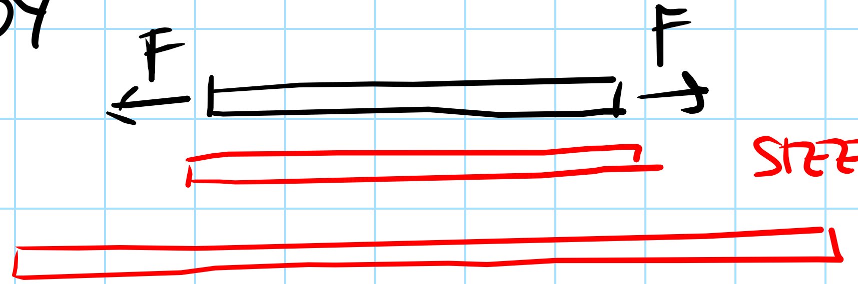
LINEAR ELASTICITY (SPECIAL CASE OF CONSTITUTIVE EQUATION)

CONSTITUTIVE EQUATION: RELATIONSHIP BETWEEN $\underset{\sim}{\sigma}$ AND $\underset{\sim}{\varepsilon}$ AT A POINT OF A BODY

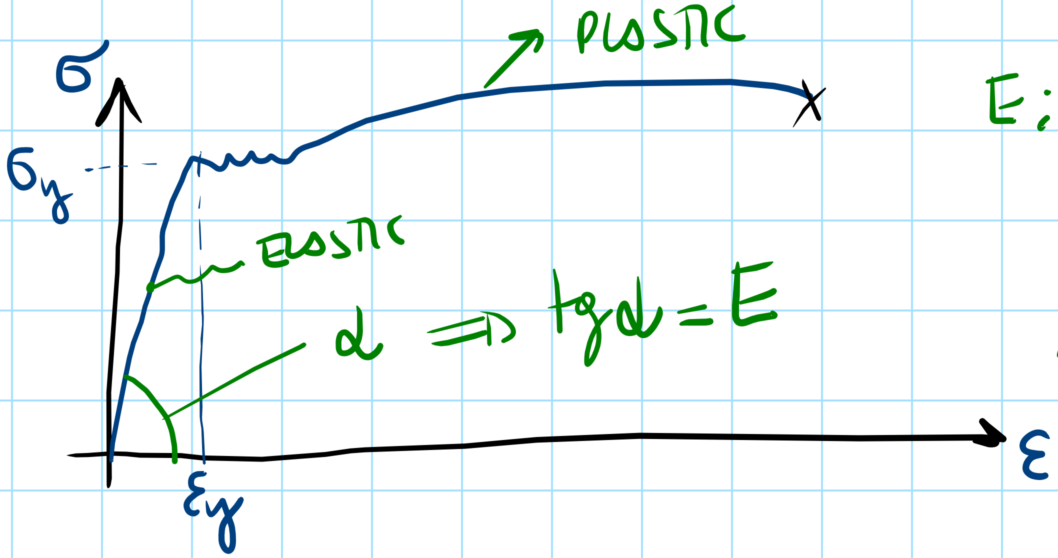


STEEL ↑ STIFFER

ELASTOMER ↓ LESS STIFF

INPUT: SAME FORCE
DIFFERENT STRAIN

UNIAXIAL TEST (STEEL)



E: YOUNG'S MODULUS (1776)

