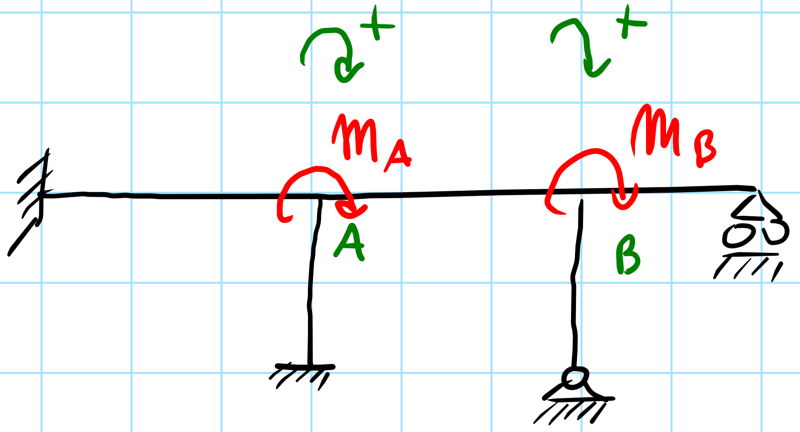
REVIEW DISPLACEMENT METHOD / STIFFNESS MATRIX

1/4/25

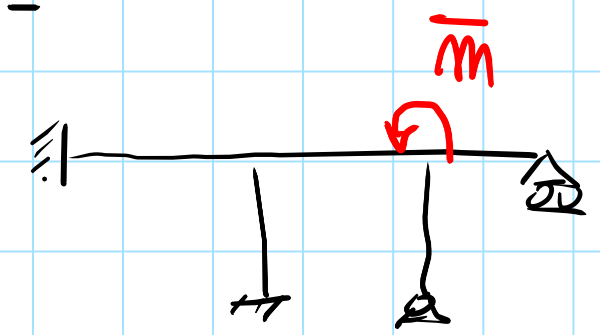


D.O.F. : φ_A, φ_B (UNKNOWNS)

DISPL. METHOD ; $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix} = \begin{bmatrix} m_A \\ m_B \end{bmatrix}$

$$\underbrace{\underset{\sim}{K}}\ \underline{q} = \underline{F}$$

$\underset{\sim}{K}$: STIFFNESS MATRIX, DEPENDS ON GEOMETRY / MECHANICAL PROPERTIES / CONSTRAINTS OF THE STRUCTURE.



SAME STRUCTURE

$$\underset{\sim}{K}\ \underline{q} = \begin{bmatrix} 0 \\ -\bar{m} \end{bmatrix}$$

PROPERTIES OF $\underset{\sim}{K}$:
- SYMMETRIC (TH. BETTI)
- POSITIVE DEFINITE (*)

(*) $\left.\begin{array}{l} \underset{\sim}{K}\underline{q}\cdot\underline{q} > 0 \quad , \underline{q} \neq \underline{0} \\ \underset{\sim}{K}\underline{q}\cdot\underline{q} = 0 \quad \text{IF ONLY IF } \underline{q} = \underline{0} \end{array}\right\}$ DEFINITION

FROM TH. CLAPEYRON $\Phi = \frac{1}{2}\underline{F}\cdot\underline{q} > 0 \Rightarrow \frac{1}{2}\underset{\sim}{K}\underline{q}\cdot\underline{q} > 0$ FOR $\underline{q} \neq \underline{0}$

WE CAN COMPUTE ELEMENTS K_{ij} <u>EITHER</u> BY WRITING DOWN EQUILIBRIUM EQUATIONS AND "EXTRACTING" THE REQUIRED COEFFICIENT :

in THE i-th EQUIL EQ ...... $K_{i2} q_2 + K_{i3} q_3 + \dots\dots$

K_{i3}

<u>OR</u> USING AN ALTERNATIVE APPROACH :

