INTRODUCTION TO THE CALCULUS OF VARIATIONS

MANY ENGINEERING PROBLEMS CAN BE SOLVED LOOKING FOR EXTREMAL VALUES OF AN INTEGRAL-DIFFERENTIAL EXPRESSION.

MINIMA
MAXIMA

HYPOTHESES: 1D domain (x), $y=f(x)$ is known, an integral-differential expression involving $y=f(x)$ can be written as:

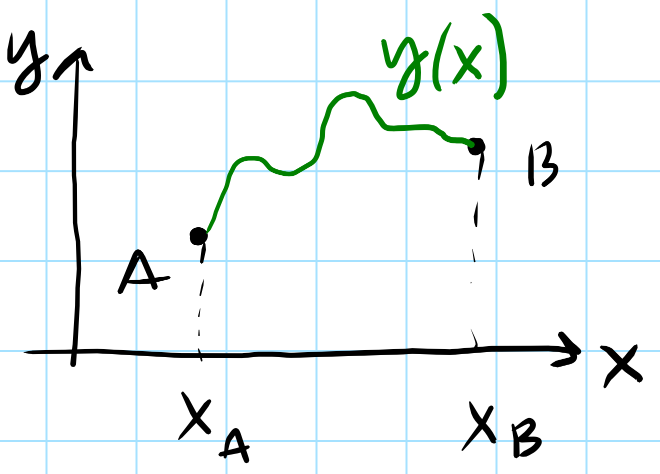
$$I(y) = \int_{x_A}^{x_B} F(x; y, y', y'' \dots)\,dx$$

↑ defines the problem

$I(y)$: FUNCTIONAL = FUNCTION OF A "FUNCTION"

i.e. $y(x) \rightarrow I(y) \in \mathbb{R}$

EX

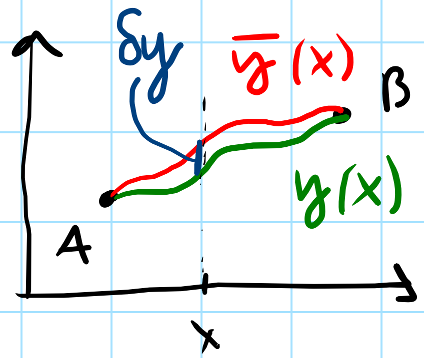


LENGTH OF THE CURVE

$$I = \int_A^B \sqrt{dx^2+dy^2} = \int_{x_A}^{x_B} \underbrace{\sqrt{1+y'^2(x)}}_{F(x;y')}\,dx$$

PROBLEM: FIND $y(x)$ THAT MINIMIZES I AMONG FUNCTIONS CONTINUOUS AND WITH $y'(x)$ CONTINUOUS

CONCEPT OF VARIATION



$\bar{y}(x)$ IS CLOSE TO $y(x)$ $\Rightarrow$ $\boxed{\delta y(x) = \bar{y}(x) - y(x)}$

NOTE THAT $\delta y(x_A) = \delta y(x_B) = 0$

PROPERTIES OF δy:

$\rightarrow \frac{d}{dx}(\delta y) = \frac{d}{dx}\left(\bar{y}(x) - y(x)\right) = \bar{y}' - y' = \delta y'$

$\rightarrow \int_{x_A}^{x_B} \delta y \, dx \Rightarrow \delta \int_{x_A}^{x_B} y \, dx$ ⊛

FROM ⊛

$$I = \int_{x_A}^{x_B} F(x; y, y') \, dx \rightarrow \delta I = \int_{x_A}^{x_B} \delta F(x; y, y') \, dx \text{ , but}$$

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATION

IF $\int_{x_A}^{x_B} y(x)\, \delta y \, dx = 0 \quad \forall \delta y$

$\Rightarrow$ $y(x) = 0$ IN THE DOMAIN $[x_A, x_B]$.

