Some Remarks on Fréchet Spaces and Convergence of Distributions

Bais V. and Manganello F.

Università di Trieste

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Contents

1 Basic Definitions and Notations

For the sake of clarity we briefly summarize the main notations which will be used in the following and some definitions that are assumed to be known by the reader.

- $\mathbb{F}$ will denote a generic field among $\mathbb{R}$ and $\mathbb{C}$.
- $\Omega \subseteq \mathbb{R}^d$ open set;
- for a given $f \in C^0(\Omega, \mathbb{R})$ we define $\mathrm{Supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}$;
- $\mathcal{D}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega, \mathbb{R}) : \ \mathrm{Supp}(f) \ \textit{compact}\}$;
- in a generic metric space (X, d) we denote the set $B_d^X(f, r) := \{g \in X : d(f, g) < r\}$ i. e. the open ball of center $f \in X$ and radius $r > 0$.

Definition 1.1. *A function $T : \mathcal{D}(\Omega) \to \mathbb{F}$ is called a distribution on Ω if:*

1. *T is $\mathbb{F}$ linear;*
2. *for all K, compact set in Ω, there exists $C_K > 0$, $m_K \in \mathbb{N}$ such that:*

$$|T(\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\mathrm{Supp}(\varphi) \subseteq K$*;*

The set of distributions on Ω will be denoted $\mathcal{D}'(\Omega)$.

2 Introduction

The main purpose of this work is to provide a brief yet exhaustive proof of the following claim.

Proposition. *Let $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}'(\Omega)$ a sequence of distributions on Ω such that for all $\varphi \in \mathcal{D}(\Omega)$ there exists in $\mathbb{F}$ $\lim_{n \to +\infty} T_n(\varphi)$. Then the functional:*

$$\begin{aligned} T : \mathcal{D}(\Omega) &\to \mathbb{F} \\ \varphi &\mapsto \lim_{n \to +\infty} T_n(\varphi) \end{aligned}$$

is a distribution on Ω.

The main reference for this work is:

[Bon01] "Cours d'analyse: thérie des distributions et analyise de Fourier" by Jean-Michel Bony (Les Éditions de l'École Polytechnique, 2001).

The proof relies on a generalized version of Banach-Steinhaus theorem that applies for Fréchet spaces and not only Banach spaces. In particular [Bon01] gives the proof of the above mentioned version of Banach-Steinhaus theorem, while the definition of distribution we use in this work is due to [Hör63].

3 Fréchet Spaces

Definition 3.1. *Let E be a vector space on $\mathbb{F}$. We call a seminorm on E an application:*

$$P : E \to [0, +\infty)$$

such that:

1. *$P(f + g) \leq P(f) + p(g) \quad \forall f, g \in E$;*
2. *$P(\lambda f) = |\lambda| P(f) \quad \forall \lambda \in \mathbb{F},\ \forall f \in E$.*

Property 1. is called subadditivity or triangular inequality, property 2. is also called homogeneity. In addition we remark that a generic seminorm P on a vector space E turns out to be a norm on E if in addition it fulfills:

3. $P(f) = 0 \Leftrightarrow f = 0 \quad \forall f \in E.$

Definition 3.2. *A vector space E on $\mathbb{F}$ endowed with an increasing family of seminorms $(P_j)_{j\in\mathbb{N}}$ such that:*

$$f = 0 \Leftrightarrow P_j(f) = 0 \;\forall j \in \mathbb{N}$$

is called a locally convex metrizable space.

Definition 3.3. *Let $(P_j)_{j\in\mathbb{N}}$ and $(Q_k)_{k\in\mathbb{N}}$ be two increasing sequences of seminorms on a vector space E over $\mathbb{F}$. We say that the two sequences of seminorms define the same structure of locally convex metrizable space or that they are equivalent if:*

$$\forall j \in \mathbb{N}, \exists k \in \mathbb{N}, \exists C \in \mathbb{R} \; s.t. \; P_j \leq C Q_k,$$
$$\forall k \in \mathbb{N}, \exists j \in \mathbb{N}, \exists C' \in \mathbb{R} \; s.t. \; Q_k \leq C' P_j.$$

Remark 3.4. The structure of locally convex metrizable space on a vector space E over $\mathbb{F}$ is fully determined once a generic sequence $(Q_i)_{i\in\mathbb{N}}$ of seminorms is given. Indeed, an increasing sequence of seminorms can be recasted as follows:

$$\forall j \in \mathbb{N} \quad P_j := \sum_{i\leq j} Q_i.$$

For any $j \in \mathbb{N}$ both subadditivity and homogeneity are preserved when summing over a finite set of indices: $i \in \{1, \ ..., \ j\}$.

