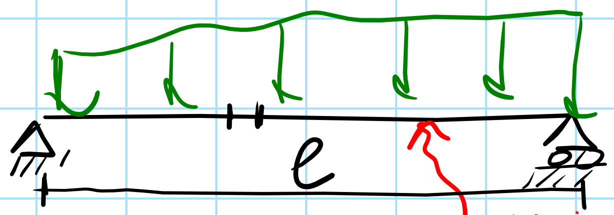
DYNAMICS OF CONTINUOUS SYSTEMS MODELLED AS A GENERALISED 1 D.O.F.



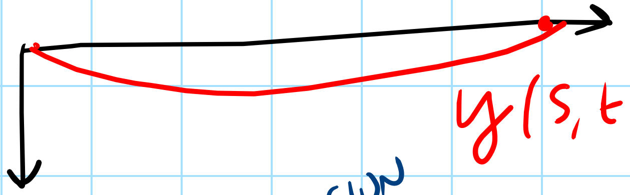
$q(s,t)$: EXTERNAL DYNAMIC LOAD

s

ℓ

EJ : BENDING STIFFNESS

μ : MASS PER UNIT LENGTH



$y(s,t)$: DYNAMIC "ELASTIC LINE"

DIMENSIONLESS

DISPL. (UNKNOWN)

$$y(s,t) = \underbrace{\psi(s)}_{\text{SHAPE FUNCTION}}\, x(t)$$

SHAPE FUNCTION (MUST FULFILL SOME CONDITIONS)

$\dot{y} = \psi(s)\,\dot{x}(t)$: SEPARAT. OF VARIABLES

$\chi = -y'' = -\psi''(s)\,x(t)$: CURVATURE

$M(s,t) = -EJ\,y'' = -\psi''(s)\,x(t)\,EJ$

HOW TO DETERMINE THE EQ. OF MOTION

$$\frac{d}{dt}\left[E_{KIN} + E_{POT}\right] = \text{POWER OF EXT LOAD } q$$

$$E_{KIN} = \frac{1}{2}\int_0^\ell \mu\, ds\, \dot{y}^2 = \frac{1}{2}\underbrace{\int_0^\ell \mu\, \psi^2(s)\, ds}_{m_{eq}\text{ : EQUIV. MASS}}\, \dot{x}^2(t) = \frac{1}{2} m_{eq}\, \dot{x}^2$$

$$E_{POT} = E_{ELASTIC} = \frac{1}{2}\int_0^\ell M\,\chi\, ds = \frac{1}{2}\int_0^\ell (-\psi'' x\, EJ)(-\psi'' x)\, ds$$

$$= \frac{1}{2}\underbrace{\int EJ\,\psi''^2\, ds}_{K_{eq}\text{ : EQUIV. STIFFNESS}}\; x^2 = \frac{1}{2} K_{eq}\, x^2$$

POWER EXT LOAD q : $\displaystyle\int_0^\ell q\, ds\, \psi\dot{x} = \underbrace{\int_0^\ell q\,\psi\, ds}_{F_{eq}\text{ : EQUIV. FORCE}}\; \dot{x}$

(FV)

$$\frac{d}{dt}\left[\frac{1}{2} m_{eq}\, \dot{x}(t)^2 + \frac{1}{2} K_{eq}\, x(t)^2\right] = F_{eq}\, \dot{x}(t)$$

$$m_{eq}\, \ddot{x}(t)\, \cancel{\dot{x}(t)} + K_{eq}\, \cancel{\dot{x}(t)}\, x(t) = F_{eq}\, \cancel{\dot{x}(t)} \Rightarrow \boxed{m_{eq}\, \ddot{x}(t) + K_{eq}\, x(t) = F_{eq}}$$

1 D.O.F. GOVERNING EQUAT.
(GENERALISED) (NO DISSIPATION)

$\psi(s)$: SHAPE FUNCTION TO BE CHOSEN CAREFULLY (SEE BOUNDARY CONDITIONS)

FREE MOTION : $m_{eq}\, \ddot{x}(t) + K_{eq}\, x(t) = 0 \Rightarrow \omega^2 = \dfrac{K_{eq}}{m_{eq}} = \dfrac{\int_0^\ell EJ\, \psi''^2\, ds}{\int_0^\ell \mu\, \psi^2\, ds}$

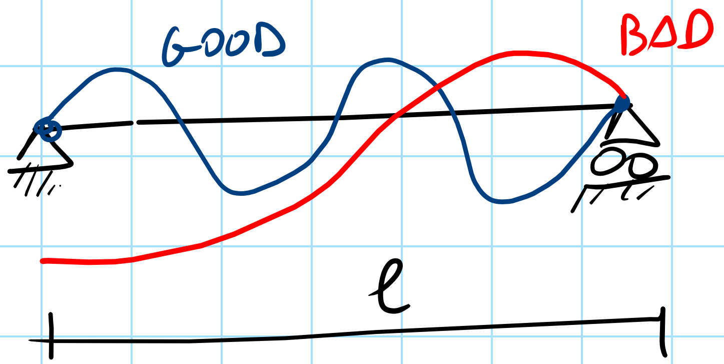
ω : CIRCULAR FREQUENCY

RAYLEIGH RATIO

(DEPENDS ON ψ)

