Leray (Weak solutions)

Mild solutions (Kato)

Moodle

Nonlinear Schrö-
dinger equations.

Scattering in Energy
space.

Cazenave

# Morawetz inequality.

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} f(x)\,dx$$

$$\mathcal{F}^* f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi x} f(\xi)\,d\xi$$

$\mathcal{F}, \mathcal{F}^*$ isometries in $L^2(\mathbb{R}^d)$

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}} \hat{f}(\xi)\, \hat{g}(\xi).$$

$f \in L^1(\mathbb{R}^d, \mathbb{R}^d)$

$\hat{f} = (\hat{f}_1, \dots, \hat{f}_d)$

$\partial_t u - \Delta u = 0$

$\Delta u = \sum_{j=1}^{d} \partial_j^2 u$

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u\big|_{t=0} = u_0 \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \end{cases}$$

$$0 = \partial_t \hat{u} - \widehat{\Delta u} = \begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} = 0 \\ \hat{u}\big|_{t=0} = \hat{u}_0 \end{cases}$$

$$e^{t|\xi|^2} \left( \partial_t \hat{u} + |\xi|^2 \hat{u} \right) = 0$$

$$\partial_t \left( e^{t|\xi|^2} \hat{u} \right) = 0$$

$$e^{t|\xi|^2} \hat{u}(t,\xi) = \hat{u}_0(\xi)$$

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{u}_0(\xi)$$

$$u(t,x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-t|\xi|^2} \hat{u}_0(\xi) e^{ix\xi} d\xi$$

$$\hat{G}(t,\xi) = e^{-t|\xi|^2}$$

$$G(t,x) = (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$$e^{-\varepsilon\frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-\frac{|x|^2}{2\varepsilon}} dx$$

$$\varepsilon = 2t$$

$$e^{-t|\xi|^2} = (2\pi)^{-\frac{d}{2}} \int e^{-i\xi x} \underbrace{(2t)^{-\frac{d}{2}} e^{-\frac{x^2}{4t}}}_{G(t,x)} dx$$

$$u(t,x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{G}(t,\xi)\, \hat{u}_0(\xi)\, e^{ix\xi} d\xi$$

$$\hat{u}(t,\xi) = \hat{G}(t,\xi)\, \hat{u}_0(\xi)$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \qquad G(t,\cdot) * u_0$$

$$u(t,x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G(t,x-y)\, u_0(y)\, dy$$

$$G(t,x) = (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$$= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy$$

1 Heat kernel $K_t(x-y)$

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}t}$$

$u(t,x) = K_t * u_0$ ~~$\neq$~~

$$u(t) = \underbrace{e^{t\Delta}}\, u_0 \left(= K_t * u_0\right)$$

<u>Teorem</u> For any $q \geq p \geq 1$

$$\| K_t * f \|_{L^q(\mathbb{R}^d)} \leq C_{qp}\, t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \| f \|_{L^p(\mathbb{R}^d)}$$

(~~t~~ this result false for $q < p$)

Pf Young's convolution inequality

$$\| K_t * f \|_{L^q} \leq \| K_t \|_{L^a} \| f \|_{L^p}$$

$$\frac{1}{q} + 1 = \frac{1}{a} + \frac{1}{p}$$

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$$\frac{1}{a} = \frac{1}{q} + 1 - \frac{1}{p}$$

$$= \frac{1}{(4\pi t)^{\frac{d}{2}}} \left| e^{-\frac{|x|^2}{4t}} \right|_{L^a} |f|_{L^p}$$

$$= \frac{1}{(4\pi t)^{\frac{d}{2}}} \left(t^{\frac{1}{2}}\right)^{+\frac{d}{a}} \left| e^{-\frac{|x|^2}{4}} \right|_{L^a} |f|_{L^p}$$

$$= \frac{\left| e^{-\frac{|x|^2}{4}} \right|_{L^a}}{(4\pi)^{\frac{d}{2}}} \; t^{-\frac{d}{2}\left(-\frac{1}{a}+1\right)} \; |f|_{L^p}$$

$$t^{\frac{d}{2}\left(\frac{1}{a}-1\right)} = t^{\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}$$

$$= t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

Sobolev Spaces based on $L^2(\mathbb{R}^d)$

$\xi \in \mathbb{R}^d$

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2} \quad \text{Japanese bracket}$$

$s \in \mathbb{R}$

$$H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) :$$
$$\langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^d)$$
$$\|f\|_{H^s} = \| \langle \xi \rangle^s \hat{f} \|_{L^2} \}$$

When $s \in \mathbb{N}$ this is the same as taking

$$\sum_{|\alpha| \le s} \|\partial_x^\alpha f\|_{L^2}$$

Homogeneous Sobolev spaces

$$\dot{H}^s(\mathbb{R}^d)$$

$u \in \mathcal{S}'(\mathbb{R}^d)$ s.t. $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$

and $\underbrace{\left| \, |\xi|^s \hat{u} \, \right|_{L^2}}_{|u|_{\dot{H}^s}} < +\infty$

$$\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$$

if and only if $s > -\frac{d}{2}$

$$\int_{\mathbb{R}} |\hat{u}(\xi)|^2 |\xi|^{2s} \, dx < +\infty$$

and here if $s \leq -\frac{d}{2}$ and $\hat{u}(0) \neq 0$

<u>Lemma</u> For $s > -\frac{d}{2}$ $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot{H}^s(\mathbb{R}^d)$