Remark 3.5. A normed space $(X, \|\cdot\|)$ is a particular case of locally convex metrizable space where all seminorms are chosen to be equal to $\|\cdot\|$.

From now on, for the sake of simplicty we will denote $(E, (P_j)_{j\in\mathbb{N}})$ the locally convex metrizable space E over a field $\mathbb{F}$ with its increasing sequence of seminorms $(P_j)_{j\in\mathbb{N}}$.

Definition 3.6. *Given $(E, (P_j)_{j\in\mathbb{N}})$, $f \in E$, $r > 0$, $j \in \mathbb{N}$, the set:*

$$SB_j^E(f, r) := \{g \in E : \; P_j(g - f) < r\} \tag{1}$$

is called a semiball relative to P_j of center f and radius r in E.

Definition 3.7. *Given $(E, (P_j)_{j\in\mathbb{N}})$, $f \in E$, $r > 0$, $j \in \mathbb{N}$, the set:*

$$\overline{SB}_j^E(f, r) := \{g \in E : \; P_j(g - f) \leq r\} \tag{2}$$

is called a closed semiball relative to P_j of center f and radius r in E.

Remark 3.8. Given $(E, (P_j)_{j\in\mathbb{N}})$, if $C \subseteq E$ is such that $SB_j^E(f,r) \subseteq C$ for some $f \in E$, $r > 0$, $j \in \mathbb{N}$, clearly there exists $\overline{r}$ such that $\overline{SB}_j^E(f,\overline{r}) \subseteq C$, indeed it is enough considering $\overline{r} = \frac{r}{2}$.

Remark 3.9. Given $(E, (P_j)_{j\in\mathbb{N}})$, let $f \in E$, $r > 0$, $j \in \mathbb{N}$. Then for all $k > j$ $SB_k^E(f,r) \subseteq SB_j^E(f,r)$ and $\overline{SB}_k^E(f,r) \subseteq \overline{SB}_j^E(f,r)$.

Definition 3.10. *Let $(E, (P_j)_{j\in\mathbb{N}})$ be a locally convex metrizable space and $(\alpha_j)_{j\in\mathbb{N}}$ be any sequence of positive real numbers such that $\sum_{j=1}^{\infty} \alpha_j < \infty$. We define*

$$d : E \times E \to \mathbb{R}$$

where:

$$d(f,g) := \sum_{j=1}^{\infty} \alpha_j \min\{1, P_j(f-g)\}. \tag{3}$$

Proposition 3.11. *d defined as in Definition 3.10 is a metric on E.*

Proof.

1. $d(f,g) \geq 0 \quad \forall f,g \in E$ as by definition $P_j \geq 0 \quad \forall j \in \mathbb{N}$.
2. $d(f,g) = 0$ if and only if $P_j(f-g) = 0$ for all $j \in \mathbb{N}$. According to Definition 3.2 this is equivalent to $f = g$.
3. $\forall f,g \in E,\ d(f,g) = d(g,f)$.
4. $\forall f,g,h \in E :$

$$\begin{aligned} d(f,h) &= \sum_{j\in\mathbb{N}} \alpha_j \min\{1, P_j(f-h)\} \\ &\leq \sum_{j\in\mathbb{N}} \alpha_j \min\{1, P_j(f-g) + P_j(g-h)\} \\ &\leq \sum_{j\in\mathbb{N}} \alpha_j \min\{1, P_j(f-g)\} + \sum_{j\in\mathbb{N}} \alpha_j \min\{1, P_j(g-h)\} \\ &= d(f,g) + d(f,h). \end{aligned}$$

