13 Marzo

$u(t,x)\in\mathbb{R}^d$, $x\in\mathbb{R}^d$

$$\begin{cases} u_t + \operatorname{div}(u\otimes u) - \Delta u = -\nabla p \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$\mathbb{P}u = u$

$\mathbb{P}\nabla p = 0$

$$u\otimes v = \{u_i v_j\}_{i,j=1}^{d}$$

$$\operatorname{div}(u\otimes v)_j = \partial_i(u_i v_j)$$

$$\begin{cases} u_t - \Delta u = -\mathbb{P}\operatorname{div}(u\otimes u) \\ u = \mathbb{P}u \\ u|_{t=0} = u_0 \end{cases}$$

$\underbrace{H(\mathbb{R}^d)}$

<u>Def</u> Sia $u_0\in H(\mathbb{R}^d)$

Un campo $u\in L^2_{loc}(\mathbb{R}_+\times\mathbb{R}^d, \mathbb{R}^d)$ è una soluzione debole per Leray se

$u\in C^0_w([0,+\infty), L^2(\mathbb{R}^d,\mathbb{R}^d))$

(cioè $\forall\ \phi\in L^2(\mathbb{R}^d,\mathbb{R}^d)$

$$t \to \langle u(t), \phi \rangle \in C^0([0,+\infty), \mathbb{R}))$$

se $\operatorname{div} u(t) = 0 \quad \forall t$ e

se $\forall \phi \in C^\infty_{c\sigma}([0,+\infty) \times \mathbb{R}^d, \mathbb{R}^d)$

e $\forall t \geq 0$ si ha

$$\langle u(t), \phi(t) \rangle =$$

$$= \int_0^t \Big(\langle u(t'), \Delta \phi(t') \rangle + \langle u(t'), \partial_t \phi(t') \rangle$$

$$- \langle \operatorname{div}(u \otimes u)(t'), \phi(t') \rangle \Big) \, dt'$$

$$+ \langle u_0, \phi(0) \rangle$$

$$\langle \ , \phi(t) \rangle$$

$$u_t - \Delta u = - \mathbb{P} \operatorname{div}(u \otimes u)$$

$$\langle \partial_t u, \phi \rangle - \langle u, \Delta \phi \rangle =$$

$$= - \langle \operatorname{div}(u \otimes u), \underbrace{\mathbb{P}\phi}_{\phi} \rangle$$

$$\int_0^t \big(\langle \partial_t u, \phi \rangle - \langle u, \Delta \phi \rangle \big) dt'$$

$$= - \int_0^t \langle \operatorname{div}(u \otimes u), \phi \rangle \, dt$$

Teor Se $u_0 \in H(\mathbb{R}^d)$ $d = 2,3$

allora $\exists$ una soluzione debole nel senso di Leray.

Inoltre si ha

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(t')\|^2_{L^2}\, dt' \leq \|u_0\|^2_{L^2} \qquad \forall t \geq 0$$

Osservazione

In particolare

$$u \in L^\infty([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$$

$$\nabla u \in L^2([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$$

Teor $(d=2)$ La soluzione e' unica,

$$u \in C^0([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$$

e

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u\|^2_{L^2}\, dt' = \|u_0\|^2_{L^2}$$

mild solutions.

Lemma $d=2,3$

$$(u,v,\varphi)\in\left(C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)\right)^3$$

$$\longrightarrow \langle \operatorname{div}(u\otimes v),\varphi\rangle\in\mathbb{R}$$

si estende in modo unico in una ~~for~~ mappa trilineare

$$\left(H^1(\mathbb{R}^d,\mathbb{R}^d)\right)^3\longrightarrow\mathbb{R}$$

$$\langle \operatorname{div}(u\otimes v),\varphi\rangle\le$$

$$\le C\,\|\nabla u\|_{L^2}^{\frac{d}{4}}\,\|\nabla v\|_{L^2}^{\frac{d}{4}}\,\cdot$$

$$\cdot\,\|u\|_{L^2}^{1-\frac{d}{4}}\,\|v\|_{L^2}^{1-\frac{d}{4}}\,\|\nabla\varphi\|_{L^2}$$