<u>Proposition</u> For $s < \frac{d}{2}$ then $\dot{H}^s(\mathbb{R}^d)$ is complete

(For $s \geq d/2$ $\dot{H}^s$ is not

complete. ( $s = \frac{d}{2}$ ) )

$\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi)$

Lemma $s < \frac{d}{2}$

1) $L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subseteq L^1_{loc}(\mathbb{R}^d, d\xi)$

2) $L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subseteq \mathcal{S}'(\mathbb{R}^d)$

$$\mathcal{S}'(\mathbb{R}^d) \xrightarrow{\mathcal{F}^*} \mathcal{S}'(\mathbb{R}^d)$$

$$L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \underset{\mathcal{F}}{\overset{\mathcal{F}^*}{\rightleftarrows}} \dot{H}^s(\mathbb{R}^d)$$

$g \in L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi)$

$g \in L^1_{loc}(\mathbb{R}^d \setminus \{0\}, d\xi)$

B unit disk center the origin

$$g = \hat{u}$$

$$\int_B |g(\xi)|\, d\xi =$$

$$= \int_B |\xi|^{s} |g(\xi)| \; |\xi|^{-s} d\xi$$

$$\leq \underbrace{\left( \int_{\mathbb{R}^d} |\xi|^{2s} |g(\xi)|^2 d\xi \right)^{\frac{1}{2}}}_{|u|_{\dot{H}^s}} \left( \int_B |\xi|^{-2s} d\xi \right)^{\frac{1}{2}}$$

this is finite

if $\quad 2s < d \qquad s <$

$$L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subseteq S'(\mathbb{R}^d)$$

$$g^\vee$$

$$g = \overbrace{g \chi_B}^{L^1(\mathbb{R}^d)} + \overbrace{(1-\chi_B) g}^{\in S'(\mathbb{R}^d)}$$

$$(1-\chi_B)\,g \in L^2(\mathbb{R}^d, \langle\xi\rangle^{2s}\,d\xi) \subseteq \mathcal{S}'(\mathbb{R}^d)$$

$f$

$$\int (1-\chi_B)\,g(x) f(x)\,dx =$$

$$= \int (1-\chi_B) g(x) \langle x\rangle^{s} \langle x\rangle^{-s} f(x)\,dx$$

$$\leq \left(\int (1-\chi_B)^2 \, |g(x)|^2 \langle x\rangle^{2s}\,dx\right)^{\frac{1}{2}}$$

$$\left(\int \langle x\rangle^{-2s} |f(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$g\chi_B \in L^1(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R})$$

$$|s| < \frac{d}{2}$$

$\dot{H}^s(\mathbb{R}^d)$ is a Hilbert

and contains $\mathcal{S}(\mathbb{R}^d)$

$$u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^d)$$

$$\operatorname{div} u = \nabla \cdot u = \sum_{j=1}^{d} \partial_j u_j$$

$$\widehat{\operatorname{div} u} = -i \sum_{j=1}^{d} \xi_j \hat{u}_j$$

$$\operatorname{div} u = 0 \iff \sum_{j=1}^{d} \xi_j \hat{u}_j = 0$$

$\mathbb{P}$ Leray projection

$$\widehat{\mathbb{P}u}^j = \hat{u}^j - \frac{1}{|\xi|^2} \sum_{k=1}^{d} \xi_j \xi_k \hat{u}^k$$

If $u$ is $\operatorname{div} u = 0 \iff \sum \xi_k \hat{u}^k = 0$

$$\mathbb{P} u = u$$

$$\sum_j \xi^j \widehat{\mathbb{P}u}^j = \sum_j \xi^j \hat{u}^j - \underbrace{\frac{1}{|\xi|^2} \sum_j \xi_j^2}_{=1} \sum_k \xi_k \hat{u}^k$$

$$= \sum_j \xi^j \hat{u}^j - \sum_k \xi_k \hat{u}^k = 0$$

$$H(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d, \mathbb{R}^d) : \operatorname{div} u = 0 \}$$

$$V(\mathbb{R}^d) = \{ u \in H^1(\mathbb{R}^d, \mathbb{R}^d) : \quad " \quad \}$$

$C_c^\infty(\mathbb{R}^d)$ = compactly supported $C^\infty$ functions

$$C_{c\sigma}^\infty(\mathbb{R}^d, \mathbb{R}^d) = \{ u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) : \operatorname{div} u = 0 \}$$

__Lemma__ $C_{c\sigma}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is dense in $H(\mathbb{R}^d)$ and in $V(\mathbb{R}^d)$ for $d = 2, 3$.

$V$ is closed in $\mathbb{R}^d$

$$\operatorname{div} : H^1(\mathbb{R}^d, \mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$

$$u \longrightarrow \operatorname{div} u$$

$V = \ker \operatorname{div}$

$H(\mathbb{R}^d)$ is closed in $L^2(\mathbb{R}^d)$