□

Remark 3.12. The choice of the sequence $(\alpha_j)_{j\in\mathbb{N}}$ has no influence on the topology induced by d, in particular, choosing a different sequence $(\beta_k)_{k\in\mathbb{N}}$ yields a metric $\tilde{d}$ such that for all $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that:

$$\begin{aligned} d(x,y) < \delta_1 \Rightarrow \tilde{d}(x,y) < \epsilon \quad \forall x,y \in E, \\ \tilde{d}(x,y) < \delta_2 \Rightarrow d(x,y) < \epsilon \quad \forall x,y \in E, \end{aligned}$$

i. e. one says that d and $\tilde{d}$ are uniformly equivalent metrics. Finally uniformly equivalent metrics induce the same topology.

Definition 3.13. *A locally convex metrizable space $(E, (P_j)_{j\in\mathbb{N}})$ is a Fréchet space if it is complete with respect to the structure of metric space as given in Definition 3.10.*

Remark 3.14. Thanks to Definition 3.10 and Proposition 3.11 any locally convex metrizable space $(E, (P_j)_{j\in\mathbb{N}})$ turns out to be a metric space and hence a topological space. This allows us to define notions like open sets, neighbourhoods of a point, convergence of a sequence and continuity of a function. In addition we have that:

- a function $\psi : E \to X$ - where X is some topological space - is continuous if and only if for any sequence $(f_n)_{n\in\mathbb{N}} \subseteq E$ converging to some $f \in E$ the sequence $(\psi(f_n))_{n\in\mathbb{N}} \subseteq X$ converges to $\psi(f) \in X$;
- $V \subseteq E$ is compact if and only if it is sequentially compact.

Nevertheless handling minimums and α_js in (3) is not easy at all, so expressing the above mentioned topological notions in terms of seminorms is definitely convenient.

Lemma 3.15. *Let $(E, (P_j)_{j\in\mathbb{N}})$ be a locally convex metrizable space.*

1. *Let $(f_n) \subseteq E$ be a sequence. Then:*
   - *(f_n) is a Cauchy sequence* $\Leftrightarrow \lim_{m,n\to\infty} P_j(f_n - f_m) = 0 \quad \forall j \in \mathbb{N}$;
   - $f_n \to f \Leftrightarrow P_j(f_n - f) \to 0 \quad \forall j \in \mathbb{N}$.
2. *$V \subseteq E$ is an open neighbourhood of f if and only if $\exists j \in \mathbb{N}, \epsilon > 0$ such that $SB^E_j(f, \epsilon) \subseteq V$. In particular for all $k \in \mathbb{N}$, for all $r > 0$ $SB^E_k(f, r)$ is an open neighbourhood of f.*
3. *Seminorms $P_j : E \to \mathbb{R}$ are continuous.*

Proof.
1. One easily deduce from the Definition 3.10 of d that $d(f_n, f_m) \to 0$ and $d(f_n, f) \to 0$ if and only if $P_j(f_n - f_m) \to 0 \quad \forall j \in \mathbb{N}$ and $P_j(f_n - f) \to 0 \quad \forall j \in \mathbb{N}$, respectively.
2. Suppose $V \subseteq E$ is a neighbourhood of $f \in E$. This means that there exists $\epsilon > 0$ such that $B^E_d(f, \epsilon) \subseteq V$. Let $k \in \mathbb{N}$ be such that $\sum_{j>k} \alpha_j < \frac{\epsilon}{2}$ (notice that this can be done as the series of α_j converges) and define $\tilde{\epsilon} := \min\{\frac{\epsilon}{2\sum_{j\le k}\alpha_j}, 1\}$. Then $SB^E_k(f, \tilde{\epsilon}) \subseteq V$. Indeed, let $g \in SB^E_k(f, \tilde{\epsilon})$. Then:
$$\begin{aligned} d(f, g) &= \sum_{j\le k} \alpha_j \min\{1, P_j(f - g)\} + \sum_{j>k} \alpha_j \min\{1, P_j(f - g)\} \\ &\le \sum_{j\le k} \alpha_j P_k(f - g) + \sum_{j>k} \alpha_j \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$
   Therefore $SB^E_k(f, \tilde{\epsilon}) \subseteq V$. The converse is trivially satisfied.

