7 gennaio   Buon anno!

$$f(x) = \int_0^{x^2} \frac{\sin t}{1+t^5}\, dt \qquad P_6^{(f)}(x)$$

$$g(x) = \int_0^{x} \frac{\sin(t)}{1+t^5}\, dt \qquad P_3^{(g)}(x)$$

$h(t)$

$$P_6^{(f)}(x) = P_3^{(g)}(x^2)$$

$$h(t) = \frac{\sin(t)}{1+t^5}$$

$$g(x) = \int_0^x h(t)\, dt \qquad h(t) = P_2^{(h)}(t) + o(t^2)$$

$$= \underbrace{\int_0^{t} P_2^{(h)}(t)\, dt}_{P_3^{(g)}(x)} + \underbrace{\int_0^x o(t^2)\, dt}_{o(x^3)}$$

$$h(t) = \sin t \quad \frac{1}{1+t^5} = \left(t - \frac{t^3}{6} + o(t^3)\right)\left(1 - t^5 + o(t^5)\right) =$$

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \dots$$

$$= \left(\cancel{1}t + o(t^2)\right)\left(1 + o(t^2)\right) = t + o(t^2)$$

$$g(x) = \int_0^x t\, dt + o(x^3) = \frac{x^2}{2} + o(x^3)$$

$$f(x) = g(x^2) = \frac{x^4}{2} + o(x^6)$$

$f(x)=\frac{1}{x\lg^p(x)} \overset{?}{\in} L[2,+\infty)$ per quali $p>0$

$f(x)\in L_{loc}[2,+\infty)$ poiché $f\in C^\infty([2,+\infty))$

Non si può confrontare con le funzioni $\frac{1}{x^a}$ perché per $x>>1$

$$\frac{1}{x} > \frac{1}{x\lg^p(x)} > \frac{1}{x^a} \quad \forall a>1 .$$

Proviamo ad applicare la definizione. $\frac{1}{x\lg^p(x)}\in L[2,+\infty)$ se e solo se esiste ed è finito

$$\lim_{R\to+\infty}\int_2^R \frac{1}{x\lg^p(x)}\,dx \qquad y=\lg x \qquad dy=\frac{1}{x}\,dx$$

$$=\lim_{R\to+\infty}\int_{\lg 2}^{\lg R}\frac{1}{y^p}\,dy$$

. Il limite è finito e coincide con $\int_{\lg 2}^{+\infty}\frac{1}{y^p}\,dy$ quando $p>1$.

Se invece $f(x)=\frac{1}{x^2\lg^p(x)}$ allora

per $x\geq e$ $\quad \frac{1}{x^2\lg^p(x)}\leq\frac{1}{x^2}\Rightarrow f\in L([e,+\infty))$

$\Rightarrow f\in L([2,+\infty))$

$$\frac{1}{x\lg x\ \lg^p(\lg x)} \qquad p>0$$

$x>>1$

$$q>1 \qquad \frac{1}{x\lg^q x} < \frac{1}{x\lg x\ \lg^p(\lg x)} < \frac{1}{x\lg x}$$

$$\int_{e^e}^R \frac{1}{x\lg x\ \lg^p(\lg x)}\,dx \qquad \lg\lg(e^e)=\lg(e\overset{1}{\lg e})=1$$

$y=\lg x$

$dy=\frac{1}{x}\,dx$

$$=\int_e^{\lg R}\frac{1}{y\lg^p y}\,dy \xrightarrow{R\to+\infty} L\in\mathbb{R} \iff p>1$$

