8 Gennaio

$$\int_1^{+\infty} 1 \quad dx \quad c \quad \mathbb{R}.$$

$$\frac{1}{3} \lg(x+1) \Big]_1^R - \frac{1}{6} \lg(x^2-x+1) \Big]_1^R$$

$x^3 +$

$$+ \frac{1}{2} \int_1^R \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} \, dx$$

$\frac{1}{x^3+1}$:

$$A = \frac{x+1}{x^2+1}\Big|_{x=-1} = \frac{1}{x^2-x+1}\Big|_{x=-1} = \frac{1}{3}$$

$A+B=0 \qquad B=-\frac{1}{3}$

$$\frac{1}{x^3+1} = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+C}{x^2-x+1} =$$

$$\frac{1}{x^3+1} = \frac{\frac{1}{3}(x^2-x+1)+(x+1)(-\frac{1}{3}x+C)}{x^3+1} =$$

$$= \frac{\dots + \frac{1}{3}+C}{x^3+1}$$

$$C+\frac{1}{3}=1 \implies C=\frac{2}{3}$$

$$\int_1^R \frac{1}{x^3+1} = \int_1^R \frac{\frac{1}{3}}{x+1} - \frac{1}{3}\int_1^R \frac{x-2}{x^2-x+1}$$

$\frac{2}{3} = \frac{1}{6} + \alpha$

$\alpha = \frac{2}{3} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{1}{2}$

$$= \frac{1}{3}\lg(x+1)\Big]_1^R - \frac{1}{6}\int_1^R \frac{2x-1}{x^2-x+1} + \frac{1}{2}\int_1^R \frac{1}{x^2-x+1}\,dx$$

$$= \frac{1}{3} \lg(x+1) \Big]_1^R - \frac{1}{6} \lg(x^2-x+1) \Big]_1^R$$

$$+ \frac{1}{2} \int_1^R \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} \, dx$$

$$\frac{1}{6}\int_1^R \frac{2x-1-1}{x^2-x+1}$$

$$x^2-x+1 = x^2-2\frac{1}{2}x+1 = x^2-2\frac{1}{2}x+\frac{1}{4}+\frac{3}{4}$$
$$= \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int_1^R \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx \qquad y = x-\frac{1}{2} \qquad dy = dx$$

$$= \int_{\frac{1}{2}}^{R-\frac{1}{2}} \frac{1}{y^2+\frac{3}{4}}\,dy \qquad y = \frac{\sqrt{3}}{2}u \qquad dy = \frac{\sqrt{3}}{2}\,du$$

$$u = \frac{2}{\sqrt{3}}y$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}(R-\frac{1}{2})} \frac{\frac{\sqrt{3}}{2}\,du}{\frac{3}{4}(u^2+1)} = \frac{2}{\sqrt{3}}\int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}(R-\frac{1}{2})} \frac{du}{u^2+1} =$$

$$= \frac{2}{\sqrt{3}} \arctan u \Big]_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}(R-\frac{1}{2})}$$

Nostro integrale $= \frac{1}{3}\lg(x+1)\Big]_1^R - \frac{1}{6}\lg(x^2-x+1)\Big]_1^R$

$$+ \frac{1}{\sqrt{3}} \arctan x \Big]_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}(R-\frac{1}{2})} =$$

$$= \frac{1}{3} \lg \underbrace{\frac{R+1}{\sqrt{R^2-R+1}}}_{\to 1} + \frac{1}{\sqrt{3}} \underbrace{\arctan\left(\frac{2}{\sqrt{3}}\left(R-\frac{1}{2}\right)\right)}_{\to \frac{\pi}{2}}$$

$$- \frac{1}{3}\lg 2 - \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{2} - \frac{1}{3}\lg 2 - \frac{1}{\sqrt{3}} \overbrace{\arctan \frac{1}{\sqrt{3}}}^{\frac{\pi}{6}} > 0$$