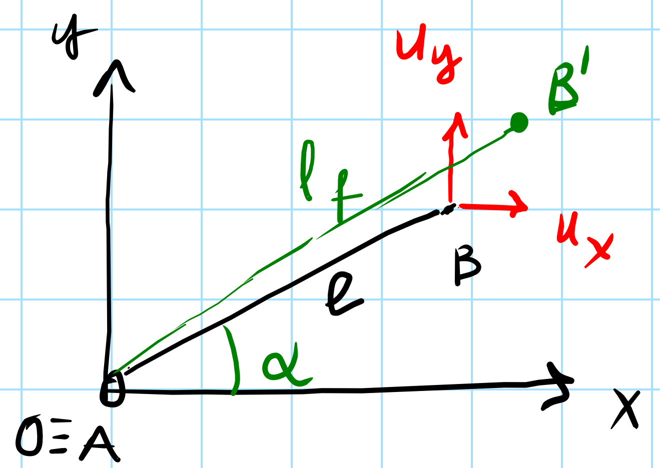
K_{ij} IS THE "FORCE" RELATED TO THE i-th D.O.F. WHEN $q_j = 1$, $q_i = 0$ $(i \neq j)$

EX:

$$\left.\begin{aligned} K_{11} q_1 + K_{12} q_2 &= F_1 \\ K_{21} q_1 + K_{22} q_2 &= F_2 \end{aligned}\right\} \Rightarrow \begin{aligned} q_1 &= 1 \\ q_2 &= 0 \end{aligned} \Rightarrow \left.\begin{aligned} K_{11} \cdot 1 &= F_1 \\ K_{21} \cdot 1 &= F_2 \end{aligned}\right\}$$

WITH THIS "SPECIAL" CHOICE $\Rightarrow$ $K_{11} = F_1$, $K_{21} = F_2$

DISPL. METHOD FOR PLANE TRUSS STRUCTURES



$$l = \sqrt{x_B^2 + y_B^2} \quad ; \quad l_f = \sqrt{(x_B+u_x)^2 + (y_B+u_y)^2} = l_f(u_x, u_y)$$

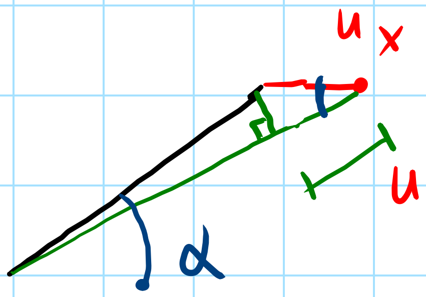
"SMALL DISPLACEMENTS", TAYLOR EXPANTION $l_f(u_x, u_y)$

$$l_f = l_f(0,0) + \frac{\partial l_f}{\partial u_x}\bigg|_{(0,0)} u_x + \frac{\partial l_f}{\partial u_y}\bigg|_{(0,0)} u_y + O\left(\sqrt{u_x^2+u_y^2}\right)$$

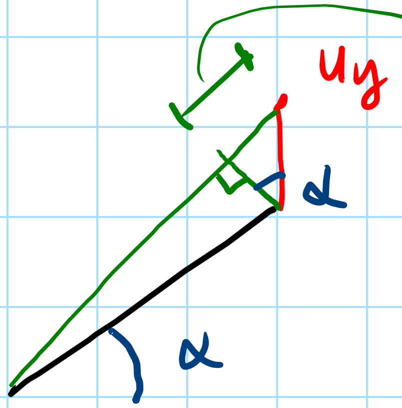
$$l_f \simeq l + \left[\frac{1}{2\sqrt{}} 2(x_B+u_x)\right]\bigg|_{(0,0)} u_x + \left[\frac{1}{2\sqrt{}} 2(y_B+u_y)\right]\bigg|_{(0,0)} u_y$$

$$l_f \simeq l + \underbrace{\frac{1}{l} x_B}_{\cos\alpha} u_x + \underbrace{\frac{y_B}{l}}_{\sin\alpha} u_y \Rightarrow l_f - l = \underbrace{\cos\alpha\, u_x}_{\Delta l_x} + \underbrace{\sin\alpha\, u_y}_{\Delta l_y}$$

