Dec 13

$$\mathbb{T}^d = \frac{\mathbb{R}^d}{2\pi \mathbb{Z}^d}$$

Let

$$f(x) = \sum_{|n| \le N} \hat{f}(n) \frac{e^{in\cdot x}}{(2\pi)^{\frac{d}{2}}}$$

$n \in (\mathbb{N} \cup \{0\})^d$
$|n| = n_1 + \cdots + n_d$
$n = (n_1, \dots, n_d)$

$\xi \in \mathbb{R}^d$

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2} = \sqrt{1 + \xi_1^2 + \cdots + \xi_d^2}$$

Japanese bracket.

$s \in \mathbb{R}$

$$|f|^2_{H^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2$$

$f \to |f|_{H^s(\mathbb{T}^d)}$ is a norm. $(P, |\cdot|_{H^s(\mathbb{R}^d)})$

$H^s(\mathbb{T}^d)$ is the completion

They are all Hilbert spaces

For $\boxed{s > \frac{d}{2}}$

$$f \in H^s(\mathbb{T}^d) \implies f \in C^0(\mathbb{T}^d) \subseteq L^\infty(\mathbb{T}^d)$$

$$H^s(\mathbb{T}^d) \overset{j}{\hookrightarrow} L^\infty(\mathbb{T}^d)$$

f a trigonometric polynomial and prove

$\exists\, C > 0$ s.t.

$$\|f\|_{L^\infty(\mathbb{T}^d)} \le C \|f\|_{H^s(\mathbb{T})} \qquad s > \frac{d}{2}$$

$\forall f$ trigonometric pol.

$$f(x) = \sum_{|n| \le N} \hat{f}(n) \frac{e^{in\cdot x}}{(2\pi)^{\frac{d}{2}}}$$

$$|f(x)| \le \sum_{|n| \le N} \frac{|\hat{f}(n)|}{(2\pi)^{\frac{d}{2}}} = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{|n| \le N} \langle n \rangle^{-s} \langle n \rangle^{s} |\hat{f}(n)|$$

$$\le \frac{1}{(2\pi)^{\frac{d}{2}}} \Big(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{-2s} \Big)^{\frac{1}{2}} \underbrace{\Big(\sum_{|n| \le N} \langle n \rangle^{2s} |\hat{f}(n)|^2 \Big)^{\frac{1}{2}}}_{|f|_{H^s(\mathbb{T}^d)}}$$

since $2s > d$

$$C_s = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{-2s} < +\infty$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^s} d\xi < \infty$$

$$|f(x)| \le K |f|_{H^s(\mathbb{T}^d)}$$

$\forall$ trigonometric polynomial f

$\Downarrow$

$\forall f \in H^s(\mathbb{T}^d)$

$$\implies |f|_{L^\infty} \le K |f|_{H^s}$$

$H^s(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, $s > \frac{d}{2}$

If instead $0 < s < \frac{d}{2}$

then $H^s(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d)$ Sobolev embedding

$$\frac{1}{p} = \frac{1}{2} - \frac{s}{d} \qquad 2 < p < \infty$$

$$f : \mathbb{T}^d \to \mathbb{R} \qquad \mathbb{T} \quad 1 \le m < d$$

$$L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^{d-1})$$

$$f(x, y) \longrightarrow f(x, 0)$$

$f|_{\mathbb{T}^{d-1}}$ $\qquad \mathbb{T}^{d-1}$ $\qquad \mathbb{T}^d = \mathbb{T}^{d-1} \times \mathbb{T}$

$\mathbb{T}^{d-1} \times \{0\}$

$$f \to f|_{\mathbb{T}^{d-1}}$$

$$H^s(\mathbb{T}^d) \to H^\sigma(\mathbb{T}^{d-1}) \quad \text{if} \quad s - \sigma > \frac{1}{2}$$

$$L^2(\mathbb{T}^d) \to H^{-1}(\mathbb{T}^{d-1} \times \{0\})$$

$T \in \mathcal{L}(E,F)$ $\qquad K(E,F)$

$T : E \longrightarrow F$ is compact

if it sends bounded sets into

precompact subsets of F

$\overline{T D_E(0,1)}$ is compact in F.

