$$E(X|\mathcal{G}) \qquad \min_{\substack{Y \ \mathcal{G}\text{-MIS} \\ Y \in L^2}} d_2(X,Y)$$

$$d_2(X,Y)^2 = E\left[(X-Y)^2\right] = E\left[\left(\underline{X-E(X|\mathcal{G})} + \underline{E(X|\mathcal{G})-Y}\right)^2\right]$$

$$= \underbrace{E\left[\left(X-E(X|\mathcal{G})\right)^2\right]}_{d_2(X,\,E(X|\mathcal{G}))^2} + \underbrace{E\left[\left(E(X|\mathcal{G})-Y\right)^2\right]}_{d_2(E(X|\mathcal{G}),\,Y)^2}$$

$$+ 2\underbrace{E\left[\left(X-E(X|\mathcal{G})\right)\left(\underbrace{E(X|\mathcal{G})-Y}_{\mathcal{G}\text{-MISURABILE}}\right)\right]}_{\langle X-E(X|\mathcal{G}),\ E(X|\mathcal{G})-Y\rangle = 0}$$

$$d_2(X,Y)^2 = d_2(X, E(X|\mathcal{G}))^2 + \underbrace{d_2(E(X|\mathcal{G}), Y)^2}_{\geq 0}$$

$$d_2(X,Y) \geq d_2(X, E(X|\mathcal{G}))$$

PER OGNI Y $\mathcal{G}$ MISURABILE, IN L^2

$$\operatorname{cov}(X,Y) = \underbrace{E[\operatorname{cov}(X,Y\mid\mathcal{G})]}_{*} + \operatorname{cov}(E(X\mid\mathcal{G}), E(Y\mid\mathcal{G}))$$

$$\operatorname{cov}(X,Y\mid\mathcal{G}) = E(XY\mid\mathcal{G}) - E(X\mid\mathcal{G})E(Y\mid\mathcal{G})$$

$$* = E[\,E(XY\mid\mathcal{G}) - E(X\mid\mathcal{G})\,E(Y\mid\mathcal{G})\,]$$

$$= E(XY) - E[\,E(X\mid\mathcal{G})\,E(Y\mid\mathcal{G})\,]$$

$$- E(X)E(Y) + E(X)E(Y)$$

$$= \operatorname{cov}(X,Y)$$

$$* = \operatorname{cov}(X,Y) - \Big\{ \underbrace{E\big[E(X|\mathcal{G})\,E(Y|\mathcal{G})\big] - \underbrace{E(X)}_{=E(E(X|\mathcal{G}))}\,\underbrace{E(Y)}_{=E(E(Y|\mathcal{G}))}}_{\operatorname{cov}(E(X|\mathcal{G}),\,E(Y|\mathcal{G}))} \Big\}$$

$$S = X_1 + \cdots + X_N$$

$$= \begin{cases} 0 & N = 0 \\ X_1 & N = 1 \\ X_1 + X_2 & N = 2 \\ \vdots & \\ X_1 + \cdots + X_n & N = n \\ \cdots & \end{cases}$$

$$E(S) = E(\underbrace{E(S|N)}_{N\mu}) = E(N\mu) = \mu E(N) = \mu\lambda$$

$$VAR(S) = E[\underbrace{VAR(S|N)}_{N\sigma^2}] + VAR[\underbrace{E(S|N)}_{N\mu}]$$

$$= \lambda\sigma^2 + \mu^2\lambda = \lambda(\mu^2+\sigma^2)$$

$$M_S(t) = E(e^{t \cdot S}) = E(E(\underbrace{e^{tS}}_{e^{tX_1} \cdots e^{tX_N}}|N))$$

(X_i) INDIP. DA N

$$= E(\underbrace{E(e^{tX_1}|\cancel{N})}_{M_X(t)} \cdots \underbrace{E(e^{tX_N}|\cancel{N})}_{M_X(t)}) = E(M_X(t)^N) =$$

$$= E\left(e^{\log(M_X(t)^N)}\right) =$$

$$= E\left(e^{N \cdot \log M_X(t)}\right) = M_N\left(\log M_X(t)\right)$$

POISSON $\quad M_N(t) = e^{\lambda(e^t - 1)}$

$$= e^{\lambda(M_X(t) - 1)}$$

$$E(N) = E(\underbrace{E(N \mid \Lambda)}_{\Lambda}) = E(\Lambda) = \frac{\alpha}{\lambda}$$

