Giov. 6

Funzione massimale di Hardy Littlewood

$f \in L^1_{loc}(\mathbb{R}^d)$ $\qquad r > 0$

$$A_r f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

$A_r f \in C^0(\mathbb{R}^d, \mathbb{C})$

$$Mf(x) = \sup_{r>0} A_r |f|$$

Mf è lower semicontinuous.

subaddictiv

$$M(f+g) \leq M(f) + M(g).$$

$$|A_r f(x)| \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|\, dx$$

$$\leq \quad \|f\|_{L^\infty}$$

$$\Rightarrow \quad Mf(x) \leq \|f\|_{L^\infty}$$

$$M : L^\infty(\mathbb{R}^d) \circlearrowleft$$

$$M : L^1 \not\longrightarrow L^1 \quad \text{in } \mathbb{R}^d$$

$K \subset\subset \mathbb{R}^d$ sia $c_0 > 0$
t.c. $B(0, c_0) \supset K$

Allora per $|x| \geq c_0$

da $B(x, 2|x|) \supset B(0, |x|) \supset K$

$$M\chi_K(x) = \sup_{r>0} \frac{|B(x,r) \cap K|}{c_d\, r^d} \geq \frac{|K|}{c_0 2^d |x|^d}$$

$\forall \; x$ t.c. $|x| > c_0$

$$M\chi_K(x) \geq C \frac{1}{|x|^d} \notin L^1(\mathbb{R}^d \setminus B_{(0,c_0)})$$

Si dimostra ora che però
$\forall \ 1 < p \leq +\infty \ \exists \ C_p$ t.c.
$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$
Ricordiamoci Cheb.
$\forall \ g \in L^1(\mathbb{R}^d)$
$$|\{x : |g(x)| > \alpha\}| \leq \frac{|g|_{L^1(\mathbb{R}^d)}}{\alpha}$$
$$\forall \alpha > 0.$$
Ovviamente se $T : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$
che soddisfa
$$\|Tf\|_{L^1} \leq A \|f\|_{L^1} \quad \forall f \in L^1(\mathbb{R}^d)$$
vale anche
$$|\{x : |Tf(x)| > \alpha\}| \leq \frac{\|Tf\|_{L^1}}{\alpha}$$
$$\leq \frac{A}{\alpha} \|f\|_{L^1} \quad \forall \alpha > 0 \text{ e } \forall f \in L^1(\mathbb{R}^d)$$
Risulta che M soddisfa

$\exists\, C_d > 0$ t.c.

$$|\{x : |Mf(x)| > \alpha\}| \leq$$

$$\leq \frac{C_d}{\alpha} \|f\|_{L^1} \quad \forall \alpha > 0$$
$e \ \forall f \in L^1(\mathbb{R}^d)$

$C_d = 3^d$ soddisfa

M è $(1,1)$ weak bounded

<u>Lemma</u> Sia $B_{x_1}, \dots, B_{x_N}$ una famiglia di palle in $\mathbb{R}^d$.

$\exists\, \{B_1, \dots, B_m\} \subseteq \{B_{x_1}, \dots, B_{x_N}\}$

a due a due disjunte t.c.

$$|B_{x_1} \cup \dots \cup B_{x_N}| \leq 3^d \sum_{j=1}^{m} |B_j| \quad]$$

$$|\{x : |Mf(x)| > \alpha\}| \le$$

$$\le \frac{C_d}{\alpha} \|f\|_{L^1} \quad \forall\, \alpha > 0$$
$$e\ \forall\, f \in L^1(\mathbb{R}^d)$$

$C_d = 3^d$

Sia $K \subset\subset \{x : |Mf(x)| > \alpha\}$

$\forall\, x \in K$ $\qquad Mf(x) = \sup\limits_{r>0} \dfrac{\int_{B(x,r)} |f|}{|B(x,r)|}$

$\exists\, r_x > 0$ t.c.

$$\int_{B(x,r_x)} |f| > \alpha\, |B(x, r_x)|$$

$$K \subset B_{x_1} \cup \cdots \cup B_{x_N}$$

$$\frac{1}{\alpha} \int_{B(x_j, r_{x_j})} |f| > |B(x_j, r_{x_j})|$$

$B_1, \dots, B_m$

$$|K| \le |B_{x_1} \cup \dots \cup B_{x_N}| \le$$

$$\le 3^d \sum_{j=1}^{m} |B_j|$$

$$\le \frac{3^d}{\alpha} \sum_{j=1}^{m} \int_{B_j} |f|$$

$$|K| \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \quad \forall K$$

$$\begin{cases} |\{Mf > \alpha\}| \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \\ \|\cancel{f}Mf\|_{L^\infty} \le \|f\|_{L^\infty} \end{cases}$$

<u>Teor</u> Sia $1 < q \le +\infty$ e $T : L^1 + L^q \to L^1_{loc}$ subadditivo

t.c. $\exists$ A_1, A_q t.c.