$E \xrightarrow{T} F \xrightarrow{S} Z$

$T \in \mathcal{L}(E,F)$

$S \in \mathcal{L}(F,Z)$ $\quad$ if T or S is compact

$\Longrightarrow$ $T \circ S$ is compact

$\kappa \in L^q(\mathbb{R}^d)$ $\kappa \neq 0$

$Tf = \kappa * f$

$\| Tf \|_{L^r(\mathbb{R}^d)} \leq \| \kappa \|_{L^q(\mathbb{R}^d)} \| f \|_{L^p(\mathbb{R}^d)}$

$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$L^p(\mathbb{R}^d) \longrightarrow L^r(\mathbb{R}^d)$

$1 < p < +\infty$

It is never a compact operator

Let $\forall \cancel{k} h \in \mathbb{R}^d$ $\tau_h f(x) = f(x-h)$

$\tau_h(\kappa * f) = \kappa * \tau_h f$

$\tau_h T f = T \tau_h f$ $\tau_h T = T \tau_h$

$h_n \xrightarrow{n\to\infty} \infty$ in $\mathbb{R}^d$

$\forall f \in L^p(\mathbb{R}^d)$ $f \neq 0$

$\| \tau_{h_n} f \|_{L^p} = \| f \|_{L^p} > 0$ $Tf \neq 0$

$\tau_{h_n} f \xrightarrow{n\to\infty} 0$ in $\sigma(L^p(\mathbb{R}^d), \underbrace{L^{p'}(\mathbb{R}^d)}_{(L^p(\mathbb{R}^d))'})$

If T is compact

$\{ T\tau_{h_n} f \}$ must have a convergent subsequence in $L^r(\mathbb{R}^d)$

Notice also $\tau_{h_n} f \longrightarrow 0 \implies T\tau_{h_n} f \longrightarrow 0$

$\| T \tau_{h_n} f \|_{L^r} = \| \tau_{h_n} T f \|_{L^r} = \| Tf \|_{L^r} > 0$

$T\tau_{h_{n_k}} f \longrightarrow g \in L^r(\mathbb{R}^d)$

$T\tau_{h_{n_k}} f \longrightarrow 0$ $\implies g = 0$

$\| T\tau_{h_{n_k}} f \|_{L^r} \longrightarrow \| g \|_{L^r} = 0$

Lemma $K(E,F)$ is closed in $\mathcal{L}(E,F)$ for the topology of the operator norm (uniform topology)

Pf Let $T \in \overline{K(E,F)}$ in $\mathcal{L}(E,F)$

$\Rightarrow \forall \varepsilon > 0 \ \exists \ S_\varepsilon \in K(E,F)$ s.t. $\|T - S_\varepsilon\| < \frac{\varepsilon}{3}$

$\overline{T D_E(0,1)}$

We will show that $\forall \varepsilon > 0 \ \exists \ f_1, \dots, f_n \in F$

s.t. $\overline{T D_E(0,1)} \subseteq \bigcup_{j=1}^{n} D_F(f_j, \varepsilon)$

$\Uparrow$

$\overline{S_\varepsilon D_E(0,1)} = \bigcup_{j=1}^{n} D_F(f_j, \frac{\varepsilon}{3})$

$y \in \overline{T D_E(0,1)}$. $\exists \ x \in D_E(0,1)$ s.t

$|y - Tx|_F < \frac{\varepsilon}{3}$

$\|Tx - S_\varepsilon x\|_F \le \underbrace{\|T - S_\varepsilon\|}_{< \frac{\varepsilon}{3}} \underbrace{\|x\|_E}_{\le 1} < \frac{\varepsilon}{3}$

$\exists \ j$ s.t $\|S_\varepsilon x - f_j\|_F < \frac{\varepsilon}{3}$

$\|y - f_j\|_F = \|(y - Tx) + (Tx - S_\varepsilon x) + (S_\varepsilon x - f_j)\|_F$

$\le \underbrace{\|y - Tx\|_F}_{< \frac{\varepsilon}{3}} + \underbrace{\|Tx - S_\varepsilon x\|_F}_{< \frac{\varepsilon}{3}} + \underbrace{\|S_\varepsilon x - f_j\|_F}_{< \frac{\varepsilon}{3}} < \varepsilon$

Lemma $T \in K(E,F) \iff T^* \in K(F', E')$

Theorem $T \in \mathcal{L}(E,H)$ with H a ~~Holbet~~ hilbert space

$T \in \mathcal{K}(E,H) \Leftrightarrow \exists \{T_n\}$ of operators
$\Leftarrow$ ✓ of finite rank s.t. $T_n \to T$
in $\mathcal{L}(E,F)$ in norm.