STEEL 210 GPa

ALUMINUM 70 GPa

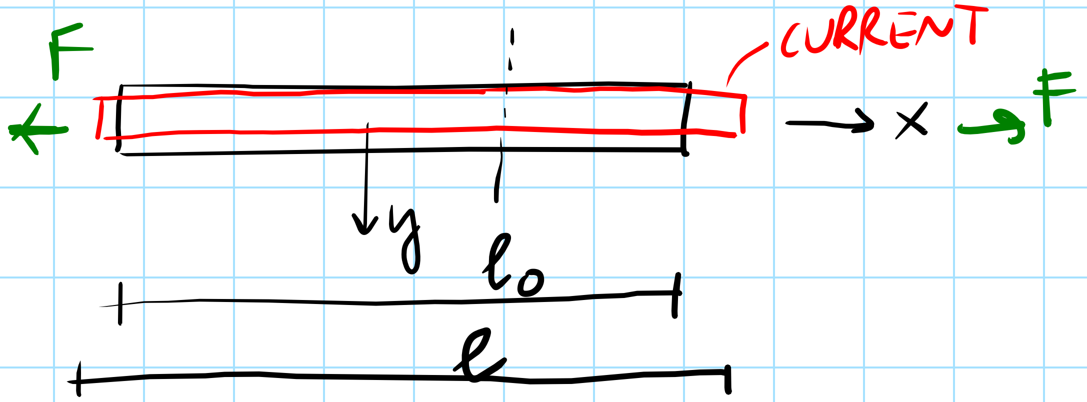
$\sigma = E\varepsilon$

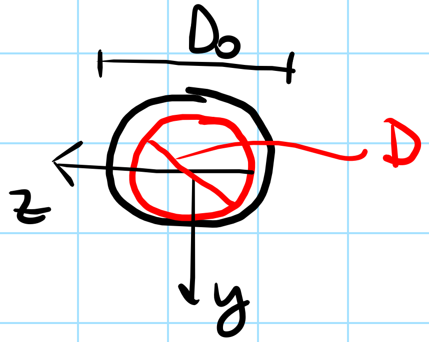
EX

DATA: $\sigma_y = 420$ MPa

$E = 210$ GPa

UNKNOWN: $\varepsilon_y = \dfrac{\sigma_y}{E} = \dfrac{420}{210 \cdot 10^3} = 2 \cdot 10^{-3}$





$$\varepsilon_{xx} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} > 0$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \frac{D - D_0}{D_0} < 0$$

$\nu > 0$

$$\boxed{\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}}$$

POISSON'S RATIO

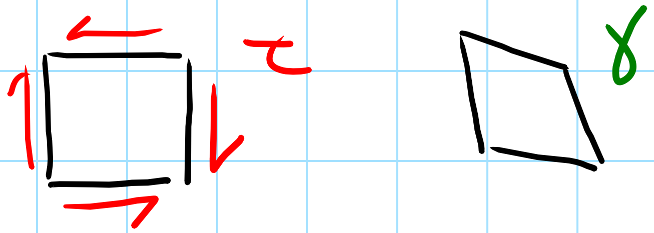
(DEGREE OF SHRINKING)

$\nu = 0.25 \div 0.35$ STEEL

$\nu \to 0.5$ RUBBER (INCOMPRESSIBLE)

$\nu < 0$ AUXETIC MATERIALS

ANOTHER MODULUS IS THE SHEAR MODULES



$\tau = G\gamma$ IN LINEAR ISOTROPIC ELASTICITY

$\tau = \mu\gamma$

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

$$\lambda = \frac{\nu E}{(1+2\nu)(1+\nu)}$$

OTHER LAMÉ PARAM. IN L. ISO ELAST.

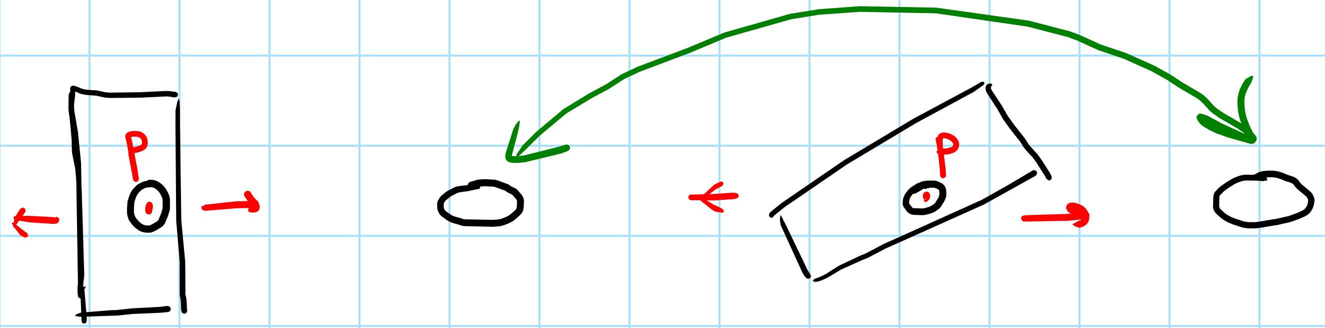
THE MOST GENERAL DEFINITION OF ELASTICITY AT A POINT IS THE FOLLOWING:

