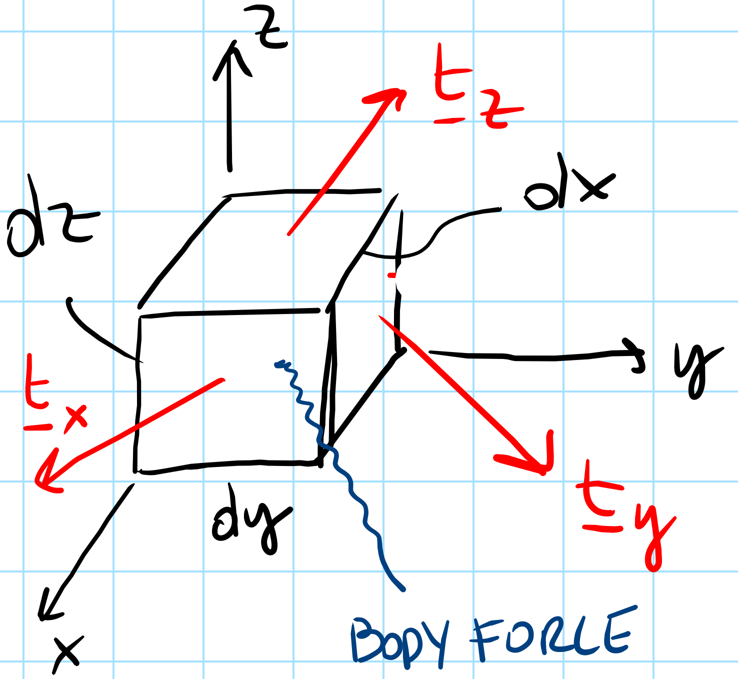
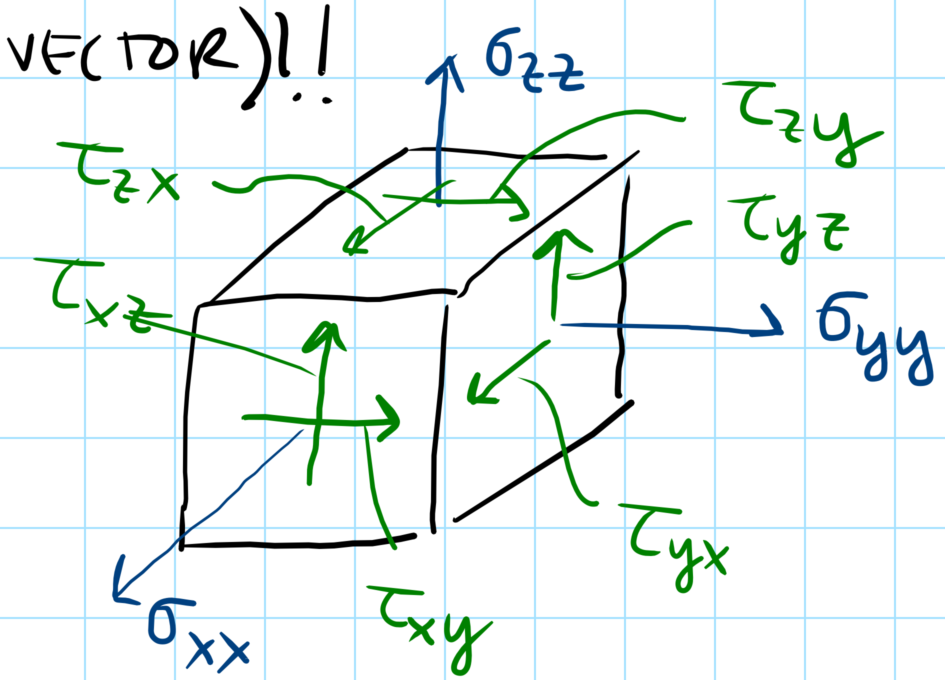
# CAUCHY'S THEOREM AND STRESS TENSOR

6/3/2025

WHAT DOES IT MEAN "TO KNOW THE STRESS STATE AT A POINT"?

