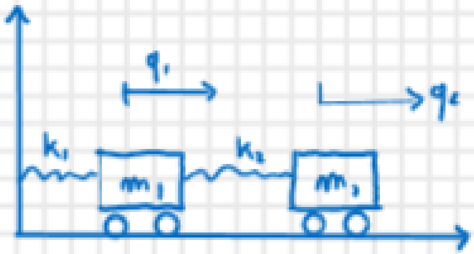
Titolo:

Riassunto Oscillazioni libere sistemi n gradi di libertà



$$\begin{cases} m_1 \ddot{q}_1(t) + (k_1 + k_2) q_1(t) - k_2 q_2(t) = 0 \\ m_2 \ddot{q}_2(t) - k_2 q_1(t) + k_2 q_2(t) = 0 \end{cases}$$

$$[A]\{\ddot{q}(t)\} + [C]\{q(t)\} = \{0\}$$

$$\downarrow$$

- $\{q\} = \sum_{i=1}^{n} \{\psi^i\} C_i \,\mathrm{sen}(\omega_i t + \varphi_i)$

$$\{q\} = \sum_{i=1}^{n} \{\psi^i\} (A_i \,\mathrm{sen}\, \omega_i t + B_i \cos \omega_i t)$$

$$\downarrow$$

$$\left(-\omega_i^2 [A] + [C]\right)\{\psi^i\} = \{0\}$$

$$\downarrow$$

$$\det\left|-\omega_i^2 [A] + [C]\right| = 0 \longrightarrow \omega_i^2$$

$$\omega_1^2 \le \omega_2^2 \le \dots \le \omega_n^2$$

Se il sistema oscillasse secondo i esimo modo di vibrare

$$\{q\} = \{\psi^i\}(A_i \,\mathrm{sen}\, \omega_i t + B_i \cos \omega_i t)$$

$$X = \begin{bmatrix} \psi_1^1 & \cdot & \cdot & \psi_1^n \\ \psi_2^1 & & & \psi_2^n \\ \vdots & & & \\ \psi_n^1 & & & \psi_n^n \end{bmatrix}$$

Per ciascuna colonna abbiamo gli autovettori

Titolo:

Considerazioni energetiche

- $\{q\} = \{\psi^i\} \operatorname{sen} \omega_i t$
- $\{\dot{q}\} = \omega_i \{\psi^i\} \cos \omega_i t$

- Fissiamo $C_i = 1$, $\varphi_i = 0$
- Vibra secondo i modi

$$U = \frac{1}{2} \{q\}^T [C] \{q\} = \{\psi^i\}^T [C] \{\psi^i\} \operatorname{sen}^2 \omega_i t$$

$$T = \frac{1}{2} \{\dot{q}\}^T [A] \{\dot{q}\} = \omega_i^2 \{\psi^i\}^T [A] \{\psi^i\} \cos^2 \omega_i t$$

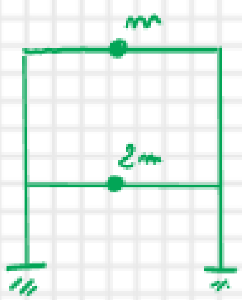
Non abbiamo dissipazione di energia $\quad T + U = \text{cost}$

$$T_{max} + 0 = U_{max} + 0$$

$$\omega_i^2 \{\psi^i\}^T [A] \{\psi^i\} = \{\psi^i\}^T [C] \{\psi^i\} \Rightarrow \omega_i^2 = \frac{\{\psi^i\}^T [C] \{\psi^i\}}{\{\psi^i\}^T [A] \{\psi^i\}}$$

Rapporto di Rayleig $\Rightarrow \dfrac{\{q^T\} [C] \{q\}}{\{q\}^T [A] \{q\}}$

Esempio



$$\omega_1^2 = \frac{2-\sqrt{2}}{2} \frac{K}{m} \qquad \{\psi^1\} = \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix}$$

$$\omega_2^2 = \frac{2+\sqrt{2}}{2} \frac{K}{m} \qquad \{\psi^2\} = \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}$$