$$\underset{\sim}{\sigma}(t) = g\left(\underset{\sim}{\varepsilon}(t)\right)$$

RESPONSE FUNCTION OF THE MATERIAL

- ELASTIC HOMOGENEOUS BODY: $g(\ )$ IS INDEPENDENT OF THE POINT

- ISOTROPIC MATERIAL AT A POINT: $g(\ )$ IS INDEPENDENT OF DIRECTION



SAME DEFORMATION FOR THE TWO TESTS

THE MOST GENERAL FORM OF LINEAR ELASTICITY

$\underset{\sim}{\sigma} = \mathbb{C}\,\underset{\sim}{\varepsilon}$ (*) $\qquad \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

$\uparrow$ 4-TH ORDER TENSOR (ELASTIC TENSOR

• HOW MANY CONSTANTS AT MOST IN C_{ijkl}

AS $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$, $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^T$ $\Rightarrow$ C_{ijkl} HAS

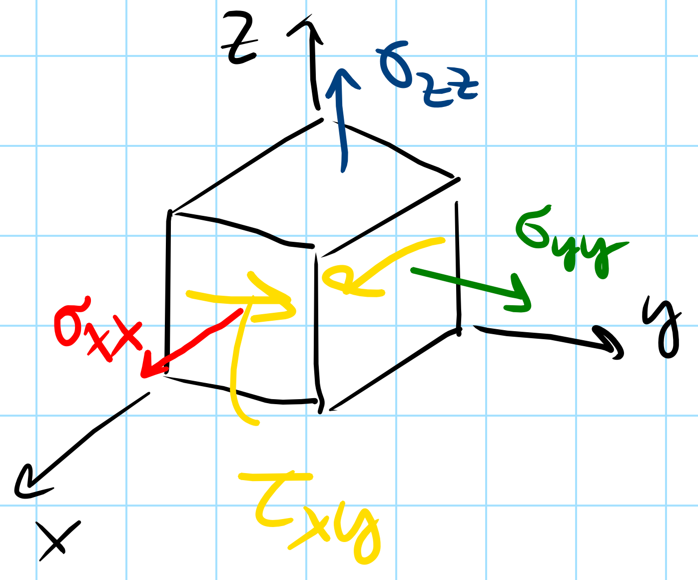
36 INDEPENDENT CONSTANTS $C_{ijkl} = C_{jikl} = C_{ijlk}$ ...

$\Rightarrow$ ISOTROPIC LINEAR ELASTICITY : 2 (E, ν) but also (G, ν) or (λ, μ)

IF $\mathbb{C}^{-1}$ EXISTS, THEN THE RELATIONSHIP (*) IS INVERTIBLE : $\underset{\sim}{\varepsilon} = \mathbb{C}^{-1}\underset{\sim}{\sigma}$

SURELY, $\mathbb{C}^{-1}$ EXISTS IF $\mathbb{C}$ IS POSITIVE DEFINITE

ISOTROPIC LINEAR ELASTICITY (ENGINEERING VIEWPOINT) (GENERALIZED HOOKE'S LAW)



$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right]$$

$$\varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{zz})\right]$$

$$\varepsilon_{zz} = -\nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \;;\; \gamma_{xz} = \frac{\tau_{xz}}{G} \;;\; \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$(2\varepsilon_{xy} = \gamma_{xy})$$

$$\boxed{\underset{\sim}{\varepsilon} = \frac{1}{E}\left[(1+\nu)\underset{\sim}{\sigma} - \nu(\mathrm{tr}\,\underset{\sim}{\sigma})\underset{\sim}{I}\right]}$$