AT A POINT 'P', WE KNOW THE STRESS STATE IF THE STRESS VECTORS $\underline{t}(P,\underline{m})$ ARE KNOWN FOR EVERY $\underline{m}$ (THIS MEANS TO KNOW AN INFINITE NUMBER OF VECTOR)!!




WE CAN SHOW by BALANCE OF MOMENTS THAT

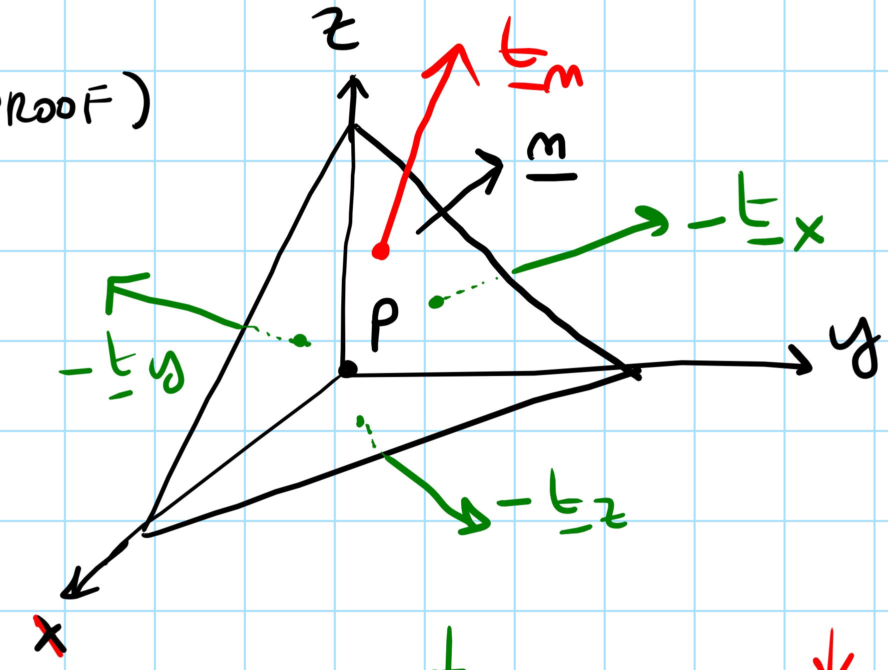
$$\left.\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned}\right\} \text{COMPLEMENTARITY OF SHEAR STRESSES}$$

# CAUCHY'S THEOREM (TETRAHEDRON THEOREM)

UNDER THE HYPOTHESIS THAT FUNCTIONS $\underline{b}(P)$ and $\underline{f}(P)$ ARE BOUNDED

IN $P \in \Omega$, $\underline{t}(P, \underline{m})$ $\forall \underline{m}$ ARE KNOWN IF THE TRACTION VECTORS RELATED TO 3 ORTHOGONAL PLANES ARE KNOWN (IN THE NEIGHBORHOOD OF P).

[PROOF]




WITH EQUIL. OF FORCES $\Rightarrow$

$$\underline{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

$$\boxed{\underline{t}_m = \underline{t}_x m_x + \underline{t}_y m_y + \underline{t}_z m_z}$$

$$\left[\underline{t}_m\right. = \begin{bmatrix} t_{xx} \\ t_{xy} \\ t_{xz} \end{bmatrix} m_x + \begin{bmatrix} t_{yx} \\ t_{yy} \\ t_{yz} \end{bmatrix} m_y + \begin{bmatrix} t_{zx} \\ t_{zy} \\ t_{zz} \end{bmatrix} m_z = \begin{bmatrix} \downarrow & \downarrow & \downarrow \\ & & \\ & & \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = [\underset{\sim}{\sigma}][\underline{m}] = \boxed{\underset{\sim}{\sigma}\, \underline{m}}$$

STRESS TENSOR

LET US FOCUS ON THE COMPONENTS OF $[\underset{\sim}{\sigma}]$

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \boxed{\tau_{zx}} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \boxed{\tau_{xz}} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \end{bmatrix}$$