Per induzione

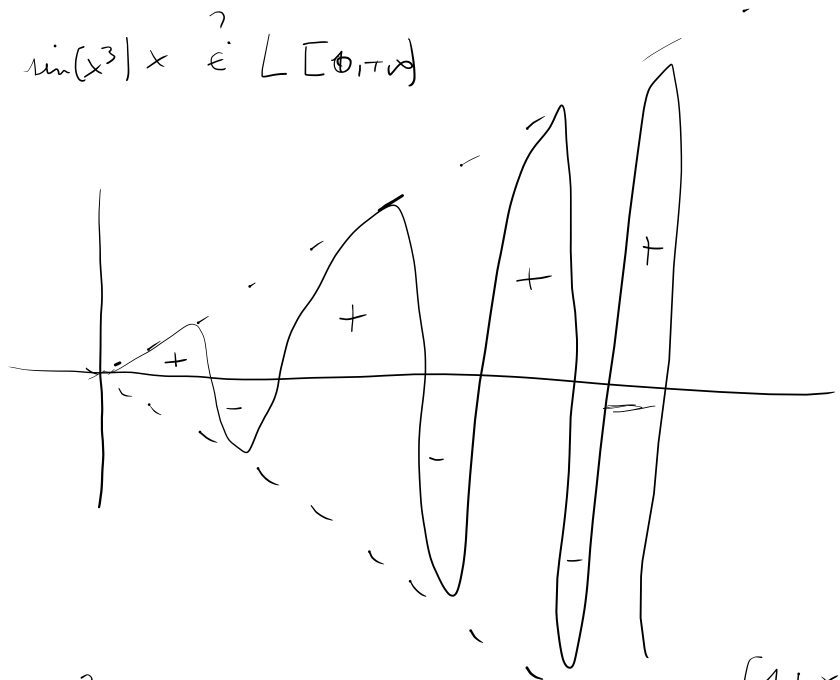
$$\frac{1}{x\lg x\ \lg(\lg x)\ \lg(\lg(\lg x))\cdot\lg^p(\lg(\lg(\lg x)))}$$

$$\in L[e^{e^{e^e}},+\infty) \iff p>1$$

$$\frac{1}{x\lg x\ \lg\circ\lg(x)\ \lg\circ\lg\circ\lg(x)\ \cdots\cdots\ \underbrace{\lg\circ\cdots\circ\lg}_{n-1}(x)\ \lg^p(\underbrace{\lg\circ\cdots\circ\lg}_{n-1}(x))}$$

$\in L[M,+\infty)$ per $M>>1$ $\iff p>1$

$\sin(x^3)\, x \overset{?}{\in} L[0,+\infty)$



$x^3 = n\pi \qquad x = \sqrt[3]{n\pi} \qquad (1+x)^{\frac{1}{3}}$

$(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}} = n^{\frac{1}{3}}\left(1+\frac{1}{n}\right)^{\frac{1}{3}} - n^{\frac{1}{3}} =$

$= n^{\frac{1}{3}}\left(1 + \frac{1}{3}\frac{1}{n} + o\left(\frac{1}{n}\right)\right) - n^{\frac{1}{3}}$

$= \frac{1}{3}\frac{1}{n^{2/3}} + o\left(n^{-\frac{2}{3}}\right)$

$\lim_{R\to+\infty} \int_0^R \sin(x^3)\, x\, dx$

$\int_0^R \sin(x^3)\, x\, dx = \qquad y = x^3 \qquad x = y^{\frac{1}{3}}$

$dy = 3x^2\, dx$

$= \int_0^{R^3} \sin(y) \frac{1}{y^{\frac{1}{3}}} dy \qquad x\,dx = \frac{1}{3}\frac{1}{x} dy = \frac{1}{3}\frac{1}{y^{\frac{1}{3}}} dy$

$\lim_{R\to+\infty} \int_0^R \sin(x^3)\, x\, dx = \lim_{R\to+\infty} \int_0^{R^3} \frac{\sin y}{y^{\frac{1}{3}}} dy$

$= \lim_{M\to+\infty} \int_0^M \frac{\sin(y)}{y^{\frac{1}{3}}} dy \in \mathbb{R}$

è integrabile.

