18 Dicembre
Limiti 2

8)
$$\lim_{x\to 0} \frac{(4+x^2)^{\frac{1}{2}} - (8+3x^2)^{\frac{1}{3}}}{x\sin(2x)+\cos(2x+2x^2)-e^{3x^4}} \qquad \frac{0}{0}$$

Calcoliamo polinomi di McLaurin di numeratore e denominatore.

Obiettivo

$$\text{Num} = \alpha x^N + o(x^N)$$

$$\text{Denom} = \beta x^M + o(x^M)$$

$$(1+y)^\alpha = 1 + \alpha y + \binom{\alpha}{2} y^2 + \binom{\alpha}{3} y^3 + o(y^3)$$

$$(4+x^2)^{\frac{1}{2}} = \left(4\left(1+\frac{x^2}{4}\right)\right)^{\frac{1}{2}} = 2\left(1+\frac{x^2}{4}\right)^{\frac{1}{2}} =$$

$$= 2\left[1 + \frac{1}{2}\frac{x^2}{4} + \binom{\frac{1}{2}}{2}\frac{x^4}{2^4} + o(x^4)\right]$$

$$= \boxed{2} + \frac{x^2}{4} + \binom{\frac{1}{2}}{2}\frac{x^4}{8} + o(x^4)$$

$$(8+3x^2)^{\frac{1}{3}} = 2\left(1+\frac{3}{8}x^2\right)^{\frac{1}{3}} = 2\left[1 + \frac{1}{\cancel{3}}\frac{\cancel{3}}{8}x^2 + \binom{\frac{1}{3}}{2}\left(\frac{3}{8}\right)^2 x^4 + o(x^4)\right]$$

$$= \boxed{2} + \frac{1}{4}x^2 + 2\binom{\frac{1}{3}}{2}\left(\frac{3}{8}\right)^2 x^4 + o(x^4)$$

$$\text{Num} = (4+x^2)^{\frac{1}{2}} - (8+3x^2)^{\frac{1}{3}} =$$

$$= \left[\cancel{2} + \cancel{\frac{x^2}{4}} + \binom{\frac{1}{2}}{2}\frac{x^4}{8} + o(x^4)\right]$$

$$- \left[\cancel{2} + \cancel{\frac{x^2}{4}} + 2\binom{\frac{1}{3}}{2}\frac{9}{64}x^4 + o(x^4)\right]$$

$$= \left[\binom{\frac{1}{2}}{2}\frac{1}{8} - \binom{\frac{1}{3}}{2}\frac{9}{32}\right]x^4 + o(x^4)$$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} = -\frac{1}{8}$$

$$\binom{\frac{1}{2}}{2}\frac{1}{8} = -\frac{1}{64}$$

$$\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2} = \frac{-\frac{1}{3}\,\frac{\cancel{2}}{3}}{\cancel{2}} = -\frac{1}{9}$$

$$\binom{\frac{1}{3}}{2}\frac{9}{32} = -\frac{1}{\cancel{9}}\frac{\cancel{9}}{32} = -\frac{1}{32}$$

$$\binom{\frac{1}{2}}{2}\frac{1}{8} - \binom{\frac{1}{3}}{2}\frac{9}{32} = -\frac{1}{64} + \frac{1}{32} = \frac{1}{64}$$

$$\text{Num} = \frac{1}{64}x^4(1+o(1))$$

$$\text{denom} = x\sin(2x) + \cos(2x+2x^2) - e^{3x^4} =$$

$$= x\sin(2x) + \cos(2x+2x^2) - 1 + 1 - e^{3x^4}$$

$$\lim_{x\to 0} \frac{\text{denom}}{\text{denum}} = \lim_{x\to 0}\left[\frac{x\sin(2x)+\cos(2x+2x^2)-1}{\frac{1}{64}x^4} + \frac{1-e^{3x^4}}{\frac{1}{64}x^4}\right]$$

$$\lim_{y\to 0}\frac{e^y-1}{y} = 1$$

$$\lim_{x\to 0}\frac{1-e^{3x^4}}{\frac{1}{64}x^4} = -\lim_{x\to 0}\frac{e^{3x^4}-1}{3x^4} \to 1 \qquad 3\cdot 64 = -3\cdot 64$$

$$64\lim_{x\to 0}\frac{x\sin(2x)+\cos(2x+2x^2)-1}{x^4} \qquad \frac{0}{0}$$

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{4!} + o(y^4)$$

$$x\sin(2x) = x\left(2x - \frac{(2x)^3}{6} + o(x^3)\right) = 2x^2 - \frac{4}{3}x^4 + o(x^4)$$

$$\cos(2x+2x^2) - 1 = -\frac{(2x+2x^2)^2}{2} + \frac{(2x+2x^2)^4}{4!} + \underbrace{o((2x+2x^2)^4)}_{o(x^4)}$$