è esempio

$$= \frac{\pi}{2} - \frac{1}{3}\lg 2 - \frac{1}{\sqrt{3}}\frac{\pi}{6} > \frac{\pi}{2} - \frac{1}{3} - \frac{1}{\sqrt{3}}\frac{\pi}{6}$$

$$> \frac{\pi}{2} - \frac{1}{3} - \frac{\pi}{6} = \frac{\pi}{3} - \frac{1}{3} =$$

$$= \frac{1}{3}(\pi - 1) > 0$$

$$\arctan(x)\,\frac{\sin(x)}{1+x} \overset{?}{\in} L[0,+\infty)$$

Questo ci ricordo $\frac{\sin x}{x} \in L[0,+\infty)$

$$\frac{\sin x}{1+x} - \frac{\sin x}{x} = \sin x\left(\frac{1}{1+x} - \frac{1}{x}\right) =$$

$$= -\frac{\sin x}{(1+x)x} \quad \text{è ass integrabile}$$

perchè è in $C^0([0,+\infty))$, e

$$\frac{|\sin x|}{(1+x)x} < \frac{1}{x^2} \Rightarrow \frac{|\sin x|}{(1+x)x} \in L[1,\infty)$$

$$\Rightarrow \frac{-\sin x}{(1+x)x} \in L[1,+\infty) \Rightarrow \frac{-\sin x}{(1+x)x} \in L[0,+\infty)$$

$$\Rightarrow \frac{\sin x}{1+x} \in L[0,+\infty)$$

$$\arctan x\,\frac{\sin x}{1+x} =$$

$$= \frac{\pi}{2}\,\frac{\sin x}{1+x} + \underbrace{\left(\arctan x - \frac{\pi}{2}\right)}_{-\arctan\frac{1}{x}}\frac{\sin x}{1+x}$$

$$\arctan\left(\frac{1}{x}\right)\frac{|\sin x|}{1+x} \le \arctan\left(\frac{1}{x}\right)\frac{1}{1+x}$$

$$\lim_{x\to+\infty}\frac{\arctan\left(\frac{1}{x}\right)\frac{1}{1+x}}{\frac{1}{1+x^2}} = \lim_{x\to+\infty}\frac{\frac{1}{x}\,\frac{1}{1+x}}{\frac{1}{1+x^2}} = 1$$

$$\boxed{\arctan x - \frac{\pi}{2} = -\arctan\frac{1}{x}} \qquad x>0$$

$$\cancel{\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}}$$

$$\cancel{\arctan x = \int_0^x \frac{1}{1+t^2}\,dt}$$

$$\left(\arctan x - \frac{\pi}{2}\right)' = \frac{1}{1+x^2}$$

$$\left(-\arctan\frac{1}{x}\right)' = -\arctan'\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)'$$

$$= \cancel{-}\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\cancel{-}\frac{1}{x^2}\right) =$$

$$= \frac{1}{1+\frac{1}{x^2}}\,\frac{1}{x^2} = \frac{1}{x^2+1} = \frac{1}{1+x^2}$$

$\exists\, c \in \mathbb{R}$ t.c.

$$\underbrace{\arctan x - \frac{\pi}{2}}_{\downarrow} = \underbrace{-\arctan\frac{1}{x}}_{\downarrow} + c \qquad x\to+\infty$$

$$0 = 0 + c \Rightarrow c = 0$$

$$\arctan x\,\frac{|\sin x|}{1+x} \le \frac{\pi}{2}\,\frac{|\sin x|}{1+x}$$

$$\arctan x\,\frac{\sin x}{1+x} \longleftrightarrow \frac{\pi}{2}\,\frac{\sin x}{1+x}$$

$$\left(\frac{\pi}{2} - \arctan x\right)\frac{|\sin x|}{1+x} \le \left(\frac{\pi}{2} - \arctan x\right)\frac{1}{1+x}$$