3. Using triangular inequality, one can prove that $|P_j(f) - P_j(g)| \leq P_j(f - g) \quad \forall j \in \mathbb{N}, \forall f, g \in E$.

$\square$

Remark 3.16. Lemma 3.15 grants that the adjective "closed" in Definition 3.7 makes sense.

We now state and prove a characterization of linear and continuous functions between locally convex metrizable spaces. This result will be useful in the following.

Proposition 3.17. *Let $(E, (P_j)_{j\in\mathbb{N}})$ and $(F, (Q_k)_{k\in\mathbb{N}})$ be two locally convex metrizable spaces. Let $L : E \to F$ be a linear application. The following three properties are equivalent:*

1. *L is continuous at every point;*
2. *L is continuous at 0;*
3. *$\forall k \in \mathbb{N}, \exists j \in \mathbb{N}, C > 0$ such that $\forall f \in E \quad Q_k(L(f)) \leq C P_j(f)$.*

Proof. $(1 \Rightarrow 2)$. This implication follows from the definition of continuity.
$(2 \Rightarrow 3)$. By continuity of L at 0 and by linearity, the set $L^{-1}(SB_k^F(0,1))$ is an open neighbourhood of 0. Hence there exists $\epsilon > 0$ and there exists $j \in \mathbb{N}$ such that $\overline{SB}_j^E(0,\epsilon) \subseteq L^{-1}(SB_k^F(0,1))$. This means that:

$$P_j(f) \leq \epsilon \Rightarrow Q_k(L(f)) < 1. \tag{4}$$

Let $f \in E$ such that $P_j(f) > 0$. We remark that $P_j(\frac{\epsilon}{P_j(f)} f) = \epsilon$ and hence by (4) and linearity of L: $Q_k(\frac{\epsilon}{P_j(f)} L(f)) < 1$. By homogeneity of Q_k one gets:

$$Q_k(L(f)) < \frac{1}{\epsilon} P_j(f),$$

i. e. the thesis with $C = \frac{1}{\epsilon}$. Suppose now $P_j(f) = 0$. Then, either there exists $j_0 \in \mathbb{N}$ such that $j_0 > j$ and $P_{j_0}(f) > 0$ and one recasts the above argument as the implication in (4) is still valid thanks to Remark 3.9 or $f = 0$ and the thesis is trivially true.
$(3 \Rightarrow 1)$. Suppose $f_n \to f$ in E. Then by Lemma 3.15 $P_j(f_n - f) \to 0 \ \forall j \in \mathbb{N}$ and then $Q_k(L(f_n) - L(f)) \to 0 \forall k \in \mathbb{N}$. This implies that $L(f_n) \to L(f)$ in F, i.e. the thesis as E is a metric space as stated in Proposition 3.11. $\square$

4 The Fréchet space $C_K^\infty(\Omega)$

Definition 4.1. *Let $K \subset \Omega$ be a compact set. We define the sets:*

$$C_K^m(\Omega) := \{f \in C^m(\Omega, \mathbb{R}) : \mathrm{Supp}(f) \subseteq K\} \quad \forall m \in \mathbb{N};$$

$$C_K^\infty(\Omega) := \{f \in C^\infty(\Omega, \mathbb{R}) : \mathrm{Supp}(f) \subseteq K\}.$$

Proposition 4.2. *$C_K^\infty(\Omega)$ endowed with the family $(P_m)_{m\in\mathbb{N}}$ where*

$$P_m(f) := \sup_{x\in K, |\alpha|\le m} |\partial^\alpha f(x)| \quad \forall f \in C_K^\infty(\Omega), \tag{5}$$

is a locally convex metrizable space.

