X INDIPENDENTE DA $\mathcal{B}$

$\sigma(X)$, $\{B\}$ SONO INDIPENDENTI

$$P(X \in H, B) = P(X \in H) P(B)$$

PER OGNI $H \in \mathcal{B}$

DISINTEGRABILITÀ

$X \geq 0$ o $X \in L^1$

$$X = X 1_\Omega = X 1_{\cup B_i} = \sum_i X 1_{B_i}$$

2 DADI $\quad Z = X_1 + X_2$

i	$E(X_1 \mid Z = i)$
2	1
3	1.5
i	$i/2$
12	6

$$\underbrace{E(Z \mid Z=i)} = i$$

$$E[X_1 + X_2 \mid Z=i]$$

$$= \underbrace{E[X_1 \mid Z=i]} + \underbrace{E[X_2 \mid Z=i]}$$

SONO UGUALI
PER SIMMETRIA

$$U = \min(X_1, X_2) \qquad V = \max(X_1, X_2)$$

$$E[U \mid V = i] \qquad i = 1 \text{ --- } 6$$

$$E(U \mid V = 1) = 1$$

$$E(U \mid V = i) = E\left(\sum_{j=1}^{6} j \, 1_{\{U = j\}} \mid V = i\right)$$

$$= \sum_{j=1}^{i} j \cdot P(U = j \mid V = i) = \ldots$$

$$E(\text{DANNO}) =$$

$$= \underbrace{E(\text{DANNO} \mid B)}_{500} \overbrace{P(B)}^{20\%} + \underbrace{E(\text{DANNO} \mid N)}_{750} \overbrace{P(N)}^{45\%} +$$

$$\underbrace{E(\text{DANNO} \mid M)}_{1500} \overbrace{P(M)}^{35\%}$$

LANCI DI UNA MONETA

$P(\text{TESTA}) = p \qquad q = 1 - p$

X_n = N. DI LANCI PER AVERE 2 TESTE CONSECUTIVE ≥ 2

$$E(X_n) = E(X_n \mid Y = 1)\, P(Y = 1) +$$

$$+ \quad \dots \quad E(X_n \mid Y = n)\, P(Y = n) +$$

$$+ E(X_n \mid Y > n)\, P(Y > n)$$

$$P(Y=k) = q\,p^{k-1}$$

$$P(Y>n) = 1-(q+qp+qp^2+\dots+qp^{n-1})$$

$$= 1-q(1+p+\dots+p^{n-1})$$

$$= 1-\cancel{q}\,\frac{1-p^n}{\cancel{1-p}}$$

$$= p^n$$

$$E(\underbrace{X_n}\,|\,Y > n) = n$$

$X_n = n$

$$E(X_n \,|\, \underbrace{Y = 1}_{\text{HO SPRECATO 1 LANCIO}}) = 1 + E(X_n)$$

$$E(X_n \,|\, Y = h) = h + E(X_n)$$

$1 \leq h \leq n$

$$E(X_n) = (1 + E(X_n))q +$$
$$+ (2 + E(X_n))qp +$$
$$\ldots$$
$$+ (h + E(X_n))qp^{h-1} +$$
$$\ldots$$
$$+ (n + E(X_n))qp^{n-1} +$$
$$+ n \cdot p^n$$

RISOLVERE PER $E(X_n)$

$$E[Z; A] = E[X; A] \quad \forall A \in \mathcal{G}$$

SEMPRE VERA SE $P(A) = 0$

SE $P(A) > 0$

$$\begin{cases} E[Z \mid A] = E[X \mid A] \quad \forall A \in \mathcal{G} \text{ CON } P(A) > 0 \\ \end{cases}$$

$\Rightarrow$ EQUIVALE ALLA (3)

(3) $E[Z 1_A] = E[X 1_A] \quad \forall A \in \mathcal{G}$

$\Longleftrightarrow$

$E[ZY] = E[XY] \quad \forall Y$ $\mathcal{G}$-MISURABILE TALE CHE LE SPERANZE SONO DEFINITE

(3) $E\left[E(X|\mathcal{G})\, 1_A\right] = E\left[X\, 1_A\right] \quad \forall A \in \mathcal{G}$

$A = \Omega \qquad 1_\Omega \equiv 1$

$E[X \mid \alpha Y + \beta Z] \neq \alpha E[X|Y] + \beta E[X|Z]$

LINEARITÀ

MOSTRO CHE

$$\alpha E(X|\mathcal{G}) + \beta E(Y|\mathcal{G}) = Z'$$

SODDISFA (1), (2), (3)

(1) Z' È $\mathcal{G}$-MISURABILE

(2) SÌ, L^1 È UNO SP. VETT.