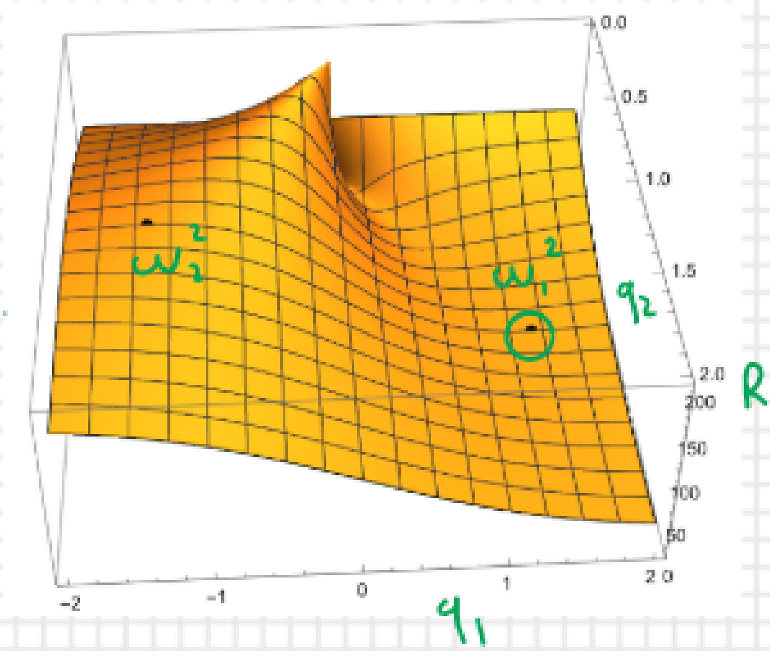
$$m = 10\, t \frac{s^2}{m} \qquad k = 1000\, t/m$$

$$\omega_1^2 = 29,29 \left(\frac{rad}{s}\right)^2 \qquad \omega_2^2 = 171,71 \left(\frac{rad}{s}\right)^2$$

Titolo:

Rapporto di Rayleigh $= \dfrac{\{q_1\ q_2\}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} k \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}}{\{q_1\ q_2\}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}} =$

$$= 100\ \frac{2q_1^2 + q_2^2 - 2q_1q_2}{2q_1^2 + q_2^2}$$



Ortogonalità dei modi di vibrare

- $([C] - \omega_i^2[A])\{\psi^i\} = \{0\}$ $\quad i$
- $([C] - \omega_j^2[A])\{\psi^j\} = \{0\}$ $\quad j$ $\qquad i \neq j$

$\{\psi^j\}^T([C] - \omega_i^2[A])\{\psi^i\} = \{0\}$ $\quad$ 1)

$\{\psi^i\}^T([C] - \omega_j^2[A])\{\psi^j\} = \{0\}$ $\quad$ 2)

2 trasposta $\quad \{\psi^j\}^T([C] - \omega_j^2[A]\{\psi^i\} = \{0\}$

Titolo:

Differenza tra la 1 e la 2^T

$$(-\omega_i^2 + \omega_j^2)\{\psi^j\}^T[A]\{\psi^i\} = \{0\}$$

$i \neq j \quad \omega_i \neq \omega_j$

- $\{\psi^j\}^T[A]\{\psi^i\} = \{0\}$ 3) condizione di ortogonalità

Sostituendo la 3 in 1

- $\{\psi^j\}^T[C]\{\psi^i\} = \{0\}$

Gli autovettori $\{\psi\}$ sono ortogonali rispetto alle matrici $[A]$ e $[C]$

Consideriamo il caso $i = j$

$\{\psi^i\}^T[A]\{\psi^i\} = m_i^* \neq 0$

$\{\psi^i\}^T[C]\{\psi^i\} = k_i^* \neq 0$

massa equivalente a quella di un sistema ad 1 GL con pulsazione ω_i

sistema 1 GL $\omega^2 = \frac{k}{m} \Rightarrow \omega_i^2 = \frac{k_i^*}{m_i^*} = \frac{\{\psi^i\}^T[C]\{\psi^i\}}{\{\psi^i\}^T[A]\{\psi^i\}}$

Titolo:

Es.

$$\{\varphi^2\}^T \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \{\varphi^1\} = 0 \qquad \{\varphi^2\}^T [\ C\ ] \{\varphi^1\} = 0$$

$$\{1\ \ -\sqrt{2}\} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 0 \qquad \{1\ \ -\sqrt{2}\} [\ C\ ] \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 0$$

$$\{\varphi^1\}^T \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \{\varphi^1\} = m_1^* \qquad \{\varphi^1\}^T [\ C\ ] \{\varphi^1\} = K_1^*$$

$$\{1\ \ +\sqrt{2}\} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 4m \qquad \{1\ \ +\sqrt{2}\} [\ C\ ] \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 1{,}17\,K$$

$$\{\varphi^2\}^T \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \{\varphi^2\} = m_2^* \qquad \{\varphi^2\}^T [\ C\ ] \{\varphi^2\} = K_1^*$$

$$\{1\ \ -\sqrt{2}\} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix} = 4m \qquad \{1\ \ -\sqrt{2}\} [\ C\ ] \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix} = 6{,}82\,K$$

$$\omega_1^2 = \frac{K_1^*}{m_1^*} = \frac{1{,}17\,K}{4m} \simeq 29{,}283 \qquad \omega_2^2 = \frac{6{,}82\,k}{4\,m} \simeq 170{,}71$$