$$\lim_{x\to+\infty}\frac{\left(\frac{\pi}{2} - \arctan x\right)\cancel{\frac{1}{1+x}}}{\frac{1}{x^{\cancel{2}}}} = \lim_{x\to+\infty}\frac{\frac{\pi}{2} - \arctan x}{x^{-1}}$$

$$= \lim_{x\to+\infty}\frac{-\frac{1}{1+x^2}}{-x^{-2}} = \lim_{x\to+\infty}\frac{x^2}{1+x^2} = 1$$

$$\int \frac{1}{x} \sin \frac{1}{x} \qquad [1, +\infty]$$

$$\left| \sin \frac{1}{x} \right| \leq \frac{1}{x}$$

$$\frac{1}{x} \left| \sin \frac{1}{x} \right| \leq \frac{1}{x^2} \in L[1, +\infty)$$

E4 8/7/2024

$f(x) = 1 + 3x + x^3$. Verificare che è biettiva $\mathbb{R} \to \mathbb{R}$

e se $g : \mathbb{R} \to \mathbb{R}$ è la funzione inversa

$\left(g(f(x)) = x \qquad f(g(x)) = x \; \forall x \right)$

Calcolare P_1 per g.

1) f biettiva. $\lim_{x \to +\infty \atop -\infty} f(x) = \lim_{x \to +\infty \atop -\infty} x^3 = +\infty \atop -\infty$

$\Rightarrow \quad f(\mathbb{R}) = \mathbb{R}$

$f'(x) = \left(x^3 + 3x + 1 \right)' = 3x^2 + 3 \geq 3 \qquad \forall x \in \mathbb{R}$

$\Rightarrow$ f è strettamente crescente $\Rightarrow$ f è iniettiva

Questo dimostra che f è biettiva.

2) $g(y) \qquad P_1(y) = g(0) + g'(0)\, y$

$g(y) = x \iff f(x) = y$

$f(x_0) = 0 \iff g(0) = x_0$

$x^3 + 3x + 1 = 0$

$g'(0) = ?$

$g'(y) = \dfrac{1}{f'(x)} \qquad y = f(x)$

$0 = f(x_0)$

$g'(0) = \dfrac{1}{f'(x_0)} = \dfrac{1}{3 + 3x_0^2}$

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$$\begin{cases} \overbrace{y'' + 2y}^{L[y]} = e^x \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$y_g = y_h + y_p$

$p(r) = r^2 + 2 = 0$ $\qquad r = \pm i\sqrt{2}$, $p(1) = 3$

$y_h = A \sin(\sqrt{2}x) + B \cos(\sqrt{2}x)$

$r = \alpha \pm i\beta$

$e^{\alpha x} \sin(\beta x)$, $e^{\alpha x} \cos(\beta x)$

$y_p = C e^x$

$L[C e^x] = C\, L[e^x] = C\, p(1)\, e^x = C\, 3\, e^x = e^x$

$\Rightarrow C = \frac{1}{3}$

$y = A \sin(\sqrt{2}x) + B \cos(\sqrt{2}x) + \frac{1}{3} e^x$

$y(0) = 1$ $\qquad y'(0) = 0$

$$\begin{cases} B + \frac{1}{3} = 1 & B = \frac{2}{3} \\ \sqrt{2}A + \frac{1}{3} = 0 & A = -\frac{1}{\sqrt{2}\,3} \end{cases}$$

$\text{th}(x)\,\dfrac{\sin x}{x^3} \overset{?}{\in} L(0,+\infty)$

$(0,+\infty) = (0,1] \cup [1,+\infty)$

e separatamente analizzo l'integrabilità nei due intervalli.

