17 dicembre

Teorema Data una equazione differenziale ordinaria a coefficienti costanti di ordine n

$$L[y] = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y =$$

$$= Q(x)\, e^{\alpha x} \cos(\beta x) \qquad \alpha, \beta \in \mathbb{R}$$

dove ~~Ω~~ $Q(x)$ è un polinomio con

$\deg Q(x) = N$

allora posto $p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$

1) se $p(\alpha + i\beta) \neq 0$ esiste una soluzione

$$y_p = A(x)\, e^{\alpha x} \sin(\beta x) + B(x)\, e^{\alpha x} \cos(\beta x)$$

dove $A(x)$ e $B(x)$ sono polinomi di grado $\leq N$

2) se $p(\alpha + i\beta) = 0$ e la molteplicità di $\alpha + i\beta$ come radice di $p(r)$ è di ordine $m \geq 1$.

esiste una soluzione

$$y_p = x^m A(x)\, e^{\alpha x} \sin(\beta x) + B(x)\, x^m e^{\alpha x} \cos(\beta x)$$

dove $A(x)$ e $B(x)$ sono polinomi di grado $\leq N$.

Es $\quad y''' + 2y'' = x\, e^{0x} \qquad \alpha = 0,\ \beta = 0$

$p(r) = r^3 + 2r^2 = r^2(r+2)$

$r = 0$ è una radice di molteplicità 2,

$$y_p = x^2 (ax + b) = ax^3 + bx^2$$

$$L[ax^3 + bx^2] =$$

$$= (ax^3 + bx^2)''' + 2(ax^3 + bx^2)'' =$$

$$= 6a + 12ax + 4b =$$

$$= 12ax + 6a + 4b = x$$

$$\begin{cases} 12a = 1 \\ 6a + 4b = 0 \end{cases} \qquad a = \frac{1}{12} \qquad \frac{1}{2} + 4b = 0 \qquad b = -\frac{1}{8}$$

$$y_p = x^2 \left(\frac{1}{12} x - \frac{1}{8} \right)$$

$$y''' + 2y'' = x + x^2 e^{-2x}$$

$$L[y_1] = x$$

$$L[y_2] = x^2 e^{-2x}$$

$$y_p = y_1 + y_2$$

$$L[y_p] = L[y_1 + y_2] = L[y_1] + L[y_2] = x + x^2 e^{-2x}$$

$$y''' + 2y'' = x^2 e^{-2x}$$

$$p(r) = r^3 + 2r^2 = r^2(r+2)$$

$p(-2) = 0$ la moltiplicità di $-2 = 1$

$$y_p = x\,(ax^2 + bx + c)\, e^{-2x}$$

$$= (ax^3 + bx^2 + cx)\, e^{-2x}$$

$$y'_p = (3ax^2 + 2bx + c)\, e^{-2x} - 2(ax^3 + bx^2 + cx)\, e^{-2x}$$

$$y''_p = (6ax + 2b)e^{-2x} - 4(3ax^2 + 2bx + c)e^{-2x} + 4(ax^3 + bx^2 + cx)\, e^{-2x}$$

$$y'''_p = 6a\, e^{-2x} - 2(6ax + 2b)e^{-2x} - 4(6ax + 2b)e^{-2x} + 8(3ax^2 + 2bx + c)e^{-2x} + 4(3ax + 2b + c)e^{-2x} - 8(ax^3 + bx^2 + cx)\, e^{-2x}$$

$$= \cancel{e^{-2x}}\,[6a - 12ax - 4b - 24ax - 8b + 24x^2 + 16bx + 8c + 12ax + 8b + 4c - 8ax^3 - 8bx^2 - 8cx] = x^2 \cancel{e^{-2x}}$$

$$L[y] = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$$

$$L[u e^{rx}] =$$

$$(u e^{rx})' = u' e^{rx} + u r e^{rx}$$

$$(u e^{rx})'' = u'' e^{rx} + 2 u' r e^{rx} + u r^2 e^{rx}$$

$$(u e^{rx})''' = u''' e^{rx} + 3 u'' r e^{rx} + 3 u' r^2 e^{rx} + r^3 e^{rx} u$$