$$VAR(N) = E(\underbrace{VAR(N \mid \Lambda)}) + VAR(E(N \mid \Lambda))$$

$$= E(\Lambda) + VAR(\Lambda)$$

$$= \frac{\alpha}{\lambda} + \frac{\alpha}{\lambda^2}$$

$$P(N=n) = \int_0^{+\infty} \underbrace{P(N=n \mid \Lambda = x)\, f_\Lambda(x)\, dx}_{P(N=n,\ \Lambda = x)\, dx}$$

$$= \int_0^{+\infty} \frac{x^n}{n!} e^{-x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$\Lambda \sim GAMMA(\alpha, \lambda)$$

$$= \int_0^{+\infty} \frac{\lambda^\alpha}{n!\,\Gamma(\alpha)} \overbrace{x^{n+\alpha-1}}^{x^{\alpha'-1}} \overbrace{e^{-(\lambda+1)x}}^{e^{-\lambda' x}}\, dx$$

$$= \frac{\lambda^{\alpha}}{n!\,\Gamma(\alpha)} \frac{\Gamma(\alpha')}{\lambda'^{\,\alpha'}} \int_0^{+\infty} \frac{\lambda'^{\,\alpha'}}{\Gamma(\alpha')} x^{\alpha'-1} e^{-\lambda' x}\, dx = 1$$

PDF of GAMMA(α', λ')

$$\alpha' = n + \alpha > 0$$

$$\lambda' = \lambda + 1 > 0$$

$$= \frac{\lambda^{\alpha}}{n!\,\cancel{\Gamma(\alpha)}} \frac{\Gamma(\alpha+n)}{(\lambda+1)^{\alpha+n}}$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$(\alpha+n-1)\,\underbrace{\Gamma(\alpha+n-1)}$$

$$(\alpha+n-2)\,\Gamma(\alpha+n-2)$$

$$\ldots = (\alpha+n-1)(\alpha+n-2)\ldots\alpha\,\cancel{\Gamma(\alpha)}$$

$$P(N=n) = \frac{(\alpha+n-1)(\alpha+n-2)\cdots\alpha}{n!}\left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\left(\frac{1}{\lambda+1}\right)^{n}$$