Proof. For all $m \in \mathbb{N}$ homogeneity and subadditivity of P_m are obtained by linearity of derivative and subadditivity of absolute value. Moreover one has that that:

$$P_m(f) = 0 \ \forall m \in \mathbb{N} \Rightarrow |\partial^\alpha f(x)| = 0 \ \forall x \in K, \forall \alpha \in \mathbb{N}^d,$$

then clearly $f \equiv 0$. □

Proposition 4.3. *The locally convex metrizable space $(C_K^\infty(\Omega), (P_j)_{j\in\mathbb{N}})$ is a Fréchet space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq C_K^\infty(\Omega)$ be a Cauchy sequence. According to Lemma 3.15, one has that:

$$\forall j \in \mathbb{N} \qquad \lim_{m,n\to+\infty} P_j(f_m - f_n) = 0,$$

and thanks to the definition of P_j as in Proposition 4.2 and linearity of derivation one has:

$$\forall \alpha \in \mathbb{N}^d \qquad \sup_{x\in K} |\partial^\alpha f_m(x) - \partial^\alpha f_n(x)| \to 0 \quad \text{as } m, n \to +\infty.$$

that is to say that for all multi-indices $\alpha \in \mathbb{N}^d$ the sequence $(\partial^\alpha f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the usual uniform norm, and hence for all $\alpha \in \mathbb{N}^d$ there exists $g_\alpha \in C_K^0(\Omega)$ such that $\partial^\alpha f_n \to g_\alpha$ uniformly on K. We now recall the classical result on uniform limits of derivatives[1]: if some sequence $(\varphi_n) \subseteq C^1(\Omega, \mathbb{R})$ is such that $\varphi_n \to \psi$ uniformly on Ω, $\partial_i\varphi_n$ are continuous and $\partial_i\varphi_n \to \xi_i$ uniformly on Ω then $\psi \in C^1(\Omega, \mathbb{R})$ and $\partial_i\psi = \xi_i$. Applying inductively this results to $\partial^\alpha f_n$ and g_α one gets that $g \in C_K^\infty(\Omega)$ and

$$\forall j \in \mathbb{N} \quad P_j(f_n - g) \to 0,$$

which is the thesis. □

5 Banach-Steinhaus Theorem for Fréchet Spaces

Definition 5.1. *Let $(E, (P_j)_{j\in\mathbb{N}})$ be a locally convex metrizable space, $C \subseteq E$ is symmetric if:*

$$f \in C \Rightarrow -f \in C.$$

Lemma 5.2. *$(E, (P_j)_{j\in\mathbb{N}})$ be a locally convex metrizable space, $C \subseteq E$ convex, symmetric and with nonempty interior, then C is a neighbourhood of* 0.

[1] See for instance [Giu89], Theorem 13.3.

Proof. C has non empty interior, hence there exist $\epsilon > 0, g \in E, j \in \mathbb{N}$ such that $SB_j^E(g,\epsilon) \subseteq C$. We claim that $SB_j^E(0,\epsilon) \subseteq C$. Indeed, let $h \in SB_j^E(0,\epsilon)$. Then $P_j(h) < \epsilon$ and:

$$P_j((g+h)-g) = P_j(h) < \epsilon,$$
$$P_j((g-h)-g) = P_j(h) < \epsilon,$$

hence $g+h, g-h \in SB_j^E(g,\epsilon)$ and hence $g+h \in C$ and $h-g = -(g-h) \in C$ by symmetry of C. Finally $h = \frac{(h-g)+(h+g)}{2} \in C$ as C is convex. $\square$

Proposition 5.3 (Banach-Steinhaus)**.** *Let $(E,(P_j)_{j\in\mathbb{N}})$ be a Fréchet space and let $(F,(Q_k)_{k\in\mathbb{N}})$ be a locally convex metrizable space. Let $(L_n)_{n\in\mathbb{N}}$ a family of linear continuous functions from E to F. Suppose that for every $f \in E$, $L_n(f)$ converges in F to a limit that we denote $L(f)$. Then:*

1. *The family $(L_n)_{n\in\mathbb{N}}$ is equicontinuous, that is: for every $k \in \mathbb{N}$ there exist a constant $C > 0$ and an index $j \in \mathbb{N}$ such that:*

$$Q_k(L_n(f)) \leq CP_j(f) \quad \forall n \in \mathbb{N}, \forall f \in E. \tag{6}$$