Inoltre se $\operatorname{div} v=0$ allora

$$\langle \operatorname{div}(u\otimes v),v\rangle=0$$

Dim

$$\|fg\|_{L^r}\le\|f\|_{L^p}\,\|g\|_{L^q}$$

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$$

$$\langle \operatorname{div}(u \otimes v), \varphi \rangle =$$

$$= \langle \operatorname{div}(u \otimes v)_j , \varphi_j \rangle$$

$$= \langle \partial_i (u_i v_j), \varphi_j \rangle$$

$$= - \langle u_i v_j , \partial_i \varphi_j \rangle$$

$$| \langle u_i v_j , \partial_i \varphi_j \rangle | \leq |u\, v|_{L^2} \, |\nabla \varphi|_{L^2}$$

$$\leq \|u\|_{L^4} \, \|v\|_{L^4} \, \|\nabla \varphi\|_{L^2}$$

$$\leq \|u\|_{L^2}^{1-\frac{d}{4}} \, \|v\|_{L^2}^{1-\frac{d}{4}} \, \|\nabla u\|_{L^2}^{\frac{d}{4}} \, \|\nabla v\|_{L^2}^{\frac{d}{4}}$$

$$\|\nabla \varphi\|_{L^2} .$$

$$\langle \operatorname{div}(u \otimes v), v \rangle$$

$$= \langle \partial_i (u_i v_j), v_j \rangle =$$

$$= - \langle u_i , v_j \, \partial_i v_j \rangle =$$

$$= -\frac{1}{2} \langle u_i , \partial_i (v_j v_j) \rangle$$

$$= + \frac{1}{2} \langle \underbrace{\partial_i u_i}_{\operatorname{div} u = 0}, |v|^2 \rangle$$

Dim.

Teor di Leray.

$$\rho \in C_c^\infty(\mathbb{R}^d, [0,1])$$

$$\int \rho(x)\, dx = 1 \qquad \rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$$

$$\operatorname{div}(u \otimes u) = (u \cdot \nabla) u$$

$$\begin{cases} u_t^{(\varepsilon)} - \Delta u^{(\varepsilon)} = - \mathbb{P}(\rho_\varepsilon * u^{(\varepsilon)} \cdot \nabla u^{(\varepsilon)}) \\ u^{(\varepsilon)}(0) = \rho_\varepsilon * u_0 \end{cases}$$

$$u^{(\varepsilon)} = e^{t\Delta} \rho_\varepsilon * u_0 - \phi_\varepsilon(u^{(\varepsilon)})$$

$$* \quad \phi_\varepsilon(u)(t) = \int_0^t e^{(t-t')\Delta} \mathbb{P}(\rho_\varepsilon * u \cdot \nabla u)\, dt'$$

Prop $*$ ha esattamente una soluzione massimale, che è globale e soddisfa

$$u \in L^\infty([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$$

$$\cap L^2([0,+\infty), \dot{H}^1(\mathbb{R}^d;\mathbb{R}^d))$$

$$\cap C^0([0,+\infty), L^2(\mathbb{R}^d,\mathbb{R}^d))$$

ed inoltre

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2 \, dt' = \|\rho_\varepsilon * u_0\|_{L^2}^2$$

<u>Lemma</u> X spazio di Banach

$B : X^2 \longrightarrow X$ ~~op~~ op. bilineare

continuo. Sia

$$0 < \alpha < \frac{1}{4\|B\|}$$

$\left(\text{dove } \|B\| = \sup\left\{\|B(x,y)\| : \begin{array}{l}\|x\|=1 \\ \|y\|=1\end{array}\right\}\right)$

Allora $\forall\ x_0 \in D_X(0,\alpha)$

$\exists!\ x \in \overline{D_X(0,2\alpha)}$ t.c.

$$x = x_0 + B(x,x)$$

<u>Dim</u> $\quad x \longrightarrow x_0 + B(x,x)$

Per prima cosa si dimostra che $\overline{D_X(0, 2\alpha)}$ è spedito in se stesso.

$$\| x_0 + B(x,x) \| \leq \| x_0 \| + \| B(x,x) \| \leq$$

$$\leq \| x_0 \| + \| B \| \, \| x \|^2$$

$$< \alpha + \| B \| \, 4 \alpha^2 =$$

$$= \alpha \, (1 + \underbrace{\| B \| \, 4 \alpha}_{< 1}) < 2 \alpha$$

$$\alpha < \frac{1}{4 \| B \|}$$

$$\| B(x,x) - B(y,y) \| =$$

$$\leq \| B(x,\cancel{x}) - B(x,y) \| + \| B(x,y) - B(y,y) \|$$

$$= \|B(x, x-y)\| + \|B(x-y, y)\|$$

$$\lesssim \|B\| \big(\overbrace{\|x\| + \|y\|}^{\leq 4\alpha}\big) \|x-y\|$$

$$\leq \underbrace{4\alpha \|B\|}_{<1} \; \|x-y\| \, .$$

Dim Prop $\qquad T > 0$

$$X = L^\infty([0,T], L^2(\mathbb{R}^d, \mathbb{R}^d)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^d, \mathbb{R}^d))$$

$$B(u,v) = -\int_0^t e^{(t-t')\Delta} \, \mathbb{P} \, \underline{\mathrm{div}} (\rho_\varepsilon * v \otimes u) \, dt'$$