6 INDEPENDENT COMPONENTS

$\Rightarrow \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$

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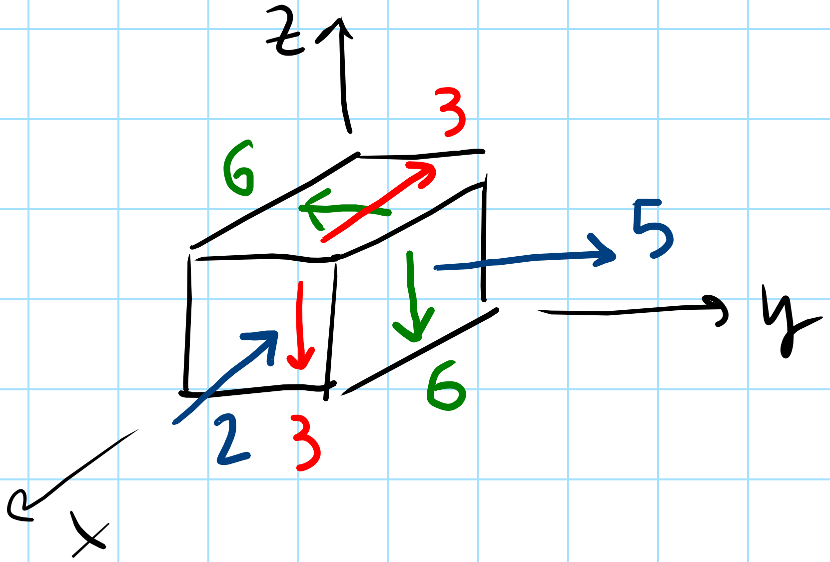
EX




$$[\underset{\sim}{\sigma}] = \begin{bmatrix} -2 & 0 & -3 \\ 0 & 5 & -6 \\ -3 & -6 & 0 \end{bmatrix} \text{ MPa}$$

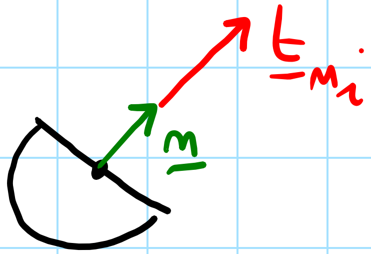
WHAT IS $\underline{t}_q$ WITH $\underline{q} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ : $\underline{t}_q = \underset{\sim}{\sigma}\,\underline{q} \Rightarrow [\underline{t}_q] = \begin{bmatrix} -2 & 0 & -3 \\ 0 & 5 & -6 \\ -3 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -5/\sqrt{3} \\ -1/\sqrt{3} \\ -9/\sqrt{3} \end{bmatrix}$ MPa

$$\sigma_{qq} = \underline{t}_q \cdot \underline{q} = \frac{1}{3}\left(-5 + (-1) + (-9)\right) = -5 \text{ MPa}$$

# EIGENVALUES AND EIGENVECTORS OF $\underset{\sim}{\sigma}$

$\sigma_i$ IS AN EIGENVALUE AND $\underline{m}_i$ IS AN ASSOCIATED EIGENVECTOR OF $\underset{\sim}{\sigma}$ IF:

$$\boxed{\underset{\sim}{\sigma}\,\underline{m}_i = \sigma_i\,\underline{m}_i} = \underline{t}_{m_i}$$




$\sigma_i$ : NORMAL COMPONENT

$\sigma_i$ : PRINCIPAL STRESS ; $\underline{m}_i$ : PRINCIPAL DIRECTION (OF STRESS)

$\Rightarrow$ 3 PRINCIPAL STRESSES , 3 PRINCIPAL DIRECTIONS

$\underset{\sim}{\sigma}$ IS REAL AND SYMMETRIC $\rightarrow$ $\sigma_i$ $(i=1,2,3)$ ARE REAL NUMBERS

$\rightarrow$ $\underline{m}_i$ ( ) ARE $\perp$

INVARIANTS OF STRESS

$$(\underset{\sim}{\sigma} - \sigma_i \underset{\sim}{I})\,\underline{m}_i = \underline{0}\,, \quad \det(\underset{\sim}{\sigma} - \sigma_i \underset{\sim}{I}) = 0 \Rightarrow \sigma_i^3 - I_1^{\sigma}\sigma_i^2 + I_2^{\sigma}\sigma_i - I_3^{\sigma} = 0$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$I_1^{\sigma} = \operatorname{tr}\underset{\sim}{\sigma}\,; \quad I_3^{\sigma} = \det\underset{\sim}{\sigma}\,; \quad I_2^{\sigma} = \frac{1}{2}\left[(\operatorname{tr}\underset{\sim}{\sigma})^2 - \operatorname{tr}\underset{\sim}{\sigma}^2\right]$$