CHANGE IN LENGTH: SUM OF 2 CONTRIBUTIONS

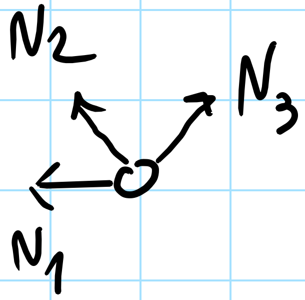


$u_x \cos\alpha = \Delta \ell_x$

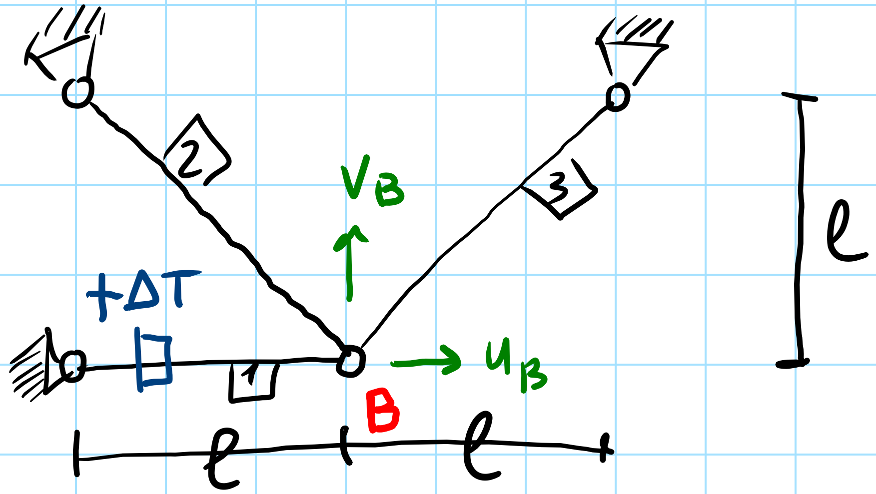


$u_y \sin\alpha = \Delta \ell_y$

$$\Delta \ell_i = \Delta \ell_i^{el} + \Delta \ell_i^{\Delta T}$$



EX



D.O.F.s: u_B, v_B

LOAD: $+\Delta T$ (BAR 1)

2 DOFs ⇒ 2 EQUIL. EQUATIONS

WHOSE UNKNOWNS ARE u_B, v_B

$A_1 = A$; $A_2 = A_3 = \sqrt{2} A$

$\ell_1 = \ell$; $\ell_2 = \ell_3 = \sqrt{2} \ell$

E : cost

NODE B

$$\left.\begin{aligned} \overset{+}{\rightarrow}:\ & -N_1 - N_2 \frac{1}{\sqrt{2}} + \frac{N_3}{\sqrt{2}} = 0 \\ +\uparrow:\ & +N_2 \frac{1}{\sqrt{2}} + N_3 \frac{1}{\sqrt{2}} = 0 \end{aligned}\right\}$$