(3) $E(Z'; A) = E(\alpha X + \beta Y; A) \ \forall A \in \mathcal{G}$

$$E(Z';A) = E(\alpha E(X|\mathcal{Y}) + \beta E(Y|\mathcal{Y});A)$$

LINEARITÀ DI $E(\cdot)$

$$= \alpha \underbrace{E(E(X|\mathcal{Y});A)}_{\substack{\| \\ E(X;A)}} + \beta \underbrace{E(E(Y|\mathcal{Y});A)}_{\substack{\| \\ E(Y;A)}}$$

(3) APPLICATA A $E(X|\mathcal{Y})$, $E(Y|\mathcal{Y})$

$$= E(\alpha X + \beta Y;A)$$

LINEARITÀ DI $E(\cdot)$

$E(XY|\mathcal{G}) = Y\,E(X|\mathcal{G})$ $\qquad$ Y È $\mathcal{G}$ MISURABILE

(MOSTRIAMOLA NEL CASO Y È LIMITATA)

MOSTRO CHE $\overbrace{Y \cdot E(X|\mathcal{G})}^{Z'}$ HA LE 3 PROPRIETÀ CHE DEFINISCONO $E(XY|\mathcal{G})$

1) Z' È $\mathcal{G}$ MISURABILE? SI $\quad Z' = \underbrace{Y}_{\mathcal{G}\text{-MIS}}\ \underbrace{E(X|\mathcal{G})}_{\mathcal{G}\text{-MIS}}$

2) $Z' \in L^1$? SI, $\quad Z' = \underbrace{Y}_{\text{LIMITATA}}\ \underbrace{E(X|\mathcal{G})}_{\in L^1} \in L^1$

3) $E(Z' W) = E(XY W)$ $\quad \forall W$ V.A. $\mathcal{G}$-MISURABILE

$$E(Z' W) = E(Y E(X|\mathcal{G}) W)$$

$$= E(E(X|\mathcal{G}) \cdot \underbrace{(YW)}_{\mathcal{G}\text{-MISURABILE}})$$