STATIONARITY OF I WITH $\boxed{y(x_A)=y_A,\ y(x_B)=y_B}$

$$\int f g' = [fg] - \int f' g$$

THE CONDITION OF STATION. OF I CAN BE WRITTEN AS

$$\boxed{\delta I = 0 \quad (\forall \delta y)} \implies \text{SOLUTION } y^*(x) \text{ OF THE PROBLEM}$$

$$\delta I = \int_{x_A}^{x_B} \delta F\, dx = \boxed{\int_{x_A}^{x_B} \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\, dx = 0 \qquad \forall \delta y} \quad (\square)$$

INTEGRATE BY PARTS: $\displaystyle \int \frac{\partial F}{\partial y'}\delta y'\, dx = \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_A}^{x_B} - \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx$

OUR STATEMENT $(\square)$

$$\int_{x_A}^{x_B}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx + \underbrace{\left[\frac{\partial F}{\partial y'}\delta y\right]_{x_A}^{x_B}}_{\to 0 \text{ BECAUSE } \delta y_B = \delta y_A = 0} = 0 \quad \forall \delta y$$

FOR THE "LEMMA"

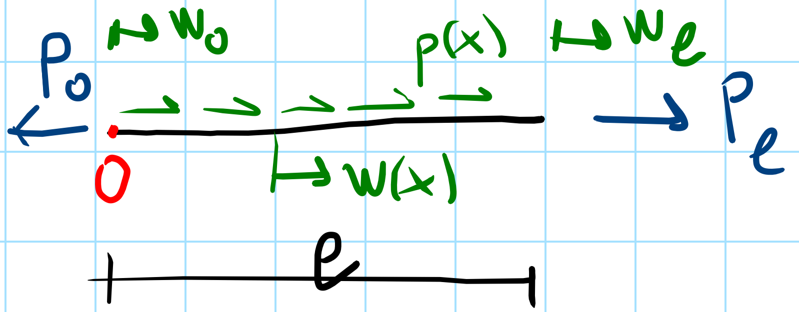
$$\implies \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

FOR ALL x

$$\boxed{\delta I = 0 \ \forall \delta y} \iff \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{WITH} \quad \begin{aligned} y(x_A) &= y_A \\ y(x_B) &= y_B \end{aligned}}$$

EQS OF EULERO-LAGRANGE OF THE VARIATIONAL PROBLEM.

EX: EQS OF EULERO-LAGRANGE FOR THE FUNCTIONAL T.P.E. OF A BAR



$$\varepsilon(x) = w'(x) \qquad \varphi = \frac{1}{2} E \varepsilon^2$$

$$\Pi(w) = \underbrace{\int_0^\ell \frac{1}{2} EA \, w'(x)^2 \, dx}_{\text{STRAIN ENERGY}} - \underbrace{\int_0^\ell p(x) w(x) \, dx - P_\ell w_\ell + P_0 w_0}_{\text{POT. OF EXTERN. LOADS}}$$

$$= \int_0^\ell \underbrace{\frac{1}{2} EA w'^2 - pw}_{F(x,w,w')} \, dx - \Big[P w \Big]_0^\ell$$

NOTE THAT IN THIS PROBLEM $w(0) = w_0$ AND $w(\ell) = w_\ell$ CAN VARY BECAUSE IN THE SKETCH THERE ARE NO CONSTRAINT, SO STATIONARITY OF $\Pi(w)$, δw CANNOT BE IMPOSED <u>ONLY</u> WITH THE EULERO-LAGRANGE EQS.

EQS EULERO - LAGRANGE ;

$$F = \frac{1}{2} EA w'^2 - p w \, ; \quad \frac{\partial F}{\partial w} - \frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right) = \overline{-p - \frac{d}{dx}\left(EA w'\right) = 0}$$

$$\therefore -p - (EA w')' = 0 \quad x \in [x_{A}, x_{B}] \; ; \text{ IF } EA \text{ const} \quad \boxed{EA \, w''(x) = -p(x)}$$

STATIONARITY OF $\Pi(w)$ CALCULATED DIRECTLY

$$\Pi(w) = \int_0^\ell \frac{1}{2} EA w'^2 - p w \, dx - [P w]_0^\ell \; . \qquad \boxed{\delta\Pi(w) = 0 \quad \forall \delta w} \Rightarrow w^*(x) \text{ SOLUTION}$$

$$\delta\Pi(w) = \int_0^\ell \Big(\underbrace{EA w' \delta w'}_{\text{BY PARTS}} - p\delta w\Big) dx - [P \delta w]_0^\ell = -\int_0^\ell \left[(EA w')' + p\right] \delta w - \left[(P - EA w')\delta w\right]$$

$$\int_0^\ell EA w' \delta w' = \left[EA w' \delta w\right] - \int_0^\ell (EA w')' \delta w \, dx$$

$$\text{(*)} \quad \underbrace{\int_0^\ell \left[(EA w')' + p\right] \delta w}_{=0} + \underbrace{\left[(P - EA w')\delta w\right]_0^\ell}_{=0} = 0 \, , \; \forall \delta w$$

