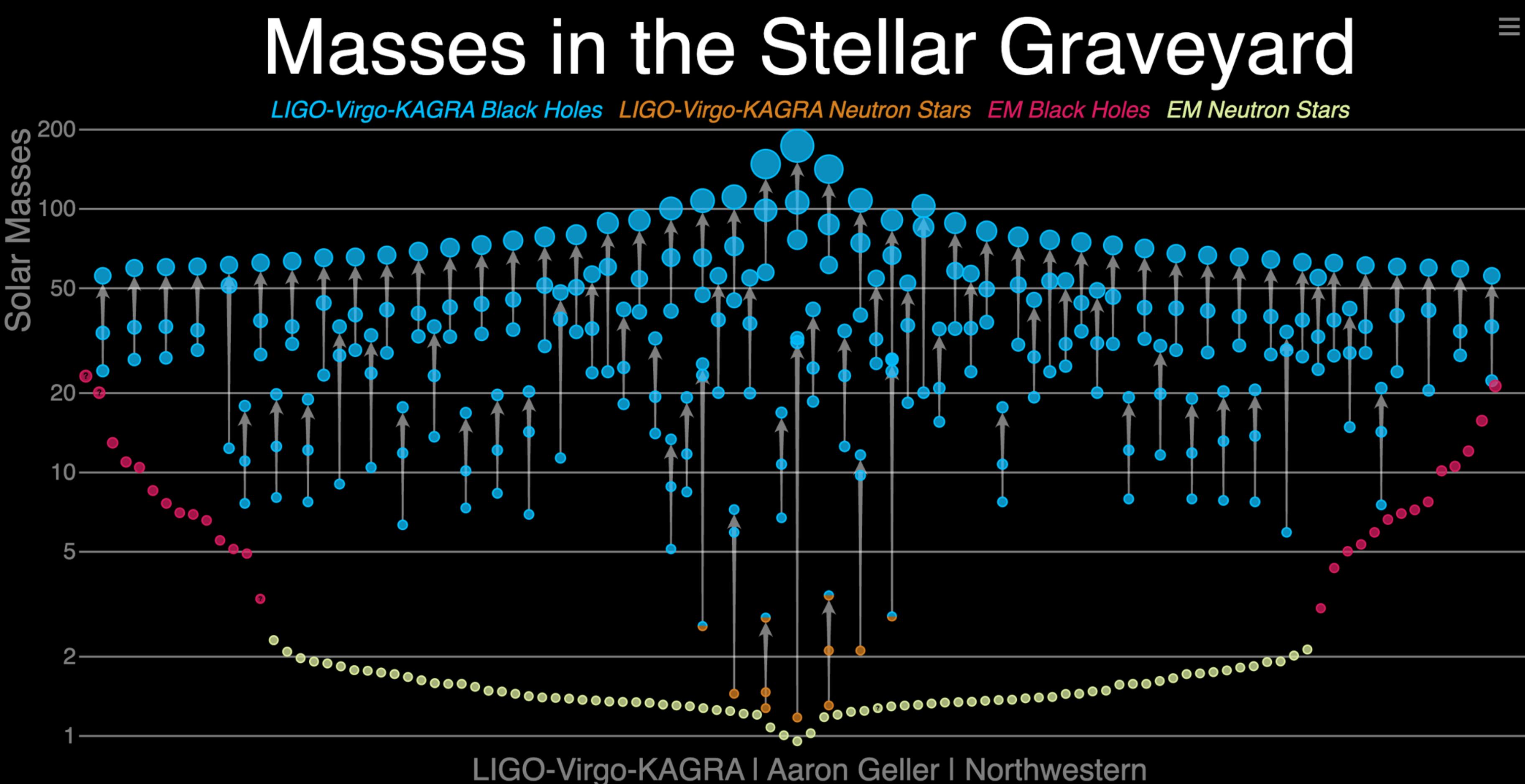
The LHS is measured from observations, and is called the ***mass function f(m).***

f(m) < M_{BH} always, since sin *i* <1 and M_c>0

Hence we have firm lower limit on BH mass from relatively simple measurements



https://www.esa.int/ESA_Multimedia/Videos/2023/03/Gaia_discovers_a_unique_black_hole



https://ligo.northwestern.edu/media/mass-plot/index.html

Summary

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- There is an upper limit to the mass of a white dwarf - we do not see WDs with masses > 1.4 $M_{\odot}$
- We will see in next lectures what the implications of this are for other phenomena in the Universe. It actually led to the discovery of dark energy!
- The collapse of massive stars produces two types of remnants - neutron stars and black holes.
- Their masses have been measured in X-ray emitting binary systems
- NS masses are clustered around 1.4 $M_{\odot}$
- The maximum limit for a stable neutron star is 3-5$M_{\odot}$
- Hard lower limits for masses of compact objects have been determined which have values much greater than this limit
- These are the best stellar mass black hole candidates - with masses of 5-15 $M_{\odot}$ they may be the collapsed remnants of very massive stars.