SO, W.R.T. $\{\underline{n}_1, \underline{n}_2, \underline{n}_3\} \Rightarrow \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix}$ (SHEAR STRESSES $= 0$)

$$\sigma_1 = \max_{\underline{n}} \{\sigma_{nn}\}$$

$$\sigma_3 = \min_{\underline{n}} \{\sigma_{nn}\}$$

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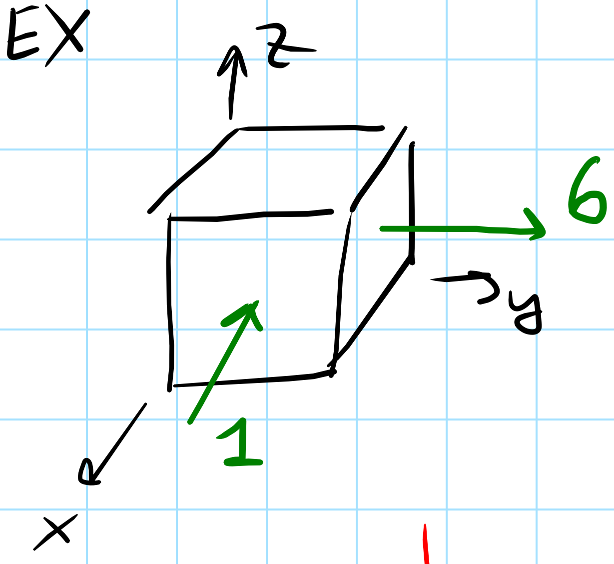
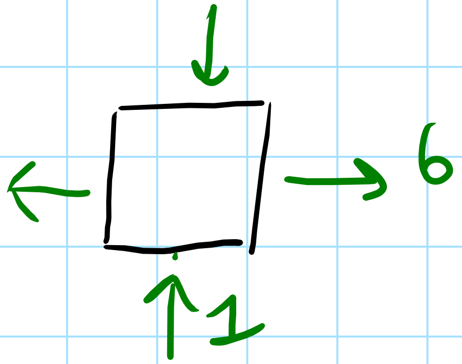
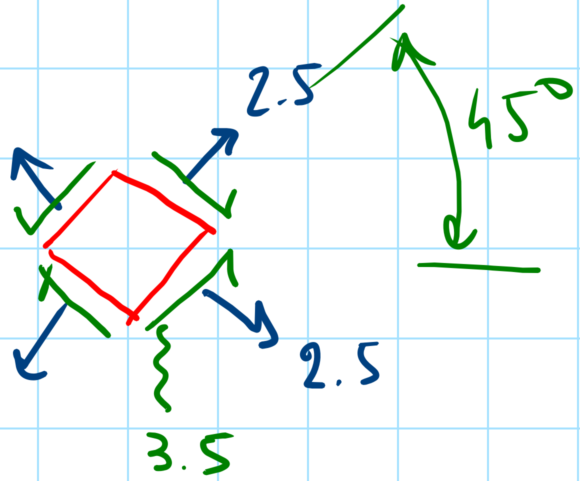
EX




$\Rightarrow$




(MAXIMUM SHEAR STRESS)