HOOKE'S LAW

$$\varepsilon_{xx} = \frac{1}{E}\left[(1+\cancel{\nu})\sigma_{xx} - \nu(\cancel{\sigma_{xx}}+\sigma_{yy}+\sigma_{zz})\right]$$

$$\varepsilon_{xy} = \frac{1}{E}\left[(1+\nu)\tau_{xy}\right] \Rightarrow \gamma_{xy} = \underbrace{\frac{2(1+\nu)}{E}}_{G}\tau_{xy}$$

ONE CAN PROOF THAT THE INVERSE RELATIONSHIP CAN BE WRITTEN

$$\underset{\sim}{\sigma} = 2G\underset{\sim}{\varepsilon} + \lambda(\mathrm{tr}\,\underset{\sim}{\varepsilon})\underset{\sim}{I}$$

LAMÉ CONSTANTS

VOIGT REPRESENTATION OF LINEAR ELASTICITY (ISOTROPIC)

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/2G & & \\ & 0 & & & 1/2G & \\ & & & & & 1/2G \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

$$\underline{\varepsilon} \qquad \mathcal{C}^{-1} \qquad \underline{\sigma}$$

$\underline{\sigma}, \underline{\varepsilon}$: ALGEBRAIC VECTORS (6 ELEMENTS)

LET US STUDY RESTRICTIONS ON MATERIAL PARAMETERS ASSUMING $\mathcal{C}^{-1}$ DEFINITE POSITIVE.

① $\frac{1}{E} > 0 \longrightarrow \boxed{E > 0}$

② $\frac{1}{E^2} - \frac{\nu^2}{E^2} > 0 \; ; \; 1 - \nu^2 > 0 , \; \nu^2 < 1 \longrightarrow -1 < \nu < 1$

③ $\frac{1}{E^3} (1+\nu)^2 (1-2\nu) > 0 \; ; \; 1 - 2\nu > 0 \longrightarrow \nu < \frac{1}{2}$

$$\boxed{-1 < \nu < \frac{1}{2}}$$

PLAYING WITH THE HOOKE' LAW WE CAN SHOW THAT :

$$\mathrm{tr}\,\underset{\sim}{\varepsilon} = \frac{1}{E}(1-2\nu)\,\mathrm{tr}\,\underset{\sim}{\sigma}$$

$$= \underbrace{\frac{1}{E}(1-2\nu)\,3}_{1/K}\,\sigma_m$$

CHANGE IN VOLUME

MEAN TENSION

$$\sigma_m = K\,\mathrm{tr}\,\underset{\sim}{\varepsilon}$$

BULK MODULUS

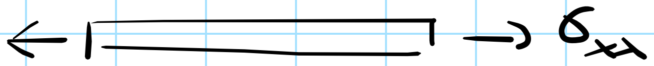
$$K = \frac{E}{3(1-2\nu)}$$

$\nu \to \frac{1}{2}$, $K \to \infty$: $\sigma_m \uparrow$ but $\mathrm{tr}\,\underset{\sim}{\varepsilon} \to 0$

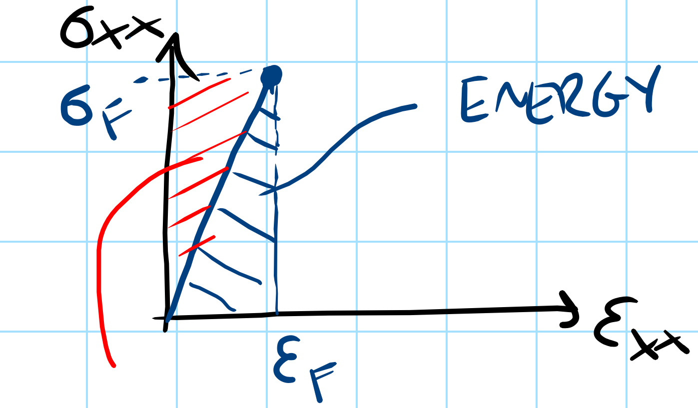
INCOMPRESSIBLE MATERIAL

SOME CONSIDERATIONS ON ENERGY: THE ELASTIC POTENTIAL

1D : $\sigma_{xx} \longrightarrow \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz}$