$$= \binom{\alpha+n-1}{n} q^{\alpha} p^{n}$$

$$p = \frac{1}{\lambda+1}$$

$$q = \frac{\lambda}{\lambda+1}$$

BINOMIALE NEGATIVA

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

$x \in \mathbb{R}$

$n \in \mathbb{N}$

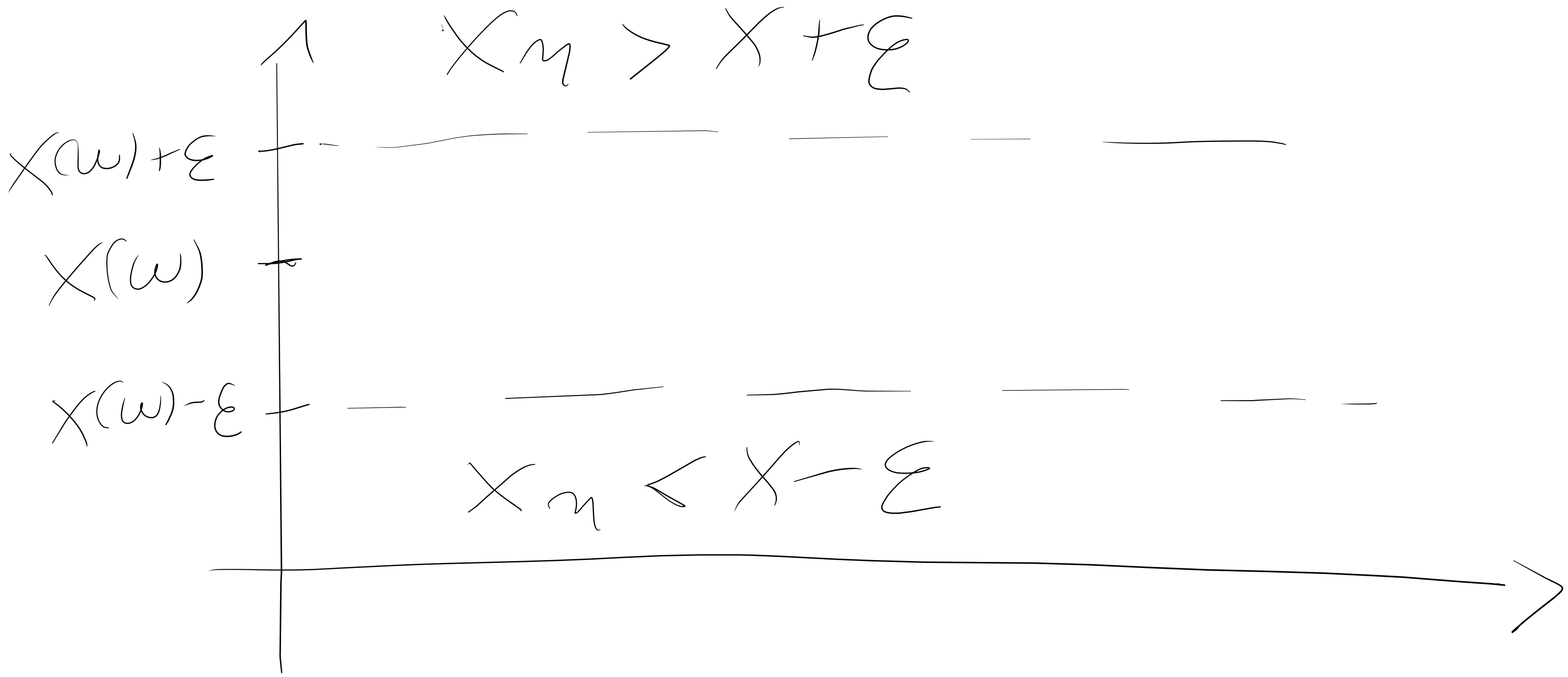
$\underline{X} = (\underset{\mathbb{R}^n}{\underline{X}'}, \underset{\mathbb{R}^m}{\underline{X}''})$ $\quad f_{\underline{X}} \quad f_{X''} \quad f_{\underline{X}'|\underline{X}''}$

$$f_{\cancel{\underline{X}}'}(\underline{x}') = \int_{\mathbb{R}^m} f_{\underline{X}}(\underline{x}', \underline{x}'')\, d\underline{x}''$$

"DISINTEGRABILITÀ NEL CONTINUO"

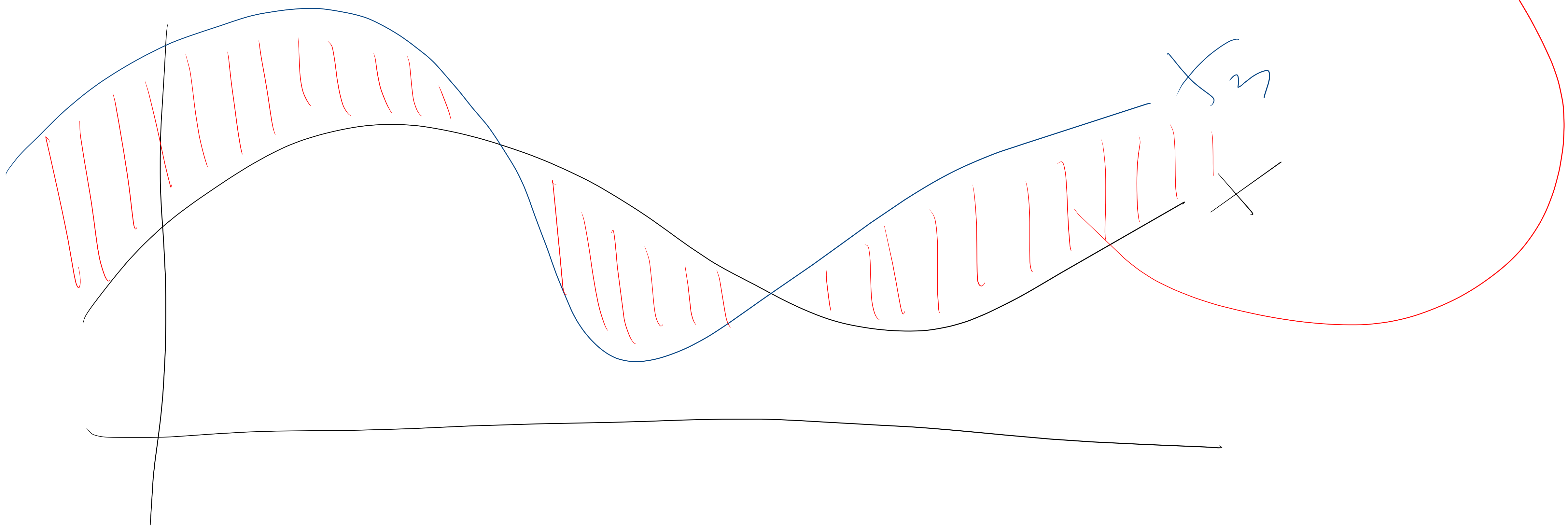
$$= \int_{\mathbb{R}^m} f_{\underline{X}'|\underline{X}''}(\underline{x}'|\underline{x}'')\, f_{\underline{X}''}(\underline{x}'')\, d\underline{x}''$$

