Dec 19

Theorem (Fredholm alternative) X is B space

$K \in \mathcal{K}(X)$, $T = 1 - K$

1) $\dim \ker T < +\infty$

2) $R(T) = (\ker T^*)^\perp$

3) $\boxed{\ker T = 0 \iff R(T) = X}$

4) $\dim \ker T = \dim \ker T^*$

Pf only of 1) and 3)

1) $N = \ker T = \ker(1-K)$

$x \in N \iff x = Kx$

$\|x\| \le 1$ $\quad x \in \overline{D_N(0,1)} \Rightarrow x \in \overline{D_X(0,1)}$

$x = Kx \in K\,\overline{D_X(0,1)}$

$\overline{D_N(0,1)} \subseteq K\,\overline{D_X(0,1)}$ — precompact

$\Rightarrow \overline{D_N(0,1)}$ is compact $\Rightarrow \dim N < +\infty$.

3) $\ker T = 0 \iff R(T) = X$

$\Longrightarrow$

Let us assume $\ker T = 0$ but $\underbrace{R(T)}_{X_1} \subsetneq X$

X_1 closed

$T : X \longrightarrow X_1$ is an isomorphism

By induction we define $\{X_n\}$ a sequence of Banach spaces

$X_{n+1} = T X_n \subsetneq X_n \ \forall n$

$T : X_n \longrightarrow X_{n+1}$

If $X_n \subsetneq X_{n-1}$ then

$T X_n \subsetneq T X_{n-1}$

$x_{n-1} \in X_{n-1} \setminus X_n$

$T x_{n-1} = T x_n \quad x_n \in X_n$

$\{x_n\}$ $\quad x_n \in X_n$

$\operatorname{dist}(x_n, X_{n+1}) \ge \frac{1}{2}$ $\quad x_n \in X_n$

$\|x_n\| = 1$

$\{x_n\}$ $\quad \{K x_n\}$ must have a convergent subsequence

$T = 1 - K \qquad K = 1 - T$

$n > m$

$K x_m - K x_n = (1-T)x_m - (1-T)x_n =$

$= x_m + \underbrace{(-x_n + T x_n - T x_m)}_{-x_{n,m} \in X_{m+1}}$

$n > m \Rightarrow n \geq m+1$

$x_n \in X_n \subseteq X_{m+1}$

$T x_n \in X_{n+1} \subset X_n \subseteq X_{m+1}$

$x_m \in X_m \Rightarrow T x_m \in X_{m+1}$

$\|K x_m - K x_n\| = \|x_m - x_{n,m}\| \geq \operatorname{dist}(x_m, X_{m+1}) \geq \frac{1}{2}$

$\{K x_n\}$ cannot contain a subsequence which is a Cauchy sequence.

$\ker T = 0 \Rightarrow R(T) = X$

$\Longleftarrow$

Let $R(T) = X$ $\qquad R(T)^\perp = \ker T^* = 0$

$T^* : X^* \circlearrowleft$ $\qquad T^* = (1-K)^* = 1 - K^*$

$K^* \in \mathcal{K}(X')$

$\ker T^* = 0 \Longrightarrow R(T^*) = X'$

$\Rightarrow \ker T = R(T^*)^\perp = 0$

$T : X \longrightarrow X$

$\ker T = R(T^*)^\perp$ $\qquad \ker T^\perp$

$\ker T^* = R(T)^\perp$ $\qquad \ker T^{*\perp}$

$K \in K(X)$

$R(T) = \ker(T^*)^\perp$

We will consider a spectral theorem for $\underline{K}$ which implies that

$$X = X_0 \oplus \bigoplus_{\lambda \in \sigma(K)\setminus\{0\}} N_g(K-\lambda)$$

$$N_g(K-\lambda) = \bigcup_{n=1}^{\infty} \ker((K-\lambda)^n)$$

where $\sigma(K|_{X_0}) = \{0\}$

K in X_0 $\quad \sigma(K|_{X_0}) = \{0\}$

$T|_{X_0} = 1 - K|_{X_0} \quad \Leftarrow \quad 1 \notin \sigma(K|_{X_0})$

$\Rightarrow T|_{X_0} = 1 - K|_{X_0}$ is an isomorphism in X_0

T and K restrict in $\underbrace{N_g(K-\lambda)}$ is going to be expressed as a direct sum and in each term of the sum K is like a Jordan block.