$$\int_2^{+\infty} \frac{\sin(2x)}{\lg(1+x)}\, dx$$

$$\int_2^{R} \frac{\sin x}{x^{\frac{1}{2}}}\, dx = \int_2^{R} \frac{(-\cos x)'}{x^{\frac{1}{2}}}\, dx =$$

$$= -\frac{\cos x}{x^{\frac{1}{2}}}\Big]_2^R - \int_2^R (\cancel{-}\cos x)\left(\cancel{-}\frac{1}{2}\right) x^{-\frac{3}{2}}\, dx$$

$$= \frac{\cos 2}{2^{\frac{1}{2}}} - \frac{\cos R}{R^{\frac{1}{2}}} - \frac{1}{2}\int_2^R \frac{\cos x}{x^{3/2}}\, dx$$

$$\int_2^R \frac{\sin(2x)}{\lg(1+x)}\, dx = \frac{1}{2}\int_2^R \frac{(-\cos(2x))'}{\lg(1+x)}\, dx$$

$$= -\frac{1}{2}\frac{\cos(2x)}{\lg(1+x)}\Big]_2^R - \frac{1}{2}\int_2^R \frac{\cos(2x)}{\lg^2(1+x)}\,\frac{1}{1+x}\, dx$$

$$\xrightarrow{R\to+\infty} \in \mathbb{R}$$

$$\frac{1}{2}\,\frac{\cos(4)}{\lg(3)}$$

$$\frac{1}{\lg^2(1+x)\,(1+x)} \in L[2,+\infty)$$

$$\lim_{x\to0^+}\int_x^{3x} \underbrace{\lg\left(1+\frac{1}{\sqrt t}+\frac1t\right)}_{f(t)}dt = \int_0^0 f(t)\,dt$$

per $t\to0^+$ il termine dominante è $\frac1t$

$$\lg\left(1+\frac1{\sqrt t}+\frac1t\right) = \lg\left(\frac1t\left(1+\sqrt t+t\right)\right) =$$

$$= \lg\frac1t + \lg(1+\sqrt t+t)$$

$$\int_x^{3x} f(t)\,dt = -\int_x^{3x}\lg(t)\,dt + \int_x^{3x}\lg(1+\sqrt t+t)\,dt$$

$\lg(1+\sqrt t+t)\in C^0([0,+\infty))$

$$\int_x^{3x}\lg(1+\sqrt t+t)\,dt = \int_0^{3x}\lg(1+\sqrt t+t)\,dt - \underbrace{\int_0^x \lg(1+\sqrt t+t)\,dt}_{G(x)}$$

$= G(3x)-G(x)$ $\quad G(x)\in C^1([0,+\infty))$

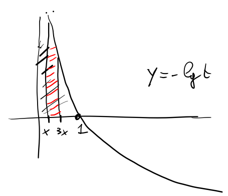
grazie al teorema fondamentale del calcolo.

$x\to0$

$G(0)-G(0)=0$

$$\lim_{x\to0^+} -\int_x^{3x}\lg(t)\,dt = 0$$

$\lg t\in L((0,1])$



$$\lim_{x\to0^+}\int_x^1 \lg t\,dt = \int_0^1 \lg t\,dt \in \mathbb{R}_-$$

$$\lim_{x\to0^+}\int_{x^3}^1 \lg t\,dt = \int_0^1\lg t\,dt\in\mathbb{R}_-$$

$$\lim_{x\to0^+}\left[\int_x^1 \lg t\,dt - \int_{x^3}^1 \lg t\,dt\right] = 0$$

$$= \lim_{x\to0^+}\left[\int_x^1 \lg t\,dt + \int_1^{x^3}\lg t\,dt\right] = \lim_{x\to0^+}\int_x^{x^3}\lg t\,dt$$

$= 0$

a, b, c $\quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

$|\lg t|\in L(0,1]$

$$\lim_{t\to0^+} \frac{\lg\frac1t}{\frac{1}{t^{1/2}}} = 0$$ e siccome $\frac{1}{t^{1/2}}\in L(0,1]$