In $[1,+\infty)$ $\quad \left|\text{th}(x)\,\dfrac{\sin x}{x^3}\right| \leq \dfrac{1}{x^3} \in L[1,+\infty)$

$\Rightarrow$ f è ass. integrabile in $[1,+\infty) \Rightarrow f \in L[1,+\infty)$

Esaminiamo $f(x) = \text{th}(x)\,\dfrac{\sin x}{x^3}$ in $(0,1]$

In $(0,1]$ si ha $f(x) > 0$

$$\lim_{x\to 0^+} \frac{f(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{\text{th}(x)\,\sin x}{x^{\cancel{3}\,2}}}{\cancel{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+} \frac{\text{th}(x)\,\sin x}{x^2} = \lim_{x\to 0^+} \frac{\text{th}\,x}{x}\,\frac{\sin x}{x} =$$

$$= \text{th}'(0)\,\sin'(0) = 1\cdot 1 = 1$$

$$\text{th}'(x)\Big|_{x=0} = 1 - \text{th}^2(x)\Big|_{x=0} = 1$$

$$\lim_{x\to 0^+} \frac{f(x)}{\frac{1}{x}} = 1 \Rightarrow \left(f \notin L(0,1] \Leftrightarrow \frac{1}{x} \notin L(0,1]\right) \checkmark$$

$$\Rightarrow f \notin L(0,1] \Rightarrow f \notin L(0,+\infty)$$

$$\int x^2 \sin^2(x)\,dx = \int x^2 \frac{1-\cos(2x)}{2}\,dx$$

$$= \frac{1}{2}\int x^2 - \frac{1}{2}\int x^2 \overbrace{\cos(2x)}^{\frac{1}{2}(\sin 2x)'}\,dx$$

$$= \frac{x^3}{6} - \frac{1}{2}\left[x^2\,\frac{\sin 2x}{2} - \int \frac{\sin(2x)}{\cancel{2}}\,\cancel{2}x\right]$$

$$= \frac{x^3}{6} - \frac{x^2}{4}\sin(2x) + \frac{1}{2}\int \underbrace{\sin(2x)}_{\left(-\frac{\cos(2x)}{2}\right)'}\,x\,dx$$

$$= \frac{x^2}{6} - \frac{x^2}{4}\sin(2x) + \frac{1}{2}\left[-\frac{\cos(2x)}{2}x + \int \frac{\cos(2x)}{2}\,dx\right]$$

$$= \frac{x^2}{6} - \frac{x^2}{4}\sin(2x) - \frac{1}{4}\cos(2x)\,x + \frac{1}{4}\,\frac{\mathrm{sen}(2x)}{2} + C$$

$$\int_1^{+\infty} \frac{1}{x^3+x^2+x+1}\, dx$$

$$x^3+x^2+x+1 = x^2(x+1)+(x+1) = (x+1)(x^2+1)$$

$$\frac{1}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \qquad A+B=0$$

$$A = \left.\frac{1}{x^2+1}\right|_{x=-1} = \frac{1}{2} \quad \Longrightarrow \quad B = -\frac{1}{2}$$

$$\frac{1}{x^3+x^2+x+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{x^3+x^2+x+1}$$

Il termine costante in alto a destra è $A + C = \frac{1}{2} + C = 1$

$$\Longrightarrow C = \frac{1}{2}$$

$$f(x) = \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}$$

$$\int_1^R f(x)\,dx = \frac{1}{2}\lg(1+x)\Big]_1^R - \frac{1}{2}\frac{1}{2}\int_1^R \frac{2x}{x^2+1} + \frac{1}{2}\int_1^R \frac{1}{x^2+1}$$

$$= \frac{1}{2}\lg(1+x)\Big]_1^R - \frac{1}{4}\lg(x^2+1)\Big]_1^R + \frac{1}{2}\arctan x\Big]_1^R$$

$$= \frac{1}{2}\left(\lg\frac{1+R}{\sqrt{R^2+1}}\right) \to 0 \quad - \frac{1}{2}\lg 2 + \frac{1}{4}\lg 2 + \frac{1}{2}\arctan R - \frac{\pi}{8}$$

$$\xrightarrow{R\to+\infty} \quad -\frac{1}{4}\lg 2 + \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} - \frac{1}{4}\lg 2 =$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \lg 2\right) > 0$$