2. *for every compact subset $K \subseteq E$, $L_n \to L$ uniformly on K, that is:*

$$\lim_{n\to+\infty} \sup_{f\in K} Q_k(L_n(f) - L(f)) = 0 \quad \forall k \in \mathbb{N}. \tag{7}$$

Proof. 1. Let $k \in \mathbb{N}$ and for every $p \in \mathbb{N}$ define the set:

$$C_p = \{f \in E : \ \forall n \in \mathbb{N}\ Q_k(L_n(f)) \leq p\}.$$

Since Q_k, L_n are continuous for every $k, n \in \mathbb{N}$, every set C_p is closed. Moreover, $\forall f \in E$, $L_n(f)$ is convergent in $\mathbb{F}$ and therefore, by continuity of Q_k, also $Q_k(L_n(f))$ is convergent and hence bounded in $[0,\infty)$. This means that every f belongs to some C_p for p big enough and $\bigcup_{p\in\mathbb{N}} C_p = E$. Since E is a Fréchet space, it is a complete metric space and hence we can apply Baire's categories theorem and find $\overline{p} \in \mathbb{N}$ such that $C_{\overline{p}}$ has non empty interior. In particular, being every C_p convex and symmetric, $C_{\overline{p}}$ is a neighbourhood of zero according to Lemma 5.2. Therefore there exist $\epsilon > 0$ and $j \in \mathbb{N}$ such that $\overline{SB}_j^E(0,\epsilon) \subseteq C_{\overline{p}}$.
We can now prove the equicontinuity of the family $(L_n)_{n\in\mathbb{N}}$. Let $f \in E$ such that $P_j(f) > 0$. Then:

$$Q_k(L_n(f)) = \frac{P_j(f)}{\epsilon} Q_k(L_n(\frac{\epsilon}{P_j(f)} f)) \leq \frac{\overline{p}}{\epsilon} P_j(f)$$

Hence we get item 1 proved with $C = \frac{\overline{p}}{\epsilon}$. If on the contrary $P_j(f) = 0$ either one can pick $j_0 > j$ such that $P_{j_0}(f) > 0$ and recovers previous argument thanks to Remark 3.9 or one has $f = 0$ and then (6) is trivially accomplished.

2. Choose a compact $K \subset E$, $\epsilon > 0$ and $k \in \mathbb{N}$. We want to show that, if n is large enough enough, then:

$$\sup_{f \in K} Q_k(L_n(f) - L(f)) \leq \epsilon. \tag{8}$$

According to item 1, one can pick $C > 0$, $j \in \mathbb{N}$ such that (6) holds true. K is compact, so there exists $\{g_\lambda\}_{\lambda \in \Lambda} \subseteq K$ such that Λ is finite and $K \subseteq \bigcup_{\lambda \in \Lambda} SB_j^E(g_\lambda, \frac{\epsilon}{3C})$. Thus for all $f \in K$ there exists g_λ such that:

$$Q_k(L_n(f) - L_n(g_\lambda)) = Q_k(L_n(f - g_\lambda)) \leq CP_j(f - g_\lambda) \leq \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}. \tag{9}$$

By continuity of the Q_k, one can replace L_n with L in (9) obtaining:

$$Q_k(L(f) - L(g_\lambda)) \leq CP_j(f - g_\lambda) \leq \frac{\epsilon}{3}. \tag{10}$$

Moreover, by the continuity of Q_k and the finiteness of the set $\{g_\lambda\}_{\lambda \in \Lambda}$ one can find $N \in \mathbb{N}$ such that:

$$Q_k(L(g_\lambda) - L_n(g_\lambda)) \leq \frac{\epsilon}{3} \quad \forall \lambda \in \Lambda \quad \forall n \geq N. \tag{11}$$

Finally by subadditivity of seminorms one can recollect (9), (10) and (11) obtaining:

$$\begin{aligned} Q_k(L_n(f) - L(f)) &\leq Q_k(L_n(f) - L_n(g_\lambda)) \\ &+ Q_k(L_n(g_\lambda) - L(g_\lambda)) \\ &+ Q_k(L(g_\lambda) - L(f)) \\ &\leq \epsilon. \end{aligned} \tag{12}$$

Since this is valid for every $n \geq N$, we get the thesis.