(3) PER $E(X|\mathcal{G})$

$$= E(X \cdot (YW))$$

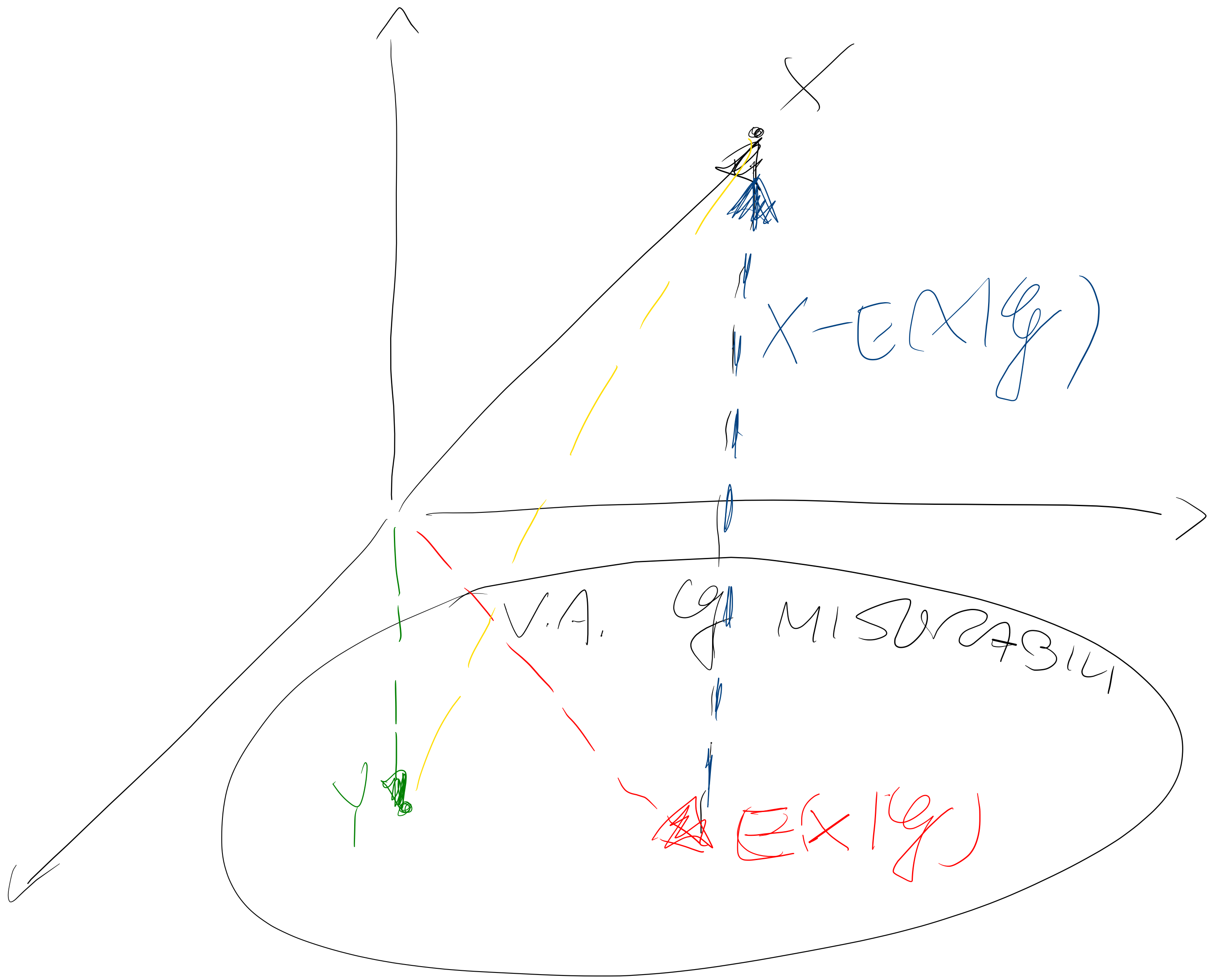
$$= E(XY \cdot W)$$

(3) $E(E(X|\mathcal{G})\,Y) = E(XY) \quad \forall\, Y\ \mathcal{G}$ MISURABILE

CIOÈ

$E((X - E(X|\mathcal{G}))\,Y) = 0 \quad \forall\, Y\ \mathcal{G}$-MIS.

$\langle X - E(X|\mathcal{G}), Y \rangle = 0 \quad \forall\, Y\ \mathcal{G}$ MIS.



$X - E(X|\mathcal{G}) \perp$

$\perp Y$

$\mathcal{G} = \sigma(Z)$

Y è $\mathcal{G}$-MISURABILE SSE $Y = h(Z)$ PER QUALCHE FUNZIONE h

$E(X|Z)$ RISOLVE IL PROBLEMA

$$\min_h d_2(X, h(Z))$$

$$\underbrace{E\left(\left(X - E(X|\mathcal{G})\right)^2\right)}_{d_2(X,\, E(X|\mathcal{G}))^2} = E\left(\left(X - Y + Y - E(X|\mathcal{G})\right)^2\right) =$$

con Y $\mathcal{G}$ mis., in L^2

$$= E\left((X-Y)^2\right) + E\left(\left(Y - E(X|\mathcal{G})\right)^2\right) +$$

$$+ 2\, E\left((X-Y)\left(Y - E(X|\mathcal{G})\right)\right)$$

$E(X|\mathcal{G})1_A = E(X|A)1_A$

A ATOMO DI $\mathcal{G}$, $P(A)>0$

(3) $E(E(X|\mathcal{G});A) = E(X;A) \quad \forall A \in \mathcal{G}$

SE A È UN ATOMO DI $\mathcal{G}$, $E(X|\mathcal{G})$ È COSTANTE SU A

$= E(X|\mathcal{G}) \cdot P(A) \Rightarrow E(X|\mathcal{G}) = E(X|A)$ SU A

Y È DISCRETA $y_1 \ldots y_n \ldots$

$\{Y = y_1\} \ldots \{Y = y_n\} \ldots$ PARTIZIONE

$$\sigma(Y) = \sigma(\{\text{PARTIZIONE}\})$$

$E(X|Y) = E(X|Y = y_i)$ SULL'ATOMO $\{Y = y_i\}$

$$\int_{\mathbb{R}^m} f_{\underline{x}' | \underline{x}''}(\underline{x}' | \underline{x}'')\, d\underline{x}' =$$

$$= \frac{\left\{ \int_{\mathbb{R}^m} f_{\underline{x}', \underline{x}''}(\underline{x}', \underline{x}'')\, dx' \right\} \cancel{f_{\underline{x}''}(\underline{x}'')}}{\cancel{f_{\underline{x}''}(\underline{x}'')}} = 1$$

$$= \frac{f_{\underline{x}'; \underline{x}''}(\underline{x}', \underline{x}'')}{f_{\underline{x}''}(\underline{x}'')}$$

$$f_{\underline{x}', \underline{x}''} = f_{\underline{x}' | \underline{x}''} \cdot f_{\underline{x}''}$$

1) $E[X \mid \underbrace{N, V}_{\sigma(N,V)}] = h(N, V) = N \cdot V$

$(X \mid N, V) \sim$ BINOMIALE (N, V)

2) $E[X \mid N] = E\big[\, E[X \mid N, V] \mid N \,\big] =$

ITERATIVA

$= E[NV \mid N] = N\, E[V \mid \cancel{N}]$

[MISURABILITÀ = COSTANZA

$$= N \cdot E[V] = \frac{N}{2}$$

3, 4

5) $VAR[X \mid N, V] = N V (1 - V)$

6) $VAR[X \mid N] = E[X^2 \mid N] - E[X \mid N]^2$

$$E[X^2 \mid N] = E[\, \underbrace{E[X^2 \mid N, V]} \mid N \,]$$

$$= E\left[VAR(X \mid N, V) + E(X \mid N, V)^2 \mid N \right] =$$

$$= E\left[NV(1-V) + N^2 V^2 \mid N \right]$$

$$= N \; E\left[V(1-V) \cancel{\mid N} \right] + N^2 \; E\left[V^2 \cancel{\mid N} \right]$$