$$(u e^{rx})^{(4)} = u^{(4)} e^{rx} + 4 u''' r e^{rx} + 6 u'' r^2 e^{rx}$$
$$+ 4 u' r^3 e^{rx} + u r^4 e^{rx}$$

$$(u e^{rx})^{(j)} = \sum_{k=0}^{j} \binom{j}{k} u^{(j-k)} r^k e^{rx}$$

$$L[u e^{rx}] =$$

$$= (u e^{rx})^{(n)} + a_{n-1} (u e^{rx})^{(n-1)} + \dots + a_1 (u e^{rx})' + a_0 u e^{rx}$$

$$= \Big[\sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} r^k + \sum_{k=0}^{n-1} a_{n-1} \binom{n-1}{k} u^{(n-1-k)} r^k +$$

$$+ \dots + a_1 u' + a_1 u r + a_0 u \Big] e^{rx}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{j}{k} u^{(j-k)} a_j r^k e^{rx}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{j}{k} u^{(k)} a_j r^{j-k} e^{rx}$$

Es 4 10 giugno 2024

$P_6(x)$ polinomio di Mc Laurin di

$$f(x) = \int_0^{x^2} \frac{\sin(t)}{1+t^5}\, dt$$

Dobbiamo trovare un polinomio $p(x)$ t.c. grado ≤ 6

t.c. $f(x) = p(x) + o(x^6)$ in 0.

Cerco il polinomio di McLaurin di ordine 4 $q_4(t)$ della funzione dentro l'integrale

$$g(t) = \sin(t) \frac{1}{1+t^5} = q_n(t) + o(t^n)$$

$$f(x) = \int_0^{x^2} q_n(t)\, dt + \int_0^{x^2} o(t^n)\, dt$$

$$= \int_0^{x^2} \sum_{j=0}^{n} \frac{g^{(j)}(0)}{j!} t^j\, dt + \underbrace{\int_0^{x^2} o(t^n)\, dt}_{o(x^{2n+2})}$$

$$= \sum_{j=0}^{n} \frac{g^{(j)}(0)}{j!} \int_0^{x^2} t^j\, dt + o(x^{2n+2})$$

$$= \sum_{j=0}^{n} \underbrace{\frac{g^{(j)}(0)}{j!} \frac{\cancel{x}^{2j+2}}{j+1}}_{(j+1)!} + o(x^{2n+2})$$

$\deg \le 2n+2 = 6$ $\boxed{n=2}$

$$G(x) = \int_0^x o(t^n)\, dt \qquad G(x) = o(x^{n+1})$$

$$G(x^2) = o((x^2)^{n+1}) = o(x^{2n+2})$$

$$\sin t \frac{1}{1+t^5} = q_2(t) + o(t^2)$$

$$\sin t = t - \frac{t^3}{6} + o(t^3) = t + o(t^2)$$

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + o(y^3)$$

$$\frac{1}{1+t^5} = 1 - t^5 + t^{10} - t^{15} + o(t^{15}) = 1 + o(t^2)$$

$$\sin(t) \frac{1}{1+t^5} = (t + o(t^2))(1 + o(t^2)) = \underset{q_2(t)}{\underset{\parallel}{t}} + o(t^2)$$

$$P_6(x) = \int_0^{x^2} q_2(t)\, dt = \int_0^{x^2} t\, dt =$$

$$= \frac{t^2}{2}\Big]_0^{x^2} = \frac{x^4}{2}$$

Si stabilisca se

$x \sin(x^3)$ e' integrabile in $[0, +\infty)$

$$\lim_{R\to+\infty} \int_0^R x \sin(x^3)\, dx \in \mathbb{R} \quad ?$$

$$\int_0^R x \sin(x^3)\, dx = \qquad y = x^3 \qquad x = y^{\frac{1}{3}}$$

$$dy = 3x^2\, dx$$

$$= \int_0^{R^3} \cancel{x} \sin(y) \frac{1}{3x^{\cancel{2}}}\, dy \qquad dx = \frac{1}{3x^2}$$