CONSTITUTIVE LAW $\left(N_i = f(\Delta \ell_i)\right)$

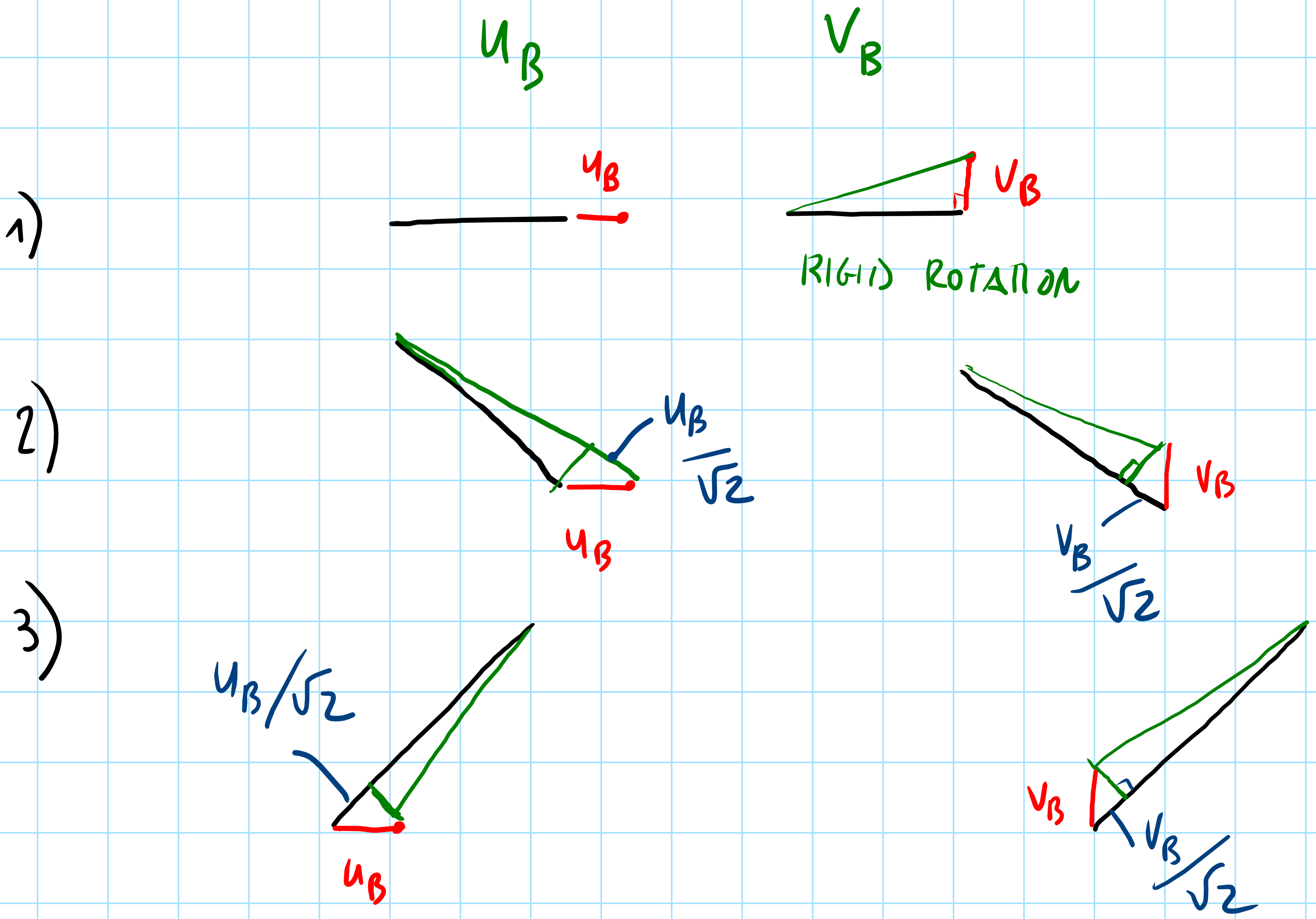
$$N_1 = \frac{EA_1}{\ell_1} \Delta \ell_1^{el} = \frac{EA_1}{\ell_1} \left(\Delta \ell_1 - \Delta \ell_1^{\Delta T}\right)$$

$$= \frac{EA_1}{\ell_1} \left(\Delta \ell_1 - \alpha \Delta T \ell\right)$$

$$N_2 = \frac{EA_2}{\ell_2} \Delta \ell_2^{(el)} \ ; \quad N_3 = \frac{EA_3}{\ell_3} \Delta \ell_3^{(el)}$$

$\Delta \ell_i = f_i(u_B, v_B)$

$u_B, v_B \longrightarrow \Delta l_i \longrightarrow N_i \longrightarrow$ EQUIL. EQS



Δl_i

$$\Delta l_1 = +u_B$$

$$\Delta l_2 = +\frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$$

$$\Delta l_3 = -\frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$$

NOTE THAT $\dfrac{EA_1}{l_1} = \dfrac{EA}{l} = K \; ; \; \dfrac{EA_2}{l_2} = \dfrac{EA_3}{l_3} = \dfrac{EA}{l} = K$

$$\left.\begin{aligned} &1)\quad -K(\Delta\ell_1 - \alpha\Delta T\ell) - K\Delta\ell_2\frac{1}{\sqrt{2}} + K\Delta\ell_3\frac{1}{\sqrt{2}} = 0 \\ &2)\quad +K\Delta\ell_2 + K\Delta\ell_3 = 0 \end{aligned}\right\}$$

