

26 settembre Definition of topological vector space. Balanced and absorbing sets. Proof that there is a basis of balanced neighborhoods. Continuous linear operators between topological vector spaces. Characterization in terms of hyperplanes of linear functionals. Metrizable topological vector spaces (only statement of theorem).

27 settembre Norms, Normed spaces and Banach spaces. Proof that $C^s(O)$ for O open subset of R^d that $0 < s < 1$ if appropriately defined is a Banach space. Partial proof that $W^{s,p}(O)$ with $1 \leq p < \infty$ is Banach space: proved the existence of a candidate to be the limit belonging to $W^{s,p}(O)$, but we did not have time to prove that it is the limit of the sequence in $W^{s,p}(O)$.

3 ottobre Completion of the proof on the $W^{s,p}(O)$ initiated last class. Definition of locally convex topological vector spaces X . Proof of the existence of a continuous seminorm, the Minkowski functional, for any open convex neighborhood of 0 in X . Sub-basis of seminorms. Space of Schwartz test functions.

4 ottobre The case of metrizable locally convex topological vector spaces. Bounded operators between normed spaces. Notions of strong convergence and of uniform convergence of sequences of operators.

10 ottobre Some example of operators converging strongly but not in norm. Spectrum $\sigma(T)$ and resolvent set $\rho(T)$ of a bounded operator T in a Banach space on C . Invertibility of $1+T$ if $\|T\| < 1$ (Neumann series). Proof that $z \in \rho(T)$ if $|z| > \|T\|$. Definition of resolvent $R_T(z)$ and proof that it depends analytically in z . Proof that the spectrum $\sigma(T)$ is non empty.

11 ottobre Analysis of the spectrum of a multiplier operator with m in $BC^0(O)$ with O open subset of R^d . Analytic form of Hahn Banach Theorem (with proof).

17 ottobre Some corollaries of the Hahn Banach Theorem. Adjoint of a bounded linear operator. Proof of three geometric forms of the Hahn-Banach Theorem.

18 ottobre Proof of a corollary of the 2nd geometric version of the Hahn Banach theorem, stating that if Y a closed proper subspace of X then there exists a nonzero functional in X' which is zero in Y . A concrete application: the Muntz-Szasz Theorem, with just a sketch of part of the proof. Bidual of a normed space and proof that the bidual map is an isometry. Statements of two lemmas about orthogonal and biorthogonal sets with in particular the orthogonality relationships between nuclei and ranges of operators.

24 ottobre Baire spaces and G_δ subspaces. Formulation and proof of the Banach and Steinhaus theorem for families of continuous operators between two Banach spaces. Definition of Fourier series, in 1 dimension and in generic d dimension. Riemann Lebesgue lemma. Dirichlet Kernel.

25 ottobre Proof that there exist continuous functions whose Fourier series are not convergent in 0 . The open map theorem. Proof of a corollary that a bijective continuous operator between two Banach spaces has bounded inverse. Application to the fact that $c_0(Z)$ has elements that are not the Fourier coefficients of functions in $L^1(T)$.

31 ottobre Closed graph theorem. Complementable spaces. Projections. Spectral projections.

November 7. More on direct sums and spectral projections. Definition of $\sigma(E, E')$ topology for E a Banach space. Proof that it is a Hausdorff. Closed convex sets in the strong topology are the same as closed convex sets in the $\sigma(E, E')$ topology. If $\dim E = \infty$ then any open subspace in the $\sigma(E, E')$ topology contains a line.

November 14 Proof that $c_0(\mathbb{N})$ is not a closed subspace of $l^\infty(\mathbb{N})$ for the $\sigma(l^\infty(\mathbb{N}), l^1(\mathbb{N}))$ topology. Proof of the theorem by Banach and Alaoglu. An example of bounded sequence in $(l^\infty(\mathbb{N}))'$ which does not have convergent subsequences in the $\sigma((l^\infty(\mathbb{N}))', l^\infty(\mathbb{N}))$ topology.

November 15 Proof of Goldstine's Theorem. Definition of reflexive Banach space. Proof of the Kakutani theorem. Reflexivity of closed subspace (only stated) and equivalence of reflexivity of E and E' (only stated). Compactness in $\sigma(E, E')$ of bounded convex closed subspaces of E if E is reflexive Existence of absolute minima for $(-\infty, +\infty]$ valued convex and lower continuous functionals ϕ such that $\lim_{x \rightarrow \infty} \phi(x) = +\infty$ defined in a closed convex subspace A of a reflexive space E .

November 21. Separable spaces. Proof that if E' is separable, then E is separable. Proof that $L^\infty(-1, 1)$ is not separable. Proof that the unit ball in E' is metrizable for the $\sigma(E', E)$ topology if and only if E is a separable Banach space. . Proof that the unit ball in E is metrizable for the $\sigma(E, E')$ topology if E' is a separable Banach space.

November 22. Proof that any bounded sequence in a reflexive Banach space E has a convergent subsequence in the $\sigma(E, E')$ topology. Definition of uniformly convex Banach space. Theorem of Milman and Pettis. Spaces L^p . Proof of the completeness of the L^p spaces.

November 28 Proof that for $2 \leq p < \infty$ the spaces L^p are uniformly convex and for $1 < p < \infty$ are reflexive. Proof of the Riesz representation theorem for $1 \leq p < \infty$.

November 29 The dual of $c_0(\mathbb{N})$ is $l^1(\mathbb{N})$. Young's convolution inequality. Mollifiers, regularization and approximations.

December 5 Proof of the density in $L^p(A)$ of $C^\infty_c(A)$ for A open in Euclidean space and $p < \infty$. Mollification. The Kolmogorov, Riesz, Frechet compactness theorem. Hilbert spaces. Cauchy Schwartz inequality. Proof of the Minkowski inequality for $\|f\|_H^2 := (f, f)_H$

December 6 An exercise from an old exam. Proof of the existence of a unique projection on a closed convex subspace of a Hilbert space H . Proof that the projection is a contraction and that it is a linear map if the convex space is a vector space. Orthonormal sets. Proof of the Bessel inequality.

December 12 Proof of Parseval identity for orthonormal families. Adjoint of a bounded operator in a Hilbert space.

December 13 Sobolev spaces $H^s(T^d)$ for any real s , proof of the immersion of $H^s(T^d)$ in $L^\infty(T^d)$ if $s > d/2$ and statement about the Sobolev immersion of $H^s(T^d)$ in $L^q(T^d)$ for $1/q = 1/2 - s/d$ when $0 < s < d/2$. Compact operators between Banach spaces. Proof that nonzero convolution operators from $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ if both p and r are in $(1, \infty)$ are always non compact. Proof that the space of compact operators is closed for the uniform topology. Proof that an operator with values in Hilbert space is compact if and only if it is the limit in the uniform topology of a sequence of finite rank operators. Characterization of finite dimensional topological Banach spaces in terms of the compactness of the unit ball centered at the origin.

December 19 The Fredholm alternative for operators of the form $1 - K$ with K a compact operator in a Banach space X . Proof that if $\dim X = \infty$ then 0 belongs to the $\sigma(K)$, that the spectrum is at most numberable, with accumulation point 0 and any nonzero element in $\sigma(K)$ is an eigenvalue.

December 20 Bounded and coercive bilinear functionals. The theorem of Lax and Milgram on the operator associated to a bounded and coercive bilinear functional. The theorem of Lax and Milgram on weak solutions of problems associated to bounded and coercive bilinear functional and to fixed functional. A concrete example in the torus.