H_B, $H_S(t)$, V $