DECADIMENTO DEL MUONE

Riprendiamo la Lagrangiana di int. debole di Fermi:

$$\mathcal{L}_{CC}^{eff} = -4 \frac{G_F}{\sqrt{2}} J_c^{\mu} J_{c\mu}^{+} \qquad G_F \simeq 1{,}66 \times 10^{-5}\ GeV^{-2}$$

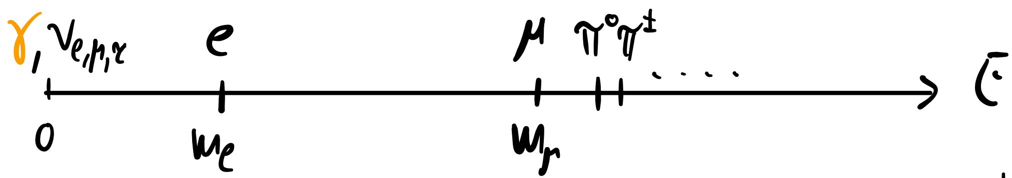
$$J_c^{\alpha} = V_{ij}\, \bar{u}_{i_L} \gamma^{\alpha} d_{j_L} + \bar{\nu}_{i_L} \gamma^{\alpha} e_{i_L}$$

Questa contiene il termine:

$$\mathcal{L}_{CC}^{eff} \supset -4 \frac{G_F}{\sqrt{2}} \left(\bar{\nu}_{\mu_L} \gamma^{\alpha} \mu_L\right)\left(\bar{e}_L \gamma_{\alpha} \nu_{e_L}\right) + h.c.$$

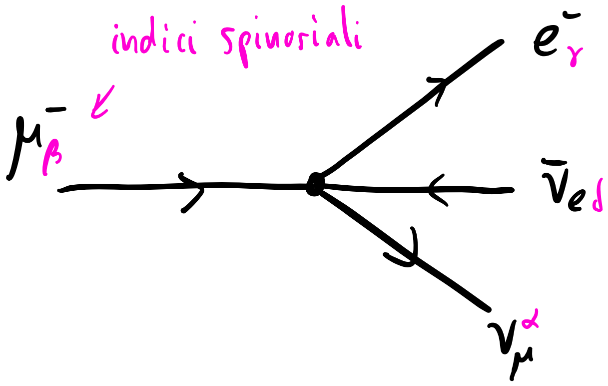
$$= -4 \frac{G_F}{\sqrt{2}} \left(\bar{\nu}_{\mu} \gamma^{\alpha} P_L \mu\right)\left(\bar{e} \gamma_{\alpha} P_L \nu_e\right) + h.c. \, , \qquad P_{R,L} \equiv \frac{1 \pm \gamma_5}{2}$$

Andando a studiare lo spettro delle particelle leggere,



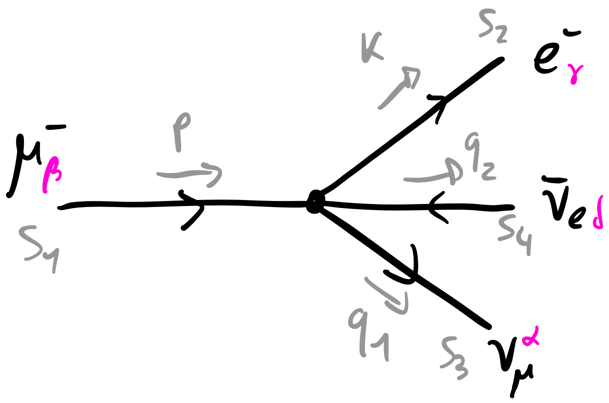
vediamo che il muone, tramite $\mathcal{L}_{CC}^{eff}$, può decadere solo nel canale: $\boxed{\mu^- \to e^- \bar{\nu}_e \nu_\mu}$

REGOLA DI FEYNMAN DA $\mathcal{L}_{CC}^{eff}$:



$$= -4i \frac{G_F}{\sqrt{2}} \left(\gamma^{\alpha} P_L\right)_{\alpha\beta} \left(\gamma_{\alpha} P_L\right)_{\gamma\delta}$$

AMPIEZZA DI DECADIMENTO



$$i\mathcal{M} = -i\,4\frac{G_F}{\sqrt{2}}\left[\bar{u}^{s_3}_{\nu_\mu}(q_1)\,\gamma^\alpha P_L\, u^{s_1}_{\mu}(p)\right]\left[\bar{u}^{s_2}_e(k)\,\gamma_\alpha P_L\, v^{s_4}_{\nu_e}(q_2)\right]$$