$$= -2(x^2+2x^3+x^4) + \frac{2^4}{4!}x^4(1+x)^4 + o(x^4)$$

$$= -2x^2 - 4x^3 - 2x^4 + \frac{2}{6}\,x^4 + o(x^4) \qquad \frac{2}{3}$$

$$= -2x^2 - 4x^3 + \left(\frac{2}{3}-2\right)x^4 + o(x^4)$$

$$= -2x^2 - 4x^3 - \frac{4}{3}x^4 + o(x^4) =$$

$$64\lim_{x\to 0}\frac{x\sin(2x)+\cos(2x+2x^2)-1}{x^4}$$

$$= 64\lim_{x\to 0}\frac{\cancel{2x^2} - \frac{4}{3}x^4 + o(x^4) - \cancel{2x^2} - 4x^3 - \frac{4}{3}x^4 + o(x^4)}{x^4}$$

$$= 64\lim_{x\to 0}\frac{-4x^3 + o(x^3)}{x^4} =$$

$$= 64\lim_{x\to 0}\frac{-4x^3}{x^4}(\cancel{1+o(1)})$$

$$\frac{x\sin(2x)+\cos(2x+2x^2)-1}{x^4} = -\frac{4}{x}(1+o(1))$$

$$\frac{\frac{x^4}{64}(1+o(1))}{x\sin(2x)+\cos(2x+2x^2)-1+1-e^{3x^4}} =$$

$$\frac{\frac{1}{64}(1+o(1))}{\underbrace{\frac{x\sin(2x)+\cos(2x+2x^2)-1}{x^4}}_{= -\frac{4}{x}(1+o(1))} + \frac{1-e^{3x^4}}{x^4}} \xrightarrow{x\to 0} 0$$

$$\frac{1-e^{3x^4}}{x^4} \to -3$$

$$\lim_{x \to 0} \frac{\sqrt{4+x^2} - \sqrt[3]{8+3x^2}}{x \sin(2x) + \cos(2x+2x^2) - e^{3x^4}}$$

$$\text{numer} = \frac{x^4}{64}\left(1+o(1)\right)$$

$$\text{denom} = x \sin(2x) + \cos(2x+2x^2) - e^{3x^4}$$

$$x \sin(2x) = x\left[2x - \frac{(2x)^3}{6} + o(x^3)\right]$$

$$= 2x^2 - \frac{4}{3}x^4 + o(x^4)$$

$$\cos(2x+2x^2) = 1 - \frac{(2x+2x^2)^2}{2} + \frac{(2x+2x^2)^4}{4!} + o(x^4)$$

$$= 1 - 2\left(x^2+2x^3+x^4\right) + \frac{2^4 x^4}{6\cdot \cancel{4}} + o(x^4)$$

$$= 1 - 2x^2 - 4x^3 - 2x^4 + \frac{2}{3}x^4 + o(x^4)$$

$$= 1 - 2x^2 - 4x^3 - \frac{4}{3}x^4 + o(x^4)$$

$$e^{3x^4} = 1 + 3x^4 + o(x^4)$$

$$\text{denom} = \left[\cancel{2x^2} - \frac{4}{3}x^4\right] + \left[\cancel{1} \cancel{-2x^2} - 4x^3 - \frac{4}{3}x^4\right] - \left[\cancel{1} + 3x^4\right] + o(x^4)$$

$$= -4x^3 + o(x^3) = -4x^3\left(1+o(1)\right)$$

$$\lim_{x\to 0} \frac{\frac{x^4}{64}}{-4x^3} = \lim_{x \to 0} \frac{1}{-4\cdot 64}\, x = 0$$

$$\lim_{x\to+\infty} \frac{2(x+3)^{\frac{3}{2}} - 2x^{\frac{3}{2}} - 9(x+3)^{\frac{1}{2}}}{\sin\left((x+2)^{-\frac{1}{2}}\right)}$$

$$\lim_{y\to 0} \frac{\sin(y)}{y} = 1 \Rightarrow \lim_{x\to+\infty} \frac{\sin\left((x+2)^{-\frac{1}{2}}\right)}{(x+2)^{-\frac{1}{2}}} = 1$$

$$= \lim_{x\to+\infty} \frac{2(x+3)^{\frac{3}{2}} - 2x^{\frac{3}{2}} - 9(x+3)^{\frac{1}{2}}}{(x+2)^{-\frac{1}{2}}}$$

$$= \lim_{x\to+\infty} \frac{2(x+3)^{\frac{3}{2}} - 2x^{\frac{3}{2}} - 9(x+3)^{\frac{1}{2}}}{x^{-\frac{1}{2}}}$$

$$(x+2)^{-\frac{1}{2}} \left(x\left(1+\frac{2}{x}\right)\right)^{-\frac{1}{2}} = x^{-\frac{1}{2}}\left(1+\frac{2}{x}\right)^{-\frac{1}{2}}$$

$$= x^{-\frac{1}{2}}(1+o(1))$$

$$\text{Numeratore} = 2(x+3)^{\frac{3}{2}} - 2x^{\frac{3}{2}} - 9(x+3)^{\frac{1}{2}}$$