$(EA w')' + p = 0$ EULERO LAGR. $x \in]0, \ell[$

$\left[(P - EA w')\delta w\right]_0^\ell = 0$

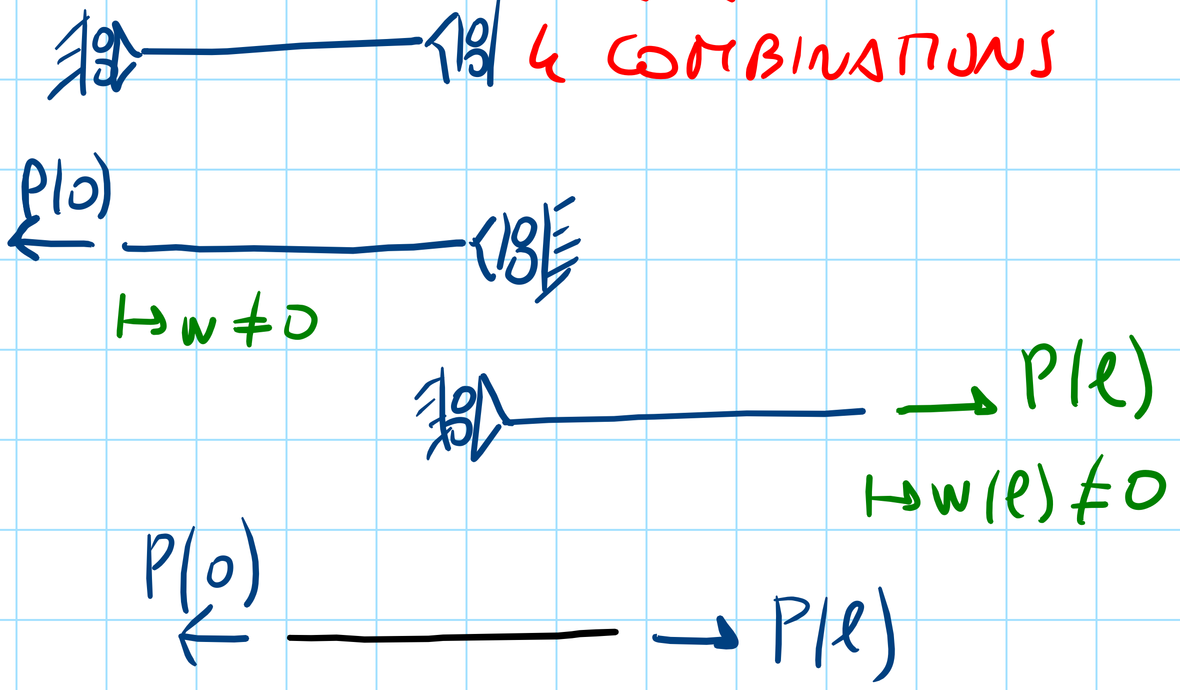
BOUNDARY CONDITIONS TO BE DISCUSSED.

LET US DISCUSS THE BOUND. CONDITIONS

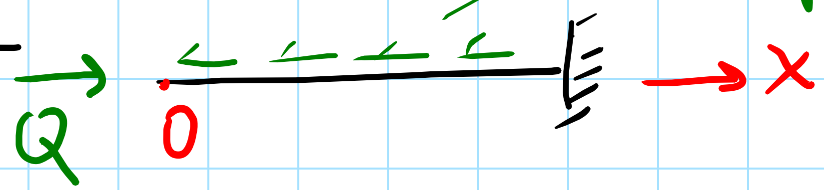
$$\left(P(\ell)-EA\,w'(\ell)\right)\delta w(\ell)-\left(P(0)-EA\,w'(0)\right)\delta w(0)=0 \qquad \forall \delta w$$