Rate di decadimento <u>non polarizzata</u>:

- Sommare sulle pol. finali
- Mediare " " iniziali

$$\frac{1}{2}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}\mathcal{M}^\dagger = 4G_F^2\sum_{\text{spin}}\left[\bar{u}^{s_3}_{\nu_\mu}(q_1)\,\gamma^\alpha P_L\, u^{s_1}_{\mu}(p)\,\bar{u}^{s_1}_{\mu}(p)\,\gamma^\beta P_L\, u^{s_3}_{\nu_\mu}(q_1)\right]$$
$$\left[\bar{u}^{s_2}_e(k)\,\gamma_\alpha P_L\, v^{s_4}_{\nu_e}(q_2)\,\bar{v}^{s_4}_{\nu_e}(q_2)\,\gamma_\beta P_L\, u^{s_2}_e(k)\right]$$

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m \quad , \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p}-m$$

$P_L\gamma^\mu = \gamma^\mu P_R$

$\{\gamma^5, \gamma^\mu\} = 0$

$$= 4G_F^2\,\mathrm{Tr}\left[\not{q}_1\gamma^\alpha P_L(\not{p}+\cancel{m_\mu})\gamma^\beta P_L\right]\mathrm{Tr}\left[(\not{k}+\cancel{m_e})\gamma_\alpha P_L\not{q}_2\gamma_\beta P_L\right]$$

$$= 4G_F^2\,\mathrm{Tr}\left[\not{q}_1\gamma^\alpha\not{p}\gamma^\beta P_L\right]\mathrm{Tr}\left[\not{k}\gamma_\alpha\not{q}_2\gamma_\beta P_L\right]$$

Tensore antisimm. di Levi-Civita

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta P_L\right] = 2\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta} + i\,\varepsilon^{\mu\alpha\nu\beta}\right)$$

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right)$$

$$= 16\, G_F^2 \left(q_1^{\alpha} p^{\beta} + q_1^{\beta} p^{\alpha} - (q_1 p) g^{\alpha\beta} + i \varepsilon^{\mu\alpha\nu\beta} q_{1\mu} p_{\nu} \right)$$

$$\left(k_{\alpha} q_{2\beta} + k_{\beta} q_{2\alpha} - (k \cdot q_2) g_{\alpha\beta} + i \varepsilon_{\rho\alpha\sigma\beta} k^{\rho} q_2^{\sigma} \right) =$$

$$= 16\, G_F^2 \Big[2(q_1 k)(p q_2) + 2(q_1 q_2)(p k) + \cancel{(-2-2+4)}(q_1 p)(k \cdot q_2)$$

$$+ i^2 (-1)^2 \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} q_{1\mu} p_{\nu} k^{\rho} q_2^{\sigma} \Big] =$$

$$= 16\, G_F^2 \left[2(q_1 k)(p q_2) \cancel{+ 2 (q_1 q_2)(p k)} + 2 (q_1 k)(p q_2) \cancel{- 2 (q_1 q_2)(p k)} \right] =$$

$$\boxed{= 64\, G_F^2 (q_1 k)(p q_2) = \frac{1}{2} \sum_{spin} \mathcal{M} \mathcal{M}^{\dagger}}$$

RATE DI DECADIMENTO

$$d\Gamma = \frac{1}{2m_\mu}\left(\frac{1}{2}\sum_{spin} \mathcal{M}\mathcal{M}^+\right)(2\pi)^4\delta^4(p-k-q_1-q_2)\frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}\frac{d^3k}{(2\pi)^3 2E_e}$$

- Vogliamo calcolare inizialmente $\frac{d\Gamma}{dE_e}$: spettro di energia dell'elettrone
- Dopodiché integreremo in dE_e per ottenere $\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \tau^{-1}$

$$d\Gamma = \frac{4G_F^2}{m_\mu (2\pi)^5} k_\mu p_\nu q_1^\mu q_2^\nu \, \delta^4(\overbrace{p-k}^{q}-q_1-q_2)\frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\frac{d^3k}{E_e}$$

$q \equiv p-k = q_1+q_2$

Integriamo sullo spazio delle fasi dei neutrini:

$$\int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2} q_1^\mu q_2^\nu \, \delta^4(q-q_1-q_2) = I^{\mu\nu}(q)$$