$\operatorname{span}\{e_1, e_2, e_3\} = V_3$

$$K = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$T = 1 - K = \begin{pmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

If $\lambda \neq 1$ T is an isomorphism in V_3

$\lambda = 1$

$$T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$X' = V_3^\perp \oplus \operatorname{Span}\{e_1^*, e_2^*, e_3^*\}$ $\qquad T^*$

$\langle e_j, e_k^* \rangle_{X \times X'} = \delta_{jk}$

T in ~~Span~~ in V_3

$$\boxed{T e_1 = 0 \quad T e_2 = -e_1 \quad T e_3 = -e_2}$$

$j < 3$

$\delta_{jk} = \langle e_j, e_k^* \rangle = \langle -T e_{j+1}, e_k^* \rangle =$

$= -\langle e_{j+1}, \underbrace{T^* e_k^*}_{-e_{k+1}^*} \rangle$

$T^* e_k^* = -e_{k+1}^*$ if $k < 3$

$T^* e_3^* = 0 \qquad \langle e_j, T^* e_3^* \rangle = \langle T e_j, e_3^* \rangle = 0$

$$T^* \quad \begin{array}{ccc} e_1^* & e_2^* & e_3^* \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$R(T) = \operatorname{Span}\{e_1, e_2\}$ $\qquad \underline{\langle e_j, e_k^* \rangle_{X \times X'} = \delta_{jk}}$

$\ker(T^*) = \operatorname{Span}\{e_3^*\}$

$R(T) = \ker^\perp(T^*)$

Theorem Let X B-space $\dim X = +\infty$

Then

1) $0 \in \sigma(K)$

2) $\lambda \in \sigma(K)$ $\lambda \neq 0 \Rightarrow \lambda$ is an eigenvalue

3) Either $\sigma(K)$ has finitely many elements
~~or~~ it has a numberable number of elements
and can be put into a sequence $\{\lambda_n\}$ $\lambda_n \to 0$

4) For any $\lambda \neq 0$ $\lambda \in \sigma(K)$

$\dim \ker(K - \lambda) < +\infty$ finite geometric dimension

$\dim N_g(K-\lambda) = \dim \underline{\bigcup_{n=1}^{\infty} \ker(K-\lambda)^n} < +\infty$

Finite algebraic dimension.

Pf 1) $0 \in \sigma(K)$ because otherwise

$K K^{-1} = 1 \in K(X)$ impossible if $\dim X = +\infty$

2) $\lambda \in \sigma(K)$ $\lambda \neq 0$. Suppose λ is not an eigenvalue

$\lambda - K = \lambda \left(1 - \frac{1}{\lambda} K\right)$

λ not an eigenvalue $\Leftrightarrow$ $\ker\left(1 - \frac{1}{\lambda}K\right) = 0 \Leftrightarrow$

$(\lambda - K)x = \lambda\left(1 - \frac{1}{\lambda}K\right)x$ $\Leftrightarrow \left(1 - \frac{1}{\lambda}K\right)$ has bounded inverse.

so $(\lambda - K)^{-1} \in \mathcal{L}(X)$ contradiction because $\lambda \in \sigma(K)$

Conclusion $\lambda \in \sigma(K), \lambda \neq 0 \Rightarrow \lambda$ is an eigenvalue.

3) $\operatorname{card} \sigma(K) = +\infty$

$\sigma(K) \subseteq \overline{D_{\mathbb{C}}(0, \|K\|)}$

Let $\{\lambda_n\}$ be a sequence in $\sigma(K)$ of distinct elements

Suppose that $\lambda_n \to \lambda \neq 0$

$K x_n = \lambda_n x_n$ $\|x_n\| = 1$ $X_n = \operatorname{Span}\{x_1, \ldots, x_n\}$

$\dim X_n = n$ $\{X_n\}$ is growing

Let $y_n \in X_n$ $\|y_n\| = 1$ s.t.