Disaccoppiamento delle eq. del moto

- Cambio di variabili $\longrightarrow$ nuove coordinate

$$\{q\} = \sum_{i=1}^{m} \{\varphi^i\} p_i = [X]\{p\}$$

$p \rightarrow$ coordinate principali

$$q_1 = \varphi_1^1 p_1 + \varphi_1^2 p_2 + \dots + \varphi_1^m p_m$$

$$q_2 = \varphi_2^1 p_1 + \varphi_2^2 p_2 + \dots + \varphi_2^m p_m$$

$$\vdots$$

$$q_m = \varphi_m^1 p_1 + \varphi_m^2 p_2 + \dots + \varphi_m^m p_m$$

$$\{p\} = [X]^{-1}\{q\}$$

$[X]$ invertibile non singolare

Titolo:

$$\{q\}=[X]\{p\} \longrightarrow \{\dot q\}=[X]\{\dot p\}$$

Eq. moto

$$[A]\{\ddot q\}+[C]\{q\}=\{0\}$$

$$\underbrace{[X]^T[A][X]}_{[L]}\{\ddot p\}+\underbrace{[X]^T[C][X]}_{[N]}\{p\}=\{0\}$$

$$\begin{bmatrix} \psi_1^1 & \psi_2^1 & \cdot & \psi_n^1 \\ \cdot & \cdot & \cdot & \cdot \\ \psi_1^n & \psi_2^n & \cdot & \psi_n^n \end{bmatrix} \Big[A \Big] \begin{bmatrix} \psi_1^1 & \cdot & \psi_1^n \\ \psi_2^1 & \cdot & \psi_2^n \\ \psi_n^1 & \cdot & \psi_n^n \end{bmatrix} = \overset{L}{\begin{bmatrix} m_1^* & 0 & \cdot & 0 \\ 0 & m_2^* & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & m_n^* \end{bmatrix}}$$

$$l_{ij}=\begin{cases} 0 & i\neq j \\ m_i^* & i=j \end{cases}$$ $[L]$ matrice diagonale

$$\begin{bmatrix} \psi_1^1 & \psi_2^1 & \cdot & \psi_n^1 \\ \cdot & \cdot & \cdot & \cdot \\ \psi_1^n & \psi_2^n & \cdot & \psi_n^n \end{bmatrix} \Big[C \Big] \begin{bmatrix} \psi_1^1 & \cdot & \psi_1^n \\ \psi_2^1 & \cdot & \psi_2^n \\ \psi_n^1 & \cdot & \psi_n^n \end{bmatrix} = \overset{N}{\begin{bmatrix} k_1^* & 0 & \cdot & 0 \\ 0 & k_2^* & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & k_n^* \end{bmatrix}}$$

$$m_{ij}=\begin{cases} 0 & i\neq j \\ k_i^* & i=j \end{cases}$$ $[N]$ matrice diagonale

Titolo:

$$m_1^* \ddot{p}_1(t) + k_1^* p_1(t) = 0$$

$$\vdots$$

$$m_n^* \ddot{p}_n(t) + k_n^* p_n(t) = 0$$

$$[X]^T [A] [X] \{\ddot{p}\} + [X]^T [C] [X] \{p\} = \{0\}$$

$$[L] \{\ddot{p}\} + [N] \{p\} = \{0\}$$

$$[L]^{-1} [L] \{\ddot{p}\} + [L]^{-1} [N] \{p\} = \{0\}$$

$$\{\ddot{p}\} + [\Omega^2] \{p\} = 0 \qquad [\Omega^2] = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \omega_n^2 \end{bmatrix}$$

$$\frac{k_1^*}{m_1^*} \qquad \frac{k_2^*}{m_2^*}$$

Soluzione del sistema

$$p_1(t) = A_1 \operatorname{sen} \omega_1 t + B_1 \cos \omega_1 t$$

$$p_2(t) = A_2 \operatorname{sen} \omega_2 t + B_2 \cos \omega_2 t \qquad \Rightarrow \qquad \{q\} = [X] \{p\}$$

$$\vdots$$

$$p_n(t) = A_n \operatorname{sen} \omega_n t + B_n \cos \omega_n t$$

Sistemi smorzati

$$[A] \{\ddot{q}\} + [B] \{\dot{q}\} + [C] \{q\} = \{0\}$$

$$[X]^T [A] [X] \{\ddot{q}\} + [X]^T [B] [X] \{\dot{q}\} + [X]^T [C] [X] \{q\} = 0$$

$$[L] \{\ddot{q}\} + [M] \{\dot{q}\} + [N] \{q\} = \{0\}$$