□

Remark 5.4. Item 2 of Proposition 5.3 corresponds to a well known fact in analysis: on a compact space, given a family of equicontinuous functions, pointwise convergence implies uniform convergence.

Corollary 5.5. *Under the hypothesis of Proposition 5.3:*

1. *the function L is linear and continuous from E to F;*
2. *let $(f_n)_{n \in \mathbb{N}} \subset E$ be such that $f_n \to f$, then $L_n(f_n) \to L(f)$ in F.*

Proof. 1. Item 1 of Proposition 5.3 ensures that for every $k \in \mathbb{N}$ one can find $C > 0$ and $j \in \mathbb{N}$ such that:

$$Q_k(L_n(f)) \leq CP_j(f) \quad \forall n \in \mathbb{N}, \quad \forall f \in E. \tag{13}$$

Since Q_k is continuous, one can pass to the limit as $n \to +\infty$ and get:

$$Q_k(L(f)) \leq CP_j(f) \quad \forall f \in \mathbb{E}. \tag{14}$$

This last inequality implies the continuity of L according to Proposition 3.17. Linearity of L follows by linearity of limit.

2. The set $(f_n)_{n\in\mathbb{N}} \cup \{f\}$ is sequentially compact and hence compact. Thus by item 2 of Proposition 5.3:

$$\lim_{n\to+\infty} \sup_{m\in\mathbb{N}} Q_k(L_n(f_m) - L(f_m)) = 0 \quad \forall k \in \mathbb{N}. \tag{15}$$

In particular one has:

$$\lim_{n\to+\infty} Q_k(L_n(f_n) - L(f_n)) = 0 \quad \forall k \in \mathbb{N}. \tag{16}$$

Moreover, for every $k \in \mathbb{N}$:

$$Q_k(L_n(f_n) - L(f)) \le Q_k(L_n(f_n) - L(f_n)) + Q_k(L(f_n) - L(f)). \tag{17}$$

The first term on right hand side of (17) goes to 0 as $n \to +\infty$ according to (16). The second term on the right goes to 0 by the continuity of Q_k and by item 1. Hence item 2 is proved.

□

6 A Characterization of $\mathcal{D}'(\Omega)$

It is well known that the space $\mathcal{D}(\Omega)$ is not a Fréchet space, as it is not metrizable (see for instance [Hör63]) so in principle one cannot exploit the results shown above. Anyway, it has been proved in Proposition 4.3 that $C^\infty_K(\Omega)$ is a Fréchet space for every compact $K \subseteq \Omega$. Thus the following Lemma is crucial as it provides a useful characterization of $\mathcal{D}'(\Omega)$ and allow Corollary 5.5 to apply thanks to Proposition 4.3.

Lemma 6.1. *Let $T : \Omega \to \mathbb{F}$ be a $\mathbb{F}$-linear function and let $(C^\infty_K(\Omega), (P_j)_{j\in\mathbb{N}})$ as in Proposition 4.3. The following two facts are equivalent:*

1. $T \in \mathcal{D}'(\Omega)$;

2. for all compact sets $K \subseteq \Omega$ the restriction of T on $C^\infty_K(\Omega)$ is a continuous map.

Proof. First of all we remark that $\mathbb{F}$ is naturally endowed of a structure of Fréchet space thanks to the usual modulus $|\cdot|$ which acts as a norm.
$(1 \Rightarrow 2)$. Let $K \subseteq \Omega$ a compact set and $(f_n)_{n\in\mathbb{N}} \subseteq C^\infty_K(\Omega)$ such that $f_n \to 0$. By Lemma 3.15 this implies that:

$$P_j(f_n) = \sup_{x\in K, |\alpha|\le j} |\partial^\alpha f_n(x)| \to 0 \quad \forall j \in \mathbb{N}. \tag{18}$$

Thus:

$$\sup_{x\in K} |\partial^\alpha f_n(x)| \to 0 \quad \forall \alpha \in \mathbb{N}^d. \tag{19}$$