ψ "WORKS" SIMILAR TO VIBRATION MODES

REQUIREMENTS ON $\psi(s)$: THEY MUST SATISFY "GEOMETRIC" BOUNDARY CONDITIONS (CONSTRAINT CONDITION)



$EJ\, \psi^{IV} = q$

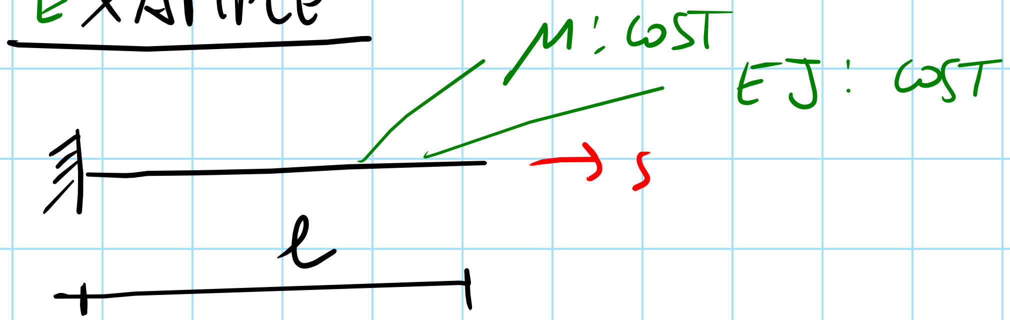
$\psi(0) = 0$
$\psi(\ell) = 0$

GEOMETRIC

$\psi''(0) = 0$
$\psi''(\ell) = 0$

ESSENTIAL

EXAMPLE



LET US STUDY FREE VIBRATION

EXACT SOLUTION : $\omega_{EXACT} = \dfrac{3.516}{\ell^2}\sqrt{\dfrac{EJ}{\mu}}$

$\psi_{EXACT} = \psi\left(\cos, \sin, \cosh, \sinh \text{ whose argument is } \left(\frac{s}{\ell}\right)\right)$

OBS : LET US CHECK DIMENSION OF ω

$$[\omega] = \frac{1}{L^2}\sqrt{\frac{FL^2 \, L}{M}} = \frac{1}{\cancel{L^2}}\sqrt{\frac{\cancel{FL^2}\,\cancel{L^2}}{\cancel{F}T^2}} = \frac{1}{T}$$

$[\omega] = \left[\frac{1}{T}\right]$

$F = M\frac{L}{T^2} \Rightarrow M = \frac{FT^2}{L}$

1) I APPROX

$$\psi_1 = \frac{3s^2}{2\ell^2} - \frac{s^3}{2\ell^3}$$

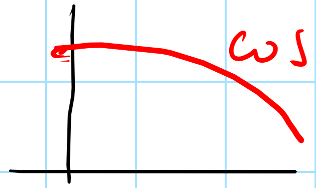
$\psi_1(0) = 0$; $\psi_1'(0) = 0$ OK VERIFIED

$m_{eq}^{(1)} = 0.236\,\mu\ell$

$K_{eq} = 3\dfrac{EJ}{\ell^3}$ $\Rightarrow \omega^{(1)} = \dfrac{3.568}{\ell^2}\sqrt{\dfrac{EJ}{\mu}}$

ERROR : 1.5 %

2) II APPROX



$$\psi_2(s) = 1 - \cos\frac{\pi s}{2\ell}$$

$\psi_2(0) = 0$; $\psi_2' = \sin\frac{\pi s}{2\ell}\cdot\frac{\pi}{2\ell} \Rightarrow \psi_2'(0) = 0$

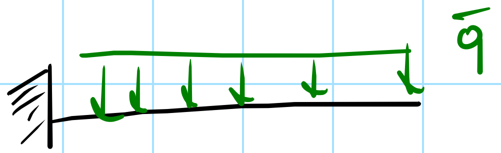
$m_{eq}^{(2)} = 0.227\,\mu\ell$

$K_{eq}^{(2)} = 3.044\dfrac{EJ}{\ell^3}$ $\Rightarrow \omega^{(2)} = \dfrac{3.664}{\ell^2}\sqrt{\dfrac{EJ}{\mu}}$

ERRORE 4 %

OBJ: THE TWO SHAPE FUNCT. SELECTED ALSO SATISFY $\psi''_k(\ell)=0$ $(k=1,2)$ (BENDING MOMENT NULL AT THE THE TIP OF THE CANTILEVER). THIS CONDITION IS NOT STRICTLY REQUIRED.

⇒ ONE POSSIBLE INITIAL GUESS FOR $\psi(s)$ COULD BE THE SOLUTION OF THE "STATIC" ELASTIC LINE.



⇒ SOLVE STATIC ELASTIC LINE "$v(s)$" ⇒ $\psi_1(s) = v(s)$