$\operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$ $n > m$, $n > n-1 \geq m$

$$\frac{K y_n}{\lambda_n} - \frac{K y_m}{\lambda_m} = \frac{(K-\lambda_n) y_n + \lambda_n y_n}{\lambda_n} - \frac{\underbrace{(K-\lambda_m) y_m}_{\in X_{n-1}} + \lambda_m y_m}{\lambda_m}$$

$$= \underset{X_{n-1}}{y_n - y_m} + \frac{(K-\lambda_n) y_n}{\lambda_n} - \frac{(K-\lambda_m) y_m}{\lambda_m}$$

$y_n = \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \alpha_n x_n$

$(K - \lambda_n) = \alpha_1(\lambda_1 - \lambda_n)x_1 \cdots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)x_{n-1} + 0 \in X_{n-1}$

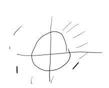
$= y_n - y_{nm}$, $y_{nm} \in X_{n-1}$

$$\left\| \frac{K y_n}{\lambda_n} - \frac{K y_m}{\lambda_m} \right\| = \|y_n - y_{nm}\| \geq \operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$$

so $\left\{ \frac{K y_n}{\lambda_n} \right\}$ has no convergent subsequences

$\lambda_n \to \lambda \neq 1$ $\frac{1}{\lambda_n} \to \frac{1}{\lambda}$ $\{K y_n\}$ needs to have a convergent subsequence

There is no accumulation point $\sigma(K)$ different from 0. $\Rightarrow$ $\operatorname{card} \sigma(K) = \operatorname{card} \mathbb{N}$



$D_{\mathbb{C}}(0, 1/n)$

$\mathbb{C}$ $\forall n$

$\sigma(K) \setminus D_{\mathbb{C}}(0, \frac{1}{n})$ is finite

$$\sigma(K) = \{0\} \cup \bigcup_{n \geq 1} \underbrace{\left(\sigma(K) \setminus D_{\mathbb{C}}(0, \tfrac{1}{n})\right)}_{\text{finite set}}$$

$s \in \mathbb{R}$

$H^s(\mathbb{T}^d) = \{f : \ldots$

$= \text{completion}\left(\{\text{Trigonometric polynomials}\}, \|\cdot\|_{H^s}\right)$

$$\|f\|_{H^s} = \sqrt{\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2}$$

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx$$

$s_1 > s_2 \Rightarrow H^{s_1} \hookrightarrow H^{s_2}$ is a compact embedding

Hilbert Schmidt operators

$T : L^2(X, d\mu) \circlearrowleft$

$$Tf(x) = \int_X K(x,y) f(y) \, d\mu(y)$$

where $K \in L^2(X \times X, d\mu)$

$\|T\|_{HS} = \|K\|_{L^2(X \times X)}$

$\|T\| \leq \|T\|_{HS}$

$$\int_X |Tf(x)|^2 d\mu(x) = \int_X \left| \int_X K(x,y) f(y) \, d\mu(y) \right|^2 d\mu(x)$$

$$\leq \int_X \|K(x,\cdot)\|^2_{L^2(X)} \|f\|^2_{L^2} \, d\mu(x)$$

$$= \|f\|^2_{L^2} \underbrace{\int_X \int_X |K(x,y)|^2 d\mu(y) \, d\mu(x)}_{\|K\|^2_{L^2(X \times X)}}$$

$\|Tf\|_{L^2} \leq \|f\|_{L^2} \|K\|_{L^2(X \times X)}$

$\|T\| \leq \|K\|_{L^2} = \|T\|_{HS}$

T H.S $\Rightarrow$ T is compact

$\exists$ a sequence $T_n \to T$ in $\mathcal{L}(X)$

in $L^2(X \times X) = L^2(X) \otimes L^2(X)$

$K_n(x,y) = \sum f$

$f_1(x) g_1(x) + \cdots + f_n(x) g_n(y)$

Rank $T_n < +\infty$