We recall from Definition 1.1 that by assumption there exist $C_K > 0$ and $m_K \in \mathbb{N}$ such that:

$$|T(f_n)| \le C_K \sum_{|\alpha|\le m_K} \sup_{x\in\Omega} |\partial^\alpha f_n(x)| \tag{20}$$

Now one remarks that the above summation is a finite sum. Moreover:

$$\mathrm{Supp}(\partial^\alpha f_n) \subseteq K \quad \forall n \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^d, \tag{21}$$

as a consequence:

$$\sup_{x\in\Omega} |\partial^\alpha f_n(x)| = \sup_{x\in K} |\partial^\alpha f_n(x)| \quad \forall n \in \mathbb{N},\ \forall \alpha \in \mathbb{N}^d. \tag{22}$$

Hence the right hand side of (20) converges to 0 as $n \to +\infty$, that is $T(f_n) \to 0$. This means that T is continuous in $0 \in C_K^\infty(\Omega)$ as a map from $(C_K^\infty(\Omega), (P_j)_{j\in\mathbb{N}})$ to $\mathbb{F}$. Finally the thesis follows from Proposition 3.17.
($2 \Rightarrow 1$). Let $K \subseteq \Omega$ be a compact set. By assumption and Proposition 3.17 there exist $C > 0$ and $j \in \mathbb{N}$ such that:

$$|T(f)| \leq CP_j(f) = C \sup_{x\in K, |\alpha|\leq j} |\partial^\alpha f(x)| \quad \forall f \in C_K^\infty(\Omega). \tag{23}$$

Setting $C_K = C$, $m_K = j$ and recalling that for all $f \in \mathcal{D}(\Omega)$ $\mathrm{Supp}(f) \subseteq K$ if and only if $f \in C_K^\infty(\Omega)$, one gets:

$$|T(f)| \leq C_K \sum_{|\alpha|\leq m_K} \sup_{x\in\Omega} |\partial^\alpha f(x)| \tag{24}$$

for all $f \in \mathcal{D}(\Omega)$ such that $\mathrm{Supp}(f) \subseteq K$, that is the thesis. □

Remark 6.2. A straightforward consequence of Lemma 6.1 is the following well known characterization of distributions.

Let $T : \mathcal{D}(\Omega) \to \mathbb{F}$ be a linear function. Then the two following conditions are equivalent.

1. *T is a distribution.*
2. *For every sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ such that:*
   - *there exists a compact set $K \subseteq \Omega$ such that $\mathrm{Supp}(f_n) \subseteq K$ for all $n \in \mathbb{N}$,*
   - *for all $\alpha \in \mathbb{N}^d$, $\partial^\alpha f_n \xrightarrow{n} 0$ uniformly,*

   one has that $T(f_n) \xrightarrow{n} 0$.

Indeed Item 2 amounts to state that the restriction of T on $C_K^\infty(\Omega)$ is continuous at 0 for every compact set $K \subseteq \Omega$.

7 A Result on Convergence of Distributions

Finally we can prove the following result.

Proposition 7.1. *Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}'(\Omega)$ a sequence of distributions on Ω such that for all $\varphi \in \mathcal{D}(\Omega)$ there exists in $\mathbb{F}$ $\lim_{n\to+\infty} T_n(\varphi)$. Then the functional:*

$$\begin{aligned} T : \mathcal{D}(\Omega) &\to \mathbb{F} \\ \varphi &\mapsto \lim_{n\to+\infty} T_n(\varphi) \end{aligned}$$

is a distribution on Ω.

Proof. The proof is in two steps.
Step 1. Let $K \subseteq \Omega$ be a compact set. According to Lemma 6.1 the sequence $(T_n)_{n\in\mathbb{N}}$ is a sequence of linear and continuous functions from the Fréchet space $C_K^\infty(\Omega)$ to the locally convex metrizable space $\mathbb{F}$. One notices that hypotheses of Proposition 5.3 are satisfied. Hence by Corollary 5.5 $T : C_K^\infty(\Omega) \to \mathbb{F}$ is continuous.
Step 2. Since Step 1 holds true for every $K \subseteq \Omega$, applying once more Lemma 6.1 T turns out to be a distribution. ☐

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