$I^{\mu\nu}(q)$ dipende solo da q ed è simmetrico in $\mu \leftrightarrow \nu$

↳ Ansatz $I^{\mu\nu}(q) = g^{\mu\nu} A(q^2) + q^\mu q^\nu B(q^2)$

funz. scalare

$$\begin{cases} g_{\mu\nu} I^{\mu\nu} = 4A + q^2 B \\ q_\mu q_\nu I^{\mu\nu} = q^2 A + (q^2)^2 B \end{cases}$$

$m_\nu = 0 \to q_1^2 = q_2^0 = 0 \to q^2 = (p-k)^2 = (q_1+q_2)^2 = +2(q_1 q_2)$

$q q_1 = (q_1+q_2)q_1 = q_1 q_2 = \frac{q^2}{2} = q q_2$

$$\bullet\ g_{\mu\nu} I^{\mu\nu} = \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}(q_1q_2)\,\delta^4(q-q_1-q_2) = \frac{q^2}{2}\int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\,\delta^4(q-q_1-q_2)$$

Nel sist. centro di massa di $\bar{\nu}_e \nu_\mu$: $\vec{q}_1 = -\vec{q}_2$

$\Rightarrow E_1 = E_2 = \omega = q^0/2$

$$= \frac{q^2}{2}\int \frac{d^3q_1}{\omega^2}\,\delta(q^0-2\omega) = \frac{q^2}{2}\int \frac{\cancel{\omega^2}\,d\omega\, d\Omega}{\cancel{\omega^2}}\,\frac{1}{2}\,\delta\!\left(\omega-\frac{q^0}{2}\right) =$$

$$= \frac{q^2}{4}\int d\Omega = \boxed{\pi q^2 = 4A + q^2 B}$$

$$\bullet\ q_\mu q_\nu I^{\mu\nu} = \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}(qq_1)(qq_2)\,\delta^4(q-q_1-q_2) = \left(\frac{q^2}{2}\right)^2\int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\,\delta^4(q-q_1-q_2)$$

$$= \frac{1}{2}\pi (q^2)^{\cancel{2}} = \cancel{q^2} A + (q^2)^{\cancel{2}} B \rightarrow \boxed{\frac{1}{2}\pi q^2 = A + q^2 B}$$

Risolvendo per A e B

$$I^{\mu\nu}(q) = \frac{\pi}{6}\left(g^{\mu\nu}q^2 + 2q^\mu q^\nu\right) \rightarrow$$ Valido in qualsiasi sistema di riferimento

Inserendo in $d\Gamma$:

$$d\Gamma = \frac{4G_F^2}{m_\mu (2\pi)^5}\,K_\mu P_\nu\,\frac{\pi}{6}\left(g^{\mu\nu}q^2 + 2q^\mu q^\nu\right)\frac{d^3K}{E_e} =$$

$$= \frac{2\pi}{3}\frac{G_F^2}{m_\mu(2\pi)^5}\left((KP)q^2 + 2(Kq)(Pq)\right)\frac{d^3K}{E_e}$$

Nel sistema di riposo del muone:

$P = (m_\mu, \vec{0})$, $\vec{k} = -\vec{q}_1 - \vec{q}_2 = -\vec{q}$, $q^0 = m_\mu - E_e$

$(KP) = m_\mu E_e$, $(Kq) = E_e(m_\mu - E_e) + |\vec{k}|^2$, $(Pq) = m_\mu^2 - m_\mu E_e$

$q^2 = (m_\mu - E_e)^2 - |\vec{k}|^2 = (m_\mu - E_e)^2 - (E_e^2 - m_e^2) = m_\mu^2 + m_e^2 - 2E_e m_\mu$

$$d^3k = |\vec{k}|^2 \, d|\vec{k}| \, d\Omega = |\vec{k}|^2 \frac{d|\vec{k}|}{d\varepsilon_e} \, d\varepsilon_e \, d\Omega = |\vec{k}| \, \varepsilon_e \, d\varepsilon_e \, d\Omega$$