$$(|X_n - X| > \varepsilon) = (X_n < X - \varepsilon \quad o \quad X_n > X + \varepsilon)$$



$p = 1$

$$d_1(X_n, X) = E[|X_n - X|]$$



$$X_n \xrightarrow{d} X$$

$$X_n \xrightarrow{d} Y$$

$$\implies X \sim Y$$

$$
\begin{tikzcd}
X_n \to X \text{ Q.C.} \arrow[dr, Rightarrow] & & X_n \xrightarrow{L^p} X \arrow[dl, Rightarrow] \\
& X_n \xrightarrow{P} X \arrow[d, Rightarrow] & \\
& X_n \xrightarrow{d} X &
\end{tikzcd}
$$

$$X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{P} X$$

$$P(\underbrace{|X_n - X| > \varepsilon}_{\substack{\parallel \\ |X_n - X|^p > \varepsilon^p}}) = P(|X_n - X|^p > \varepsilon^p) \leq$$

$$\underset{\text{MARKOV}}{\leq} \frac{E[|X_n - X|^p] \xrightarrow{n \to +\infty} 0}{\varepsilon^p} \longrightarrow 0$$

$$B = (-\infty, x]$$

$$\bar{F}_R(B) = \{x\}$$

$$P(X \in \bar{F}_R(B)) = 0 \iff P(X = x) = 0$$

$$P(X_n \in B) \longrightarrow P(X \in B)$$

PER OGNI $t \in \mathbb{R}$ IN UN INTORNO NON DEGENERE DI 0 ($(-\varepsilon, \varepsilon)$ CON $\varepsilon > 0$)

$$\lim_{n \to +\infty} E\left[e^{tX_n}\right] \to E\left[e^{tX}\right]$$

$X_n \sim$ BINOMIALE (n, p_n)

$$E(e^{tX_n}) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k}$$

$p \to p_n$

$q = q_n = 1 - p_n$

$$= \sum_{k=0}^{\infty} \binom{n}{k} (\underbrace{pe^t}_{a})^k \underbrace{q}_{b}^{\,n-k}$$

$$= (a+b)^n$$

$$= (pe^t + q)^n \qquad p = p_n$$

$$= (p_n e^t + 1 - p_n)^n \qquad t \in \mathbb{R}$$

$$\lim_{n \to +\infty} E\left(e^{t X_n}\right) = \lim_{n \to +\infty} \left(1 - p_n + p_n e^t\right)^n$$

$$n \cdot p_n \to \lambda$$

$$= \lim_{n \to +\infty} \left(1 + \frac{n p_n (e^t - 1)}{n} \right)^n \quad \to \lambda(e^t - 1)$$

RICORDIANO: SE $z_n \longrightarrow z$

$$\lim_{n \to +\infty} \left(1 + \frac{z_n}{n} \right)^n = e^z$$

$$= e^{\lambda(e^t - 1)}$$

QUESTA È LA MGF DI POISSON (λ)

$$\frac{S_n}{n} = \frac{X_1 + \text{———} + X_n}{n}$$

$$E\left(\frac{S_n}{n}\right) = \mu \qquad VAR\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$$

$$SD\left(\frac{S_n}{n}\right) = \frac{\sigma}{\sqrt{n}}$$

$$\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$$

MEDIA 0,
VARIANZA 1 PER OGNI n