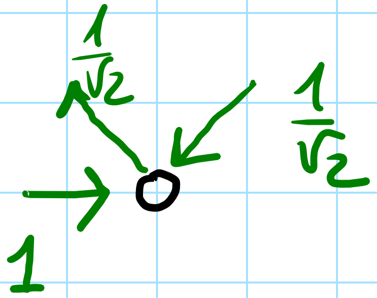
$$\begin{cases} -K(u_B - \underbrace{\alpha\Delta T\ell}_{\text{LOAD}}) - K\frac{1}{2}(u_B - v_B) - K\frac{1}{2}(u_B + v_B) = 0 \\ K\frac{1}{\sqrt{2}}(\cancel{u_B} - v_B) - K\frac{1}{\sqrt{2}}(\cancel{u_B} + v_B) = 0 \end{cases} \Rightarrow v_B = 0$$

$$-\cancel{K}u_B - \cancel{K}\frac{1}{2}u_B - \cancel{K}\frac{1}{2}u_B = -\cancel{K}\alpha\Delta T\ell$$

$$-2u_B = -\alpha\Delta T\ell$$

$$\boxed{u_B = \frac{\alpha\Delta T\ell}{2}}$$

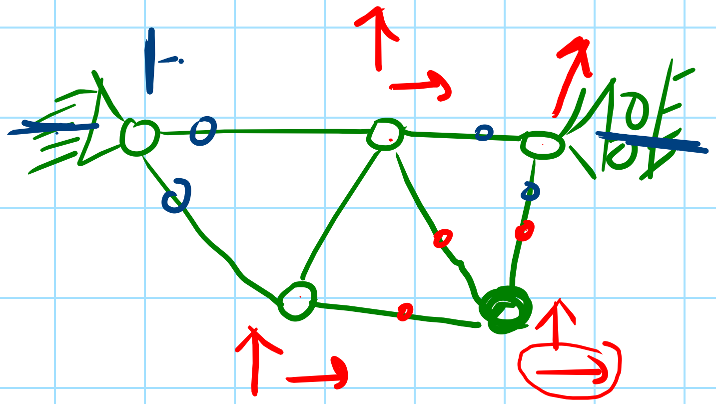
$$N_1 = K(u_B - \alpha\Delta T\ell) = -K\frac{\alpha\Delta T\ell}{2}$$



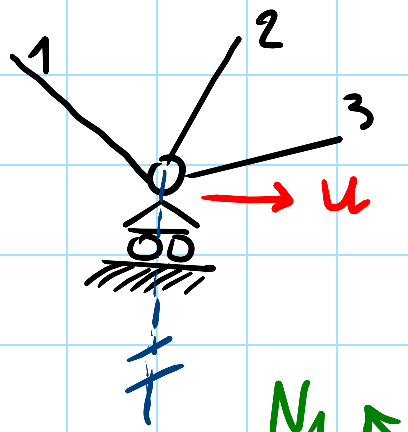
$$\left(\times\ \frac{K\alpha\Delta T\ell}{2}\right)$$

$$N_2 = K\frac{1}{\sqrt{2}}u_B = K\frac{1}{\sqrt{2}}\frac{\alpha\Delta T\ell}{2} > 0$$

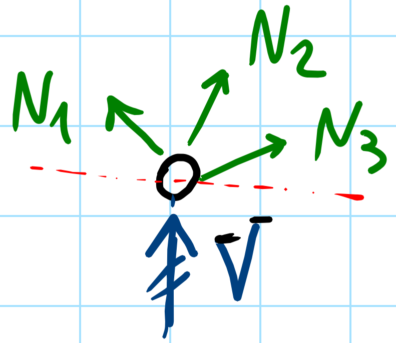
$$N_3 = K\frac{1}{\sqrt{2}}(-u_B) = -\frac{K\alpha\Delta T\ell}{2\sqrt{2}} < 0$$



THE CASE OF A ROLLER



ONLY ONE DOF : u $\Longrightarrow$ ONLY ONE EQUIL. EQ. (ALONG u)



$\Longleftrightarrow$ IN THE EQUIL. EQ V DOES NOT APPEAR

V WILL BE "SOLVED" LATER, AFTER HAVING COMPUTED N_1, N_2, N_3, BY USING THE "VERTICAL" EQUIL EQ. NOT USED BEFORE.