$$\|Tf\|_{L^q(\mathbb{R}^d)} \le A_q \|f\|_{L^q(\mathbb{R}^d)}$$

$$|\{x : |Tf(x)| > \alpha\}| \le \frac{A_1}{\alpha} |f|_{L^1(\mathbb{R}^d)}$$
$\forall\, \alpha >$

Allora $\cancel{\exists}\ \cancel{\forall}$ $1 < p < q$ $\exists\, A_p$ t.c.

$$\|Tf\|_{L^p} \le A_p |f|_{L^p} \quad \forall f \in L^p.$$

Dim

$g : \mathbb{R}^d \to \mathbb{R}$ misurabile

$\lambda(\alpha) : |\{|g(x)| > \alpha\}|$ $\forall$

$\lambda : [0, +\infty) \to [0, +\infty]$

$g \in L^p(\mathbb{R}^d)$

$$\int_{\mathbb{R}} |g(x)|^p dx =$$

$$= \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\, \alpha^{p-1} d\alpha$$

$$= \int_0^{+\infty} d\alpha\; p\, \alpha^{p-1} \int_{\{|g(x)| > \alpha\}} dx$$

$$= \int_0^{+\infty} p\, \alpha^{p-1} \lambda(\alpha)\, d\cancel{\alpha}$$

$p \in (1, q]$ $\quad f \in L^p(\mathbb{R}^d)$ $\quad \alpha > 0$

$$f(x) = f_1(x) + f_2(x)$$

$$f_1(x) = \begin{cases} f(x) & \text{se } |f(x)| \geq \frac{\alpha}{2} \\ 0 & \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{se } |f(x)| < \frac{\alpha}{2} \\ 0 & \end{cases}$$

$$f_1 \in L^1(\mathbb{R}^d)$$

$$\int_{\mathbb{R}^d} |f_1|\, dx = \int_{\{|f(x)| \geq \frac{\alpha}{2}\}} |f(x)|\, dx$$

$$\int_{\{|f(x)| \geq \alpha/2\}} \frac{|f(x)|^{p-1}}{(\alpha/2)^{p-1}} |f(x)|\, dx$$

$$\leq \frac{2^{p-1}}{\alpha^{p-1}} \int_{\mathbb{R}^d} |f|^p \, dx$$

$f_2 \in L^q$

$$\int_{\mathbb{R}^d} |f_2|^q \, dx = \int_{\{|f(x)| < \frac{\alpha}{2}\}} |f(x)|^q \, dx$$

$$\leq \left(\frac{\alpha}{2}\right)^{q-p} \int_{\mathbb{R}^d} |f(x)|^p < +\infty$$

$$|Tf| \leq |Tf_1| + |Tf_2|$$

$$\{|Tf(x)| > \alpha\} \subseteq$$

$$\subseteq \{|Tf_1| > \tfrac{\alpha}{2}\} \cup \{|Tf_2| > \tfrac{\alpha}{2}\}$$

$$|\{|Tf_1(x)| > \tfrac{\alpha}{2}\}| \leq \frac{A_1}{\alpha} |f_1|_{L^1(\mathbb{R}^d)}$$

$$\overset{*}{\leq} \frac{A_1}{\alpha} \int_{\{|f(x)| \geq \frac{\alpha}{2}\}} |f(x)| \, dx$$

$$\left|\{|Tf_2(x)| > \tfrac{\alpha}{2}\}\right| = \left|\{ |Tf_2(x)|^q > \left(\tfrac{\alpha}{2}\right)^q\}\right|$$

$$\leq \left(\frac{2}{\alpha}\right)^q \| Tf_2 \|_{L^q}^q \leq$$

$$\leq \left(\frac{2}{\alpha}\right)^q A_q^q \| f_2 \|_{L^q}^q$$

$$\overset{(**)}{=} \left(\frac{2}{\alpha}\right)^q A_q^q \int_{\{|f(x)| \leq \frac{\alpha}{2}\}} |f|^q dx$$

$$\int_{\mathbb{R}^d} |Tf|^p dx =$$

$$= \int_0^{+\infty} p\, \alpha^{p-1} \left|\{|Tf(x)| > \alpha\}\right| d\alpha$$

$$\leq \int_0^{+\infty} p\, \alpha^{p-1} \left|\{|Tf_1(x)| > \tfrac{\alpha}{2}\}\right| d\alpha +$$

$$\int_0^{+\infty} p\, \alpha^{p-1} \left|\{|Tf_2(x)| > \tfrac{\alpha}{2}\}\right| d\alpha$$

$$=: I_1 + I_2$$

$$I_1 \le 2A_1 \int_0^{\infty} p\,\alpha^{p-2} \int_{\{x:\,|f(x)|\ge \frac{\alpha}{2}\}} |f(x)|\,dx$$

$$= 2A_1 p \int_{\mathbb{R}^d} dx\,|f(x)| \int_0^{2|f(x)|} \alpha^{p-2}\,d\alpha$$