4 COMBINATIONS

$$\begin{cases} \delta w(0)=0\,, & \delta w(\ell)=0 \\ P(0)=EA\,w'(0) & \delta w(\ell)=0 \\ \delta w(0)=0\,, & P(\ell)=EA\,w'(\ell) \\ P(0)=EA\,w'(0)\,, & P(\ell)=EA\,w'(\ell) \end{cases}$$



EX



$p(x)=-\alpha X\,,\quad \alpha\in\mathbb{R}$

$EA = \text{CONST}$

$$\frac{dw}{dx}=\varepsilon=\frac{N}{EA}$$

$$\begin{cases} EA\,w''(x)=+\alpha X & x\in[0,\ell] \\ -Q=EA\,w'(0) \\ w(\ell)=0 \end{cases}$$

BOUNDARY CONDITIONS

PRINCIPLE OF STATIONARITY OF TOTAL POT. ENERGY IN 3D ELASTICITY

$\Pi(\underline{u})$ TPE

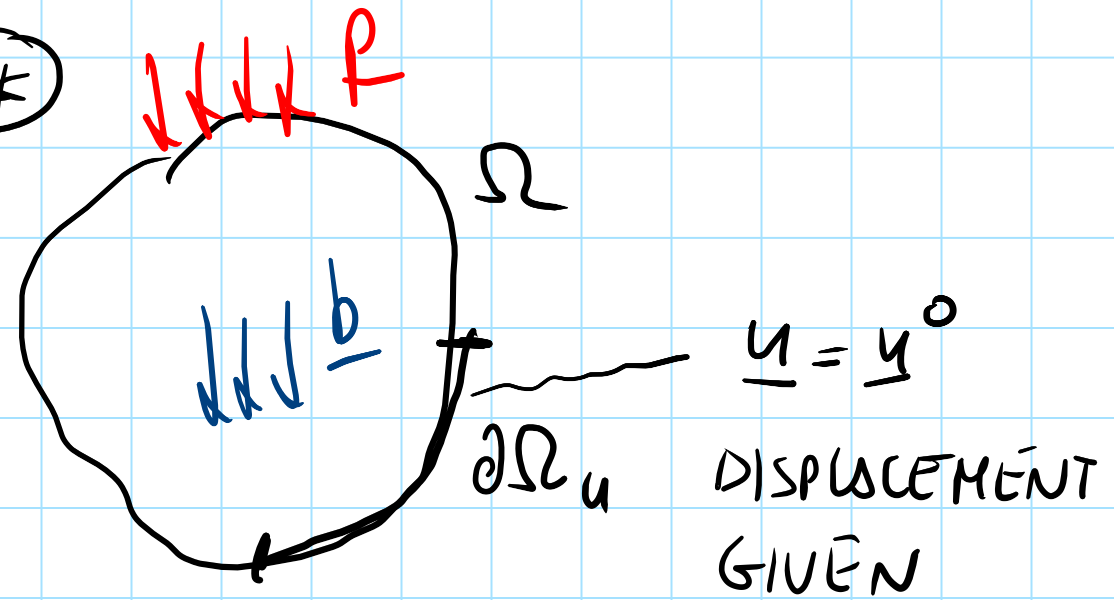
$$\boxed{\delta\Pi(\underline{u})=0 \quad \forall \delta\underline{u}}$$

WITHIN THE SET OF DISPLACEMENT FUNCTIONS $\underline{u}$ KINEMATICALLY ADMISSIBLE (*) THAT THAT SOLVES THE ELASTIC PROBLEM LET $\Pi(\underline{u})$ STATIONARY : $\boxed{\delta\Pi(\underline{u})=0 \ \forall\delta\underline{u}}$

RECALL THAT

$$\Pi(\underline{u}) = \frac{1}{8}\int_\Omega \mathbb{C}\left(\nabla\underline{u}+\nabla\underline{u}^T\right)\cdot\left(\nabla\underline{u}+\nabla\underline{u}^T\right)dV - \int_\Omega \underline{b}\cdot\underline{u}\,dV - \int_{\partial\Omega_p}\underline{p}\cdot\underline{u}\,dS$$

(*)



KINEMATICALLY ADMISSIBLE : $\underline{u}$ THAT SATISFIES AUTOMATICALLY CONDITIONS ON $\partial\Omega_u$