$$= \frac{1}{3}\int_0^{R^3} \frac{\sin y}{y^{\frac{1}{3}}}\, dy \xrightarrow{R\to+\infty} \in \mathbb{R}$$

poiché $\int_0^{+\infty} \frac{\sin y}{y^{\frac{1}{3}}}\, dy$ esiste

poiché tutte le funzioni $\frac{\sin x}{x^p}$ sono integrabili

$p > 0 \qquad [1, +\infty)$

$$f(x) = e^{-\frac{1}{x^2}} - 1 \overset{?}{\in} L(0,+\infty)$$

$$\lim_{x \to 0^+} \left[e^{-\frac{1}{x^2}} - 1 \right] = -1$$

ponendo $f(0) = -1$ otteniamo $f \in C^0([0,+\infty))$

$$f \in L_{loc}([0,+\infty))$$

$$\lim_{y \to 0} \frac{e^y - 1}{y} = 1 \qquad \lim_{x \to +\infty} \frac{e^{-\frac{1}{x^2}} - 1}{-\frac{1}{x^2}} = 1$$

$$-\frac{1}{x^2} \in L[1,+\infty) \Longrightarrow f \in L[1,+\infty)$$

$$\Longrightarrow f \in L[0,+\infty).$$

$$\int \sin^4 x \, dx = \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$= \int \left(\sin^2 x\right)^2 dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx =$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx \qquad \cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\int \frac{3}{2}\, dx - \frac{1}{2}\int \cos(2x) + \frac{1}{8}\int \cos(4x)$$

$$= \frac{3}{8} x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

$$\int_1^{+\infty} \underbrace{\frac{x}{x^3+1}}_{R(x)} dx \qquad \frac{x}{x^3+1} \in C^0([1,+\infty))$$

$$\lim_{x\to+\infty} \frac{\frac{x}{x^3+1}}{\frac{1}{x^2}} = 1 \Rightarrow \frac{x}{x^3+1} \in L[1,+\infty)$$

$$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$A = R(x)(x+1)\Big|_{x=-1} = \frac{x}{x^2-x+1}\Big|_{x=-1} = \frac{-1}{3}$$

$$A+B=0 \Rightarrow B = \frac{1}{3}$$

$$\frac{x^2}{x^3+1} = \frac{A}{x+1} + \frac{Bx^2+Cx}{x^2+x+1} \xrightarrow{x\to+\infty}$$

$$0 = A + B$$

$$\frac{x}{x^3+1} = \frac{-\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x + C^{=\frac{1}{3}}}{x^2-x+1} =$$

$$= \frac{-\frac{1}{3}(x^2-x+1) + (x+1)(\frac{1}{3}x+C)}{x^3+1}$$

$$\frac{x}{x^3+1} = \frac{\ldots\ldots - \frac{1}{3} + C}{x^3+1} \qquad C = \frac{1}{3}$$

$$\int_1^R \left[-\frac{\frac{1}{3}}{x+1} + \frac{1}{3}\frac{x+1}{x^2-x+1} \right] dx$$

$$= -\frac{1}{3}\int_1^R \frac{1}{x+1}dx + \frac{1}{3}\int_1^R \frac{x+1}{x^2-x+1} dx$$

$$= -\frac{1}{3}\left(\lg(R+1) - \lg 2\right) + \frac{1}{6}\int_1^R \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int_1^R \frac{1}{x^2-x+1}dx$$

$$= -\frac{1}{3}\lg(R+1) + \frac{\lg 2}{3} + \frac{1}{6}\lg(x^2-x+1)\Big]_1^R$$

$$+ \frac{1}{2}\int_1^R \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3}\lg\frac{\sqrt{R^2-R+1}}{R+1} \xrightarrow{R\to+\infty} 0 \quad + \frac{\lg 2}{3} - \frac{1}{6}\underbrace{\lg(1)}_{0}$$

$$+ \frac{1}{2}\int_1^R \frac{1}{x^2-x+1} dx$$