EX: IN THE STRUCTURE ABOVE THERE ARE $2m_N = 10$ EQUIL EQS

THERE ARE 7 DOFS (IN RED); THEN THE REMAINING 3 EQS ARE EMPLOYED TO CALCULATE THE 3 REACTIONS

HOW TO OBTAIN $\underset{\sim}{K}$: FROM EQUIL. EQS WE SEE THAT THE MATRIX ASSOCIATED WITH THE PROBLEM IS

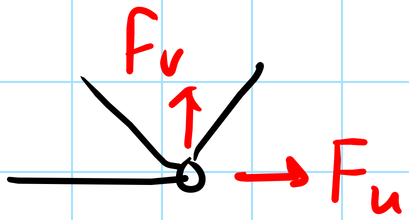
$$\begin{bmatrix} -2K & 0 \\ 0 & -\frac{2}{\sqrt{2}}K \end{bmatrix} \rightarrow$$

RECALL WE HAVE DROPPED $\frac{1}{\sqrt{2}}$ PREVIOUSLY

I GUESS THAT THE "REAL" $\underset{\sim}{K}$ IS $\begin{bmatrix} 2K & 0 \\ 0 & K \end{bmatrix}$

NOTE THAT $\underset{\sim}{K}$ WILL SOLVE THE PROBLEM

$$\underset{\sim}{K}\begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} F_u \\ F_v \end{bmatrix}$$



THE CHANGE IN SIGN IS DUE TO THIS FACT: IN BOTH EQS, IMAGINE TO HAVE FORCES F_u, F_v :

1) $(\xrightarrow{+})$ $\cdots\cdots + F_u = 0 \longrightarrow \cdots\cdots = -F_u$

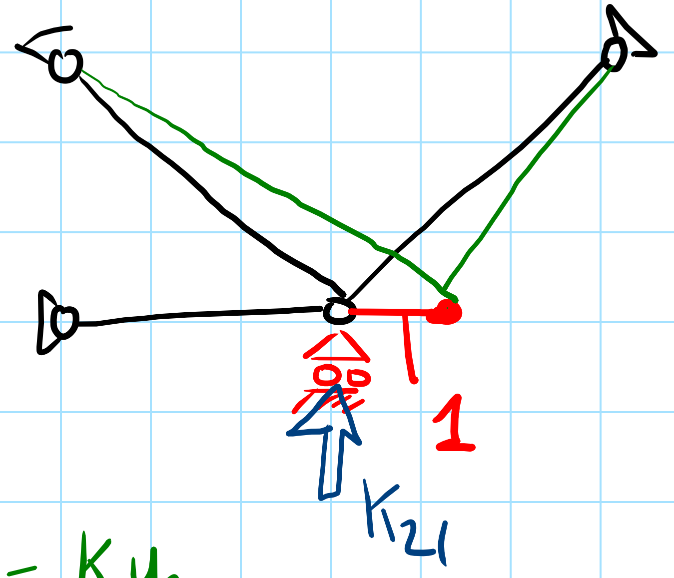
2) $(+\uparrow)$ $\frac{1}{\sqrt{2}}\cdots\frac{1}{\sqrt{2}} + F_v = 0 \longrightarrow \frac{1}{\sqrt{2}}\cdots\frac{1}{\sqrt{2}} = -F_v$

WE NEED TO CHANGE SIGN IN ORDER TO HAVE $+F_u$ AND $+F_v$

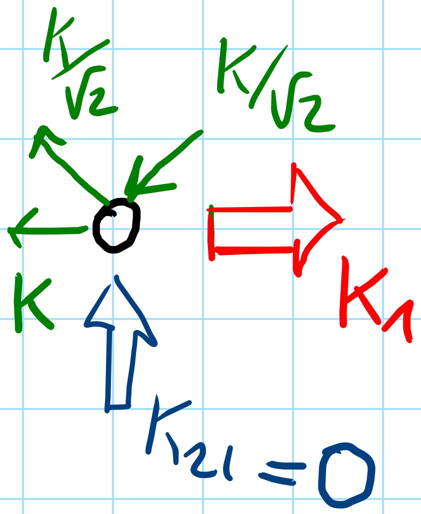
WE CANNOT DROP HERE $\frac{1}{\sqrt{2}}$!