$\sigma_{xx} = -1$

$\sigma_{yy} = +6$

$\sigma_1 = +6$

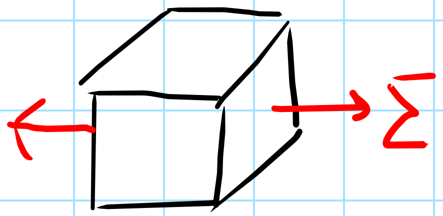
$\sigma_2 = 0$

$\sigma_3 = -1$

# RELEVANT STRESS STATES

## – UNIAXIAL STRESS STATE

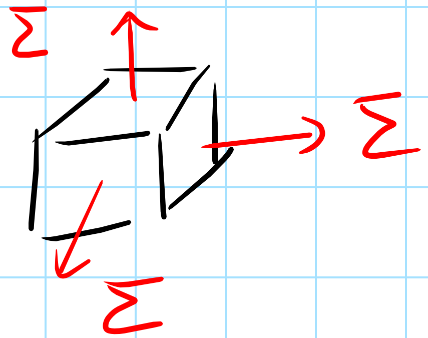
$$\sigma_1 = \Sigma, \quad \sigma_2 = \sigma_3 = 0$$




$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## – HYDROSTATIC STRESS STATE

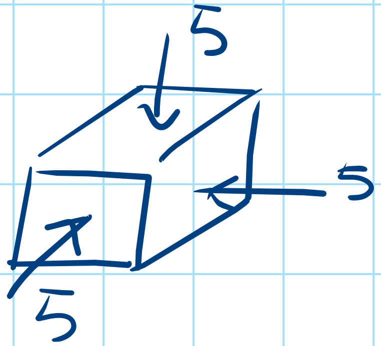
$$\sigma_1 = \sigma_2 = \sigma_3 = \Sigma$$




$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \Sigma & & 0 \\ & \Sigma & \\ 0 & & \Sigma \end{bmatrix}$$

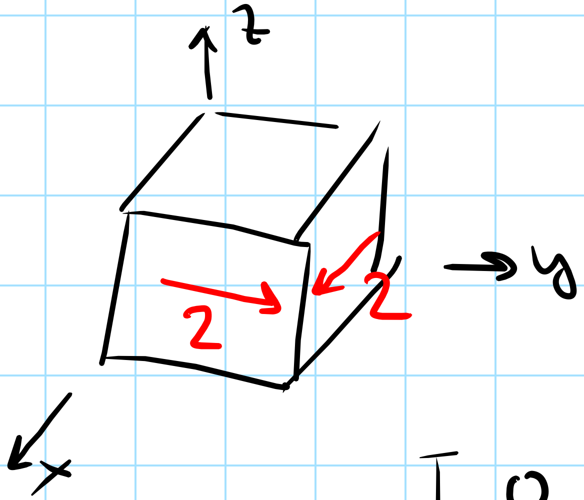
$$\underset{\sim}{\sigma} = \Sigma \underset{\sim}{I}$$

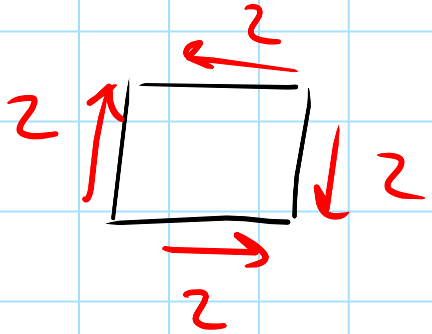
EX

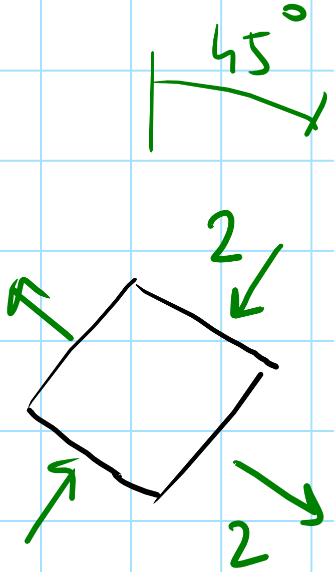



$$\underset{\sim}{\sigma} = -5 \underset{\sim}{I}$$

## – PURE SHEAR




$$[\underset{\sim}{\sigma}] = \begin{bmatrix} 0 & +2 & 0 \\ +2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|\underset{\sim}{\sigma} - \lambda \underset{\sim}{I}| =$$

$$\begin{vmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) - 2(-2\lambda) =$$

$$-\lambda^3 + 4\lambda = 0 \quad \begin{cases} \lambda = 0 \\ \lambda^2 = \pm 2^2 \end{cases}$$