$\Rightarrow \lg\frac1t\in L(0,1]$

$$\int_x^{3x}\lg t\,dt = \int_x^{3x}(t)'\lg t\,dt =$$

$$= t\lg t\Big]_x^{3x} - \int_x^{3x} t\,\frac1t\,dt$$

$$= 3x\lg(3x) - x\lg x - \int_x^{3x} dt$$

$$= 3x\lg3 + 3x\lg x - x\lg x - 2x$$

$$= \underbrace{3x\lg 3}_{\to 0,\ x\to0} + \underbrace{2x\lg x}_{\to 0,\ x\to0^+} \underbrace{- 2x}_{\to 0}$$

$$\lim_{x\to+\infty}\left(1+x-\sqrt{x^2+1}\right)^x = e^{-\frac{1}{2}}$$

$$\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e$$

$$1+x-\sqrt{x^2+1} = e^{x\lg(1+x-\sqrt{x^2+1})}$$

$$x\lg\left(1+x-\sqrt{x^2+1}\right) = x\lg\left(1+x-\sqrt{x^2\left(1+\frac{1}{x^2}\right)}\right)$$

$$= x\lg\left(1+x-x\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}\right)$$

$$= x\lg\left(1+\cancel{x}-x\left(\cancel{1}+\frac{1}{2}\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)\right) =$$

$$(1+y)^{\frac{1}{2}} = 1+\frac{1}{2}y+\binom{\frac{1}{2}}{2}y^2+o(y^2) \qquad y=\frac{1}{x^2}$$

$$= x\lg\left(1-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right)$$

$$\lim_{x\to+\infty}\frac{\lg\left(1-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right)}{-\frac{1}{2x}+o\left(\frac{1}{x}\right)}\times\left(-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right) =$$

$\to 0$

$\downarrow$

1

$$= \lim_{x\to+\infty}\left(-\frac{1}{2}+o(1)\right) = -\frac{1}{2}$$

$$\lim_{y\to0}\frac{\lg(1+y)}{y} = 1$$

$$\lim_{x\to+\infty}\left(1+x-\sqrt{x^2+1}\right)^x$$

$$\left(1+x-\sqrt{x^2+1}\right)^x = \left(\left(1+x-\sqrt{x^2+1}\right)\frac{1+x+\sqrt{x^2+1}}{1+x+\sqrt{x^2+1}}\right)^x$$

$$= \left(\frac{\cancel{1}+2x+\cancel{x^2}-(\cancel{x^2}+\cancel{1})}{1+x+\sqrt{x^2+1}}\right)^x = \left(\frac{2x}{1+x+\sqrt{x^2+1}}\right)^x$$

$$= \left(\frac{2}{1+\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}}\right)^x$$

$$= \left(\frac{2}{1+\frac{1}{x}+\left(1+\frac{1}{2}\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)}\right)^x$$

$$= \left(\frac{2}{2+\frac{1}{x}+o\left(\frac{1}{x}\right)}\right)^x = \left(\frac{1}{1+\frac{1}{2x}+o\left(\frac{1}{x}\right)}\right)^x$$

$$= \left(\left(\frac{1}{1+\frac{1}{2x}(1+o(1))}\right)^{\frac{2x}{1+o(1)}}\right)^{\frac{1+o(1)}{2}}$$

$$\lim_{x\to+\infty}\frac{1}{\left(1+\frac{1+o(1)}{2x}\right)^{\frac{2x}{1+o(1)}}} \qquad y=\frac{2x}{1+o(1)}$$

$$= \lim_{y\to+\infty}\frac{1}{\left(1+\frac{1}{y}\right)^y} = \frac{1}{e}$$

$$\lim_{x\to+\infty}\left(\left(\frac{1}{1+\frac{1}{2x}(1+o(1))}\right)^{\frac{2x}{1+o(1)}}\right)^{\frac{1+o(1)}{2}} = \left(\frac{1}{e}\right)^{\frac{1}{2}}$$