$$u^{(\varepsilon)} = e^{t\Delta} \rho_\varepsilon * u_0 - \phi_\varepsilon(u^{(\varepsilon)}) \qquad \underline{\mathrm{div}(\rho_\varepsilon * u \otimes u)}$$

$$* \quad \phi_\varepsilon(u)(t) = \int_0^t e^{(t-t')\Delta} \, \mathbb{P}(\cancel{\rho_\varepsilon * u \cdot \nabla u}) \, dt'$$

$$u = \underbrace{e^{t\Delta} \rho_\varepsilon * u_0}_{x_0} + B(u,u)$$

$$B : X \times X \xrightarrow{?} X$$

$$\|B(u,v)\|_{L^\infty((0,T),L^2)\cap L^2((0,T),\dot H^1)}$$

$$B(u,v) = -\int_0^t e^{(t-t')\Delta}\, \mathbb{P}\,\mathrm{div}(\rho_\varepsilon * v \otimes u)\,dt'$$

$$(\partial_t - \Delta)B = \underbrace{-\mathbb{P}\,\mathrm{div}(\rho_\varepsilon * v \otimes u)}_{f}$$

$$B|_{t=0} = 0$$

$$\leq \cancel{2}\ \|\cancel{\mathbb{P}}\,\mathrm{div}(\rho_\varepsilon * v \otimes u)\|_{L^2((0,T),\dot H^{-1})}$$

$$\|B\|_{L^\infty((0,T),\dot H^s)} + \|B\|_{L^2((0,T),\dot H^{s+1})}$$

$$\leq \|f\|_{L^2((0,T),\dot H^{s-1})}$$

$$\leq \|\rho_\varepsilon * v \otimes u\|_{L^2((0,T),L^2)}$$

$$\leq \sqrt{T} \; \| \rho_\varepsilon * v \; u \|_{L^\infty((0,T),L^2)}$$

$$\leq \sqrt{T} \; \| \rho_\varepsilon * v \|_{L^\infty((0,T),L^\infty)} \; \| u \|_{L^\infty((0,T),L^2)}$$

$$\leq C_\varepsilon \sqrt{T} \; \| v \|_{L^\infty((0,T),L^2)} \; \| u \|_{L^\infty((0,T),L^2)}$$

$$\leq C_\varepsilon \sqrt{T} \; \| v \|_X \; \| u \|_X$$

$$\| B(u,v) \|_X \leq C_\varepsilon \sqrt{T} \; \| u \|_X \; \| v \|_X$$

$$u = e^{t\Delta} \rho_\varepsilon * u_0 + B(u,u)$$

$$\| e^{t\Delta} (\rho_\varepsilon * u_0) \|_X \leq \| \rho_\varepsilon * u_0 \|_{L^2} \leq \| u_0 \|_{L^2}$$

$$\| u_0 \|_{L^2} < \frac{1}{4 \, \| B \|}$$

$$\| B \| \leq C_\varepsilon \sqrt{T}$$

$$\|u_0\|_{L^2} \overset{*}{<} \frac{1}{4C_0\sqrt{T}} \leq \frac{1}{4\|B\|}$$

Per ogni fissato $u_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$

$\exists \quad T = T(\|u_0\|_{L^2})$ t.c.

$*$ è vera

Dal lemma so che ~~e~~ esiste

una soluzione unica in

$\overline{D_X(0, 2\|u_0\|_{L^2})}$ di:

$$u = e^{t\Delta} \rho_\varepsilon * u_0 + B(u,u)$$

$$u = e^{t\Delta} \rho_\varepsilon * u_0 - \int_0^t e^{(t-t')} \mathbb{P} \operatorname{div}(\rho_\varepsilon * u \otimes u) \, dt'$$

Da qui ricavo che $\exists !$ soluzione in X

si noti che se u e v sono soluzioni alla $u, v \in C^0([0,T], L^2)$

so che $\{t : u(t) = v(t)\}_{0 \le t}$ è chiuso

e se supponiamo che $v(t) = u(t)$ sono uguali in $[0, t_0]$ dove $t_0 \in [0,T]$, allora

$\exists\, \varepsilon > 0$ t.c. $v(t) = u(t)$ in $[t_0, t_0 + \varepsilon)$

$\Rightarrow$ $u = v$ in $[0,T]$.

Sia ora $u \in C^0([0,T^*), L^2(\mathbb{R}^d))$ una soluzione massimale.

Allora $T^* = +\infty$

~~Sia~~ Infatti dimostreremo che

se $T^* < +\infty \implies$

$$\lim_{t \to T^{*-}} \|u(t)\|_{L^2} = +\infty$$