$$= 2\, x^{\frac{3}{2}}\left(1+\frac{3}{x}\right)^{\frac{3}{2}} - 2x^{\frac{3}{2}} - 9\, x^{\frac{1}{2}}\left(1+\frac{3}{x}\right)^{\frac{1}{2}}$$

$$(1+y)^{\alpha} = 1 + \alpha y + \binom{\alpha}{2} y^2 + o(y^2)$$

$$\left(1+\frac{3}{x}\right)^{\frac{3}{2}} = 1 + \frac{3}{2}\,\frac{3}{x} + \binom{\frac{3}{2}}{2}\frac{9}{x^2} + o(x^{-2})$$

$$\left(1+\frac{3}{x}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\,\frac{3}{x} + \binom{\frac{1}{2}}{2}\frac{9}{x^2} + o(x^{-2})$$

$$\text{num} = 2x^{\frac{3}{2}}\left[\cancel{1} + \frac{3}{2}\,\frac{3}{x} + \binom{\frac{3}{2}}{2}\frac{9}{x^2} + o\left(\frac{1}{x^2}\right)\right] - 2\cancel{x^{\frac{3}{2}}} - 9x^{\frac{1}{2}}\left[1 + \frac{1}{2}\,\frac{3}{x} + \binom{\frac{1}{2}}{2}\frac{9}{x^2} + o\left(\frac{1}{x^2}\right)\right]$$

$$= \cancel{9x^{\frac{1}{2}}} + \binom{\frac{3}{2}}{2} 18\, x^{-\frac{1}{2}} + o(x^{-\frac{1}{2}}) - \cancel{9x^{\frac{1}{2}}} - \frac{27}{2} x^{-\frac{1}{2}}$$
$$- 81\binom{\frac{1}{2}}{2} x^{-\frac{3}{2}} + o(x^{-\frac{3}{2}})$$

$$\binom{\frac{3}{2}}{2} = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2} = \frac{\frac{3}{2}\,\frac{1}{2}}{2} = \frac{3}{8}$$

$$\binom{\frac{3}{2}}{2} 18 = \frac{3}{8}\, 18 = \frac{3\cdot 9}{4} = \frac{27}{4}$$

$$\text{num} = \left(\frac{27}{4} - \frac{27}{2}\right)x^{-\frac{1}{2}} \quad \cancel{(1+o(1))}$$

$$\lim_{x\to 0} \frac{-\frac{27}{4}x^{-\frac{1}{2}}}{x^{-\frac{1}{2}}} = -\frac{27}{4}$$

Es 2 esame 10/06/2024

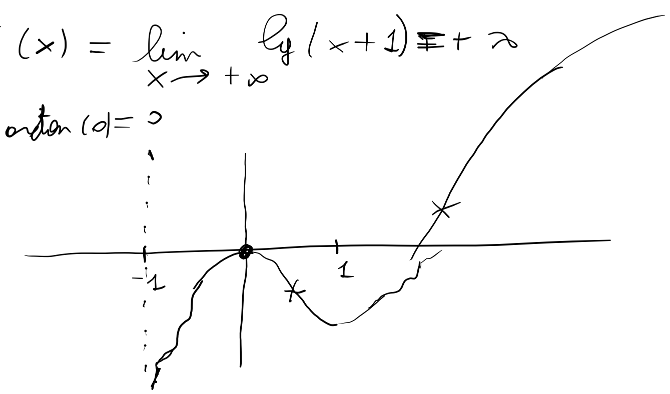
$$f(x) = \lg(x+1) - \arctan x$$

1) Dom $D(f) = (-1, +\infty)$ $\lg(x+1) = \lg\left(x\left(1+\frac{1}{x}\right)\right) = \lg x + \underbrace{\lg\left(1+\frac{1}{x}\right)}_{\downarrow 0}$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \lg(x+1) = -\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \lg(x+1) = +\infty$$

$f(0) = \lg 1 - \arctan(0) = 0$



2) $$f'(x) = \frac{1}{x+1} - \frac{1}{1+x^2} = \frac{\cancel{1}+x^2-(\cancel{1}+x)}{(x+1)(1+x^2)} =$$

$$= \frac{x^2-x}{(x+1)(1+x^2)} = \frac{x(x-1)}{(x+1)(1+x^2)} = 0 \quad \text{per } x=0,1$$

$$f'(x) > 0 \iff x(x-1) > 0 \iff \begin{matrix} x<0 \\ x>1 \end{matrix}$$

$$f'(x) = \frac{x^2-x}{x^3+x^2+x+1}$$

$$f''(x) = \frac{(2x-1)(x^3+x^2+x+1) - (x^2-x)(3x^2+2x+1)}{(x^3+x^2+x+1)^2} =$$

$$= \frac{2x^4 + \overset{1}{(2-1)}x^3 + \overset{1}{(2-1)}x^2 + \overset{1}{(2-1)}x - 1 - [3x^4 + \overset{-1}{(2-3)}x^3 - 2x^2 - x]}{(x^3+x^2+x+1)^2}$$

$$= \frac{-x^4+2x^3+3x^2+2x-1}{(x^3+x^2+x+1)^2}$$