$\sigma_{xx} = E\,\varepsilon_{xx}$



ENERGY DENSITY (PER UNIT VOLUME): ELASTIC POTENTIAL

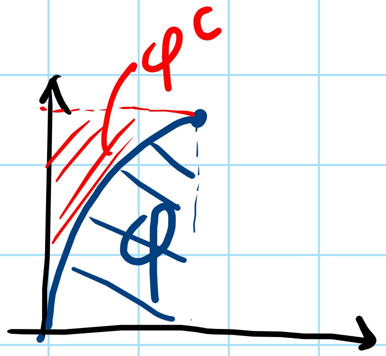
$$\varphi = \frac{1}{2}\sigma_F\varepsilon_F = \frac{1}{2}E\varepsilon_F^2 = \frac{1}{2}\frac{\sigma_F^2}{E}$$

$$\left[\begin{array}{l} \varphi(\varepsilon) = \frac{1}{2}E\varepsilon_F^2 \\ \varphi^c(\sigma) = \frac{1}{2}\frac{\sigma_F^2}{E} \end{array}\right]$$

COMPLEMENTARY ELASTIC POTENTIAL φ^c

$$\boxed{\varphi + \varphi^c = \sigma_F\varepsilon_F}$$

(IMAGINE A 1D TEST IN A NONLINEAR ELASTIC MATERIAL:



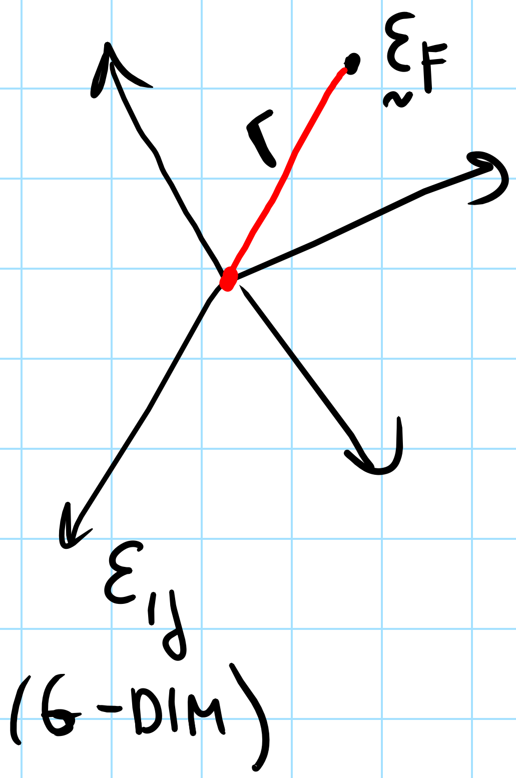
)

$$\varphi^c = \varphi - \sigma_F\varepsilon_F$$

$$\varphi^c(\sigma_F) = \varphi(\varepsilon_F) - \sigma_F\varepsilon_F$$

IN GENERAL, IMAGINE TO REACH AT A CERTAIN STAGE, THE STATUS $(\underset{\sim}{\sigma}_F, \underset{\sim}{\varepsilon}_F)$

WHAT IS $\varphi(\underset{\sim}{\varepsilon}_F)$?



$$\varphi(\underset{\sim}{\varepsilon}_F) = \int_\Gamma \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}) \cdot d\underset{\sim}{\varepsilon} \longrightarrow \frac{1}{2} \underbrace{\mathbb{C}\, \underset{\sim}{\varepsilon}_F}_{\underset{\sim}{\sigma}_F} \cdot \underset{\sim}{\varepsilon}_F$$