$$\eqsim \frac{2A_1 p}{p-1} \int_{\mathbb{R}^d} |f(x)|^p\,dx$$

$$I_2 = \int_0^{+\infty} p\,\alpha^{p-1}\, \left|\{|Tf_2(x)| \ge \alpha_2\}\right| d\alpha$$

$$\le 2^q A_q^q \int_0^{+\infty} p\,\alpha^{p-1-q} \int_{\{x:\,0<|f(x)|<\frac{\alpha}{2}\}} dx\,|f(x)|^q$$

$$= 2^q A_q^q\, p \int_{\{x:\,|f(x)|>0\}} dx\,|f(x)|^q \int_{2|f(x)|}^{+\infty} \alpha^{p-q-1}$$

$$= \frac{2^p}{p-q} A_q^q\, p \int dx\,|f(x)|^p\ \cancel{d\alpha}$$

$$\int |Tf|^p \, dx \le I_1 + I_2$$

$$\le C \quad \int |f|^p \, dx$$

<u>Teor</u> (Dis di H-L-Sob)

$\gamma \in (0, d)$

$1 < p < q < +\infty$ E. M. Stein

$$\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$$

$\exists \; C \quad$ tc.

$$\left\| \int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma} dy \right\|_{L^q(\mathbb{R}^d)}$$

$$\underbrace{\le C}_{R>0} \; \|f\|_{L^p(\mathbb{R}^d)}.$$

$$\int_{\mathbb{R}^d} f(x-y)\,|y|^{-\gamma}\,dy$$

$$= \int_{|y|\le R} f(x-y)\,|y|^{-\gamma}\,dy + \int_{|y|\ge R} dy\, f(x-y)\,|y|^{-\gamma}$$

$$\left| \int_{|y|\le R} f(x-y)\,|y|^{-\gamma}\,dy \right| \le$$

$$\le Mf(x) \int_{|y|\le R} |y|^{-\gamma}\,dy$$

$$= C_d\, R^{d-\gamma}\, Mf(x)$$

$$\left| \int_{|y|\ge R} f(x-y)\,|y|^{-\gamma}\,dy \right| \le$$

$$\gamma p' > d$$

$$\le |f|_{L^p}\ \big|\,|y|^{-\gamma}\big|_{L^{p'}\{|y|\ge R\}}$$

$$= c\,|f|_{L^p}\ R^{-\frac{d}{q}}$$

$$\left| \int_{\mathbb{R}^d} f(x-y)\,|y|^{-\gamma}\,dy \right|$$

$$\leq C\left(\|f\|_{L^p} R^{-\frac{d}{q}} + R^{d-\gamma} Mf(x)\right)$$

$$\frac{Mf(x)}{\|f\|_{L^p}} = R^{-\frac{d}{p}}$$

$$\left|\int_{\mathbb{R}^d} f(x-y)\,|y|^{-\gamma}\,dy\right|$$

$$\leq 2C\,\|f\|_{L^p}^{1-\frac{p}{q}}\,\left(Mf(x)\right)^{\frac{p}{q}}$$

$$\left\|\int_{\mathbb{R}^d} f(x-y)\,|y|^{-\gamma}\,dy\right\|_{L^q} \leq$$

$$\leq C\,\|f\|_{L^p}^{1-\frac{p}{q}}\,\left\|(Mf)^{\frac{p}{q}}\right\|_{L^q}$$

$$= C\,\|f\|_{L^p}^{1-\frac{p}{q}}\,\|Mf\|_{L^p}^{\frac{p}{q}}$$

$$\leq C_1\,\|f\|_{L^p}^{1-\frac{p}{q}}\,\|f\|_{L^p}^{\frac{p}{q}}$$

$$= C_1\,\|f\|_{L^p}$$

$$\left| \int_{|y|\leq R} f(x-y)|y|^{-\gamma} dy \right| \leq$$

$$\leq Mf(x) \int_{|y|\leq R} |y|^{-\gamma} dy$$

$\forall\ \phi \in L^1(\mathbb{R}^d)$ positiva e radiale è decrescente

$$\left| \int_{\mathbb{R}^d} f(x-y)\phi(y)\, dy \right| \leq$$

$$\leq Mf(x) \int_{\mathbb{R}^d} \phi(y)\, dy$$

$$\phi(y) = \chi_{B(0,R)} |y|^{-\gamma}$$

Si comincia con

$$\phi = \sum_j a_j \chi_{B_j}$$

$$\left| \int f(x-y) \sum_j a_j \chi_{B_j}(y)\, dy \right|$$

$$\leq \sum a_j |B_j| \frac{\int |f(x-y)| \chi_{B_j}(y)\, dy}{|B_j|}$$

$$\leq \underbrace{\sum a_j |B_j|}_{\int \phi \, dx} \qquad Mf(x)$$

Per densità ogni ϕ

$\phi \in L^1(\mathbb{R}^d)$ positiva e radiale e decrescente è approssimabile

$$\phi_n \nearrow \phi$$