LET US TRY TO COMPUTE K_{11} WITH THE "PRACTICAL" METHOD

$u_B = 1$, $V_B = 0$ $\Rightarrow$ THE FORCE ALONG DOF u_B



$N_1 = K u_B$



$\mapsto u_B > 0$ +

$$K_{11} = K + \frac{K}{2} + \frac{K}{2} = 2K$$

BECAUSE THE GREEN FORCES ARE ALREADY IN EQUILIBR. ALONG VERTICAL DIRECTION

$$0 = K_{21}$$

STRONG AND WEAK FORMULATIONS OF A DIFFERENTIAL PROBLEMS

CONSIDER THE PROBLEM OF THE ELASTIC LINE :

$$\left.\begin{array}{l}(EJ\,v'')'' = q \\ + \text{ B. CONDITIONS}\end{array}\right\}$$

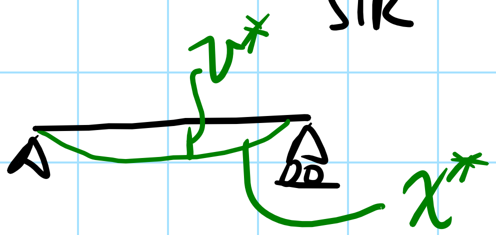
STRONG FORMULATION

DIFFERENTIAL EQUATIONS + CONDITIONS

WE CAN SOLVE ANY STRUCTURAL PROBLEM BY USING TH OF VIRTUAL WORK. FOR ELASTIC LINE :

$$\int_{STR} M(z)\,\chi^*(z)\,dz = \int_{STR} q(z)\,v^*(z)\,dz$$

STAT. ADMISS SCH. (REALE)



KINEM ADMISS SCH. (FICTITIOUS)

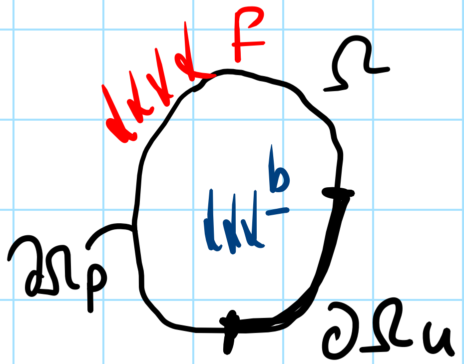
EXAMPLE OF WEAK FORMULATION

OTHER WEAK FORMULATIONS ARE :

STATIONARITY OF TOT. POT. ENERGY

+ OTHER

IN 3D LINEAR ELASTICITY:



$$\left.\begin{aligned} &\operatorname{div} \sigma + b = 0 \\ &\varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^T) \\ &\sigma = \mathbb{C}\,\varepsilon \end{aligned}\right\} \text{ in } \Omega$$

$$\left.\begin{aligned} &u = u^0 \text{ on } \partial\Omega_u \\ &\sigma\, n = p \text{ on } \partial\Omega_p \end{aligned}\right] \text{ B.C's.}$$

STRONG FORM

DISPL. FORMULATION: (u IS THE UNKNOWN)

$$\operatorname{div}\left[\frac{1}{2}\mathbb{C}(\nabla u + \nabla u^T)\right] + b = 0 \quad \Omega$$

$$\left.\begin{aligned} &u = u^0 \\ &\frac{1}{2}\mathbb{C}(\nabla u + \nabla u^T)\, n = p \end{aligned}\right| \text{ B.C.'s}$$

WEAK FORM

TH. VIRTUAL WORK

$$\int_\Omega \sigma \cdot \varepsilon^* \, dV = \int_{\partial\Omega_p} p \cdot u^* \, dS + \int_\Omega b \cdot u^* \, dV + \int_{\partial\Omega_u} r \cdot u^0 \, dS$$

WHERE $*$ DENOTES AN INDEPENDENT KINEMATIC FIELD.