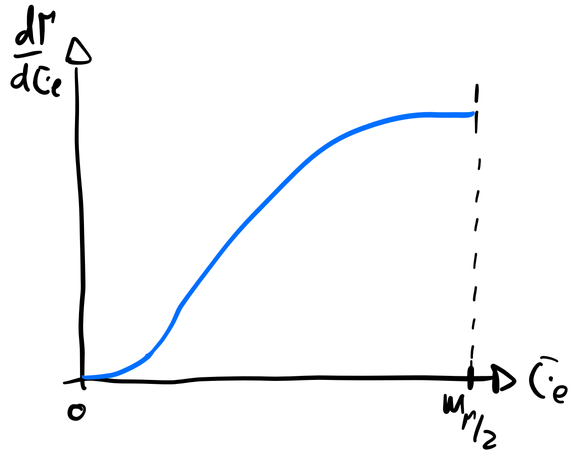
$$\Rightarrow \frac{d\Gamma}{d\varepsilon_e} = \frac{2\pi}{3} \frac{G_F^2}{m_\mu (2\pi)^5} \Big((kp) q^2 + 2 (kq)(pq) \Big) |\vec{k}| \, 4\pi$$

Infine:

$$\boxed{\frac{d\Gamma}{d\varepsilon_e} = \frac{G_F^2}{12\pi^3} \sqrt{\varepsilon_e^2 - m_e^2} \Big(\varepsilon_e (m_\mu^2 + m_e^2 - 2 m_\mu \varepsilon_e) + 2 (\varepsilon_e m_\mu - m_e^2)(m_\mu - \varepsilon_e) \Big)}$$

Prendiamo per semplicità il limite $m_e \simeq 0$:

$$\frac{d\Gamma}{d\varepsilon_e} \overset{m_e = 0}{\simeq} \frac{G_F^2}{4\pi^3} m_\mu \varepsilon_e^2 \left(m_\mu - \frac{4}{3} \varepsilon_e \right)$$



Vediamo i limiti di ε_e $(m_e \simeq 0)$

$$\varepsilon_1 = |\vec{q}_1| = |\vec{k} + \vec{q}_2| = \sqrt{\varepsilon_e^2 + \varepsilon_2^2 + 2 \varepsilon_2 \varepsilon_e \cos\theta} \in \Big[|\varepsilon_e - \varepsilon_2| , \varepsilon_e + \varepsilon_2 \Big]$$

La conservazione dell'energia mi da:

$$|\varepsilon_e - \varepsilon_2| \leq \varepsilon_1 = m_\mu - \varepsilon_e - \varepsilon_2 \leq \varepsilon_e + \varepsilon_2$$

$$|\varepsilon_e - \varepsilon_2| + \varepsilon_e + \varepsilon_2 \leq m_\mu \leq 2(\varepsilon_e + \varepsilon_2)$$

Il lato dx da il limite inf: $\varepsilon_e \geqslant \frac{m_\mu}{2} - \varepsilon_2$

Il lato sx " " sup per $\varepsilon_2 = 0 \rightarrow \varepsilon_e \leq \frac{m_\mu}{2}$

Ripetendo lo stesso per ε_2: $\varepsilon_2 \leq \frac{m_\mu}{2}$

Sfruttanddo nel limite inf per ε_2: $\varepsilon_e \geqslant 0$

$$\Rightarrow \varepsilon_e \in \left[0, \frac{m_\mu}{2} \right]$$

Integrando in E_e:

$$\Gamma(\mu \to e \nu\nu) = \int_0^{\frac{m_\mu}{2}} \frac{d\Gamma}{dE_e}\, dE_e = \int_0^{\frac{m_\mu}{2}} \frac{G_F^2}{4\pi^3} m_\mu E_e^2 \left(m_\mu - \frac{4}{3} E_e\right) dE_e$$

$$\boxed{\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = \frac{G_F^2\, m_\mu^5}{192\, \pi^3} = \tau_\mu^{-1}}$$

Sfruttando l'ANALISI DIMENSIONALE:

$\Gamma \propto |M|^2 \propto G_F^2$

$[\Gamma] = 1 \quad , \quad [G_F^2] = -4 \quad \Longrightarrow \quad \Gamma \propto \dfrac{G_F^2\, m_\mu^5}{16\, \pi^3}$ → spazio delle fasi generico in 3 part.

Per compensare il -4 da G_F^2. È l'unica scala di E se $m_e = 0$.

Dalla misura $\tau_\mu \simeq 2{,}2 \times 10^{-6}$ s e $m_\mu \simeq 105{,}7$ MeV $= 0{,}1057$ GeV

Fattore di conversione $1\ \text{GeV} \simeq 6{,}58 \times 10^{-25}\ s^{-1}$

$$G_F = \left(\frac{6{,}58 \times 10^{-25}}{\tau_\mu[s]} \; \frac{192\,\pi^3}{m_\mu^5[\text{GeV}]} \right)^{1/2} = 1{,}16 \times 10^{-5}\ \text{GeV}^{-2}$$