Example of compact operator
$T: E \to F$ continuous if $\dim R(T) < +\infty$
$\Rightarrow T$ compact

Pf

$\Rightarrow$

T compact $\overline{T D_E(0,1)}$ is compact in H

$\Rightarrow \forall \varepsilon > 0$ we have a finite cover

$\overline{T D_E(0,1)} \subseteq \bigcup_{j=1}^{n} D_H(f_j, \varepsilon)$

$G = \operatorname{Span}\{f_1, \dots, f_n\}$ is closed subspace of H

$P_G : H \to G$

$\| \underbrace{P_G \circ T}_{\text{finite rank}} - T \| < \varepsilon$

$x \in D_E(0,1) \quad Tx \in \overline{T D_E(0,1)} \Rightarrow \exists f_j$

st $\quad | Tx - \underbrace{f_j}_{\in G} |_H < \varepsilon$

$\| Tx - \underbrace{P_G \circ T x}_{\in G} \|_H \leq | Tx - f_j |_H < \varepsilon$

$\forall x \in D_E(0,1)$ we have $\| Tx - P_G \circ T x \|_H < \varepsilon$

$\Rightarrow \quad \| T - P_G \circ T \| \leq \varepsilon$

Exercise
$k \in L^q(\mathbb{T}^d) \qquad q < \infty$

$T_k f = k * f \qquad L^p(\mathbb{T}^d) \to L^r(\mathbb{T}^d)$

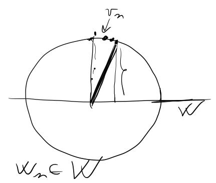
$\frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{p}$

Show that these are compact operators.

Lemma $W \subsetneq X$ W closed

Then $\exists \{v_n\}$ $\|v_n\| = 1$

s.t. $dist(v_n, W) \xrightarrow{n \to +\infty} 1$



Pf $v \in X \setminus W$ $\exists$ sequence $w_n \in W$

$\|v - w_n\| \xrightarrow{n \to +\infty} dist(v, W)$

$\phi(w) = \|v - w\|$ $\inf \{\phi(w) ; w \in W\} > 0$

$v_n = \dfrac{v - w_n}{\|v - w_n\|}$ $dist(v_n, W) \le dist(v_n, 0) = \|v_n\| = 1$

we prove $\lim_{n \to +\infty} dist(v_n, W) = 1$

I false $\liminf_{n \to +\infty} dist(v_n, W) = S < 1$

$\lim_{n \to \infty}$

$S < a < 1$

$n >> 1$

$dist(v_n, W) < a$ $\tilde{u}_n \in W$

$\|v_n - \tilde{u}_n\| < a$

$a < 1$

$\left\| \dfrac{v - w_n}{\|v - w_n\|} - \tilde{u}_n \right\| < a$

$\| v - w_n - \|v - w_n\| \tilde{u}_n \| < a \|v - w_n\| \xrightarrow{n \to +\infty} a\, dist(v, W)$

$< dist(v, W) - \varepsilon_0$

for some $\varepsilon_0 > 0$

$\Rightarrow$ $n >> 1$

$\| v - \underbrace{w_n - \|v - w_n\| \tilde{u}_n}_{\in W} \| < dist(v, W) - \varepsilon_0$

$\| v - w \| \ge dist(v, W)$ $\forall w \in W$.

Corollar V with $\overline{D_V(0, 1)}$ compact $\Leftrightarrow$ $\dim V < +\infty$

$\Rightarrow$

Pf suppose $\overline{D_V(0, 1)}$ compact but $\dim V = +\infty$

E_n an increasing sequence $\dim E_n = n$

$u_n \in E_n \setminus E_{n-1}$ $\|u_n\| = 1$

$dist(u_n, E_{n-1}) \ge \frac{1}{2}$

$\{u_n\}$ in $\overline{D_V(0, 1)}$

There exists a convergence subsequence $\forall \varepsilon > 0$

$\frac{1}{2} \le | u_{n_k} - u_{n_h} | < \varepsilon$ $\forall \varepsilon$ if $h, k >> 1$

$n_k > n_h$ $k \ne h$

$| u_{n_k} - u_{n_h} | \ge dist(u_{n_k}, E_{n_k - 1}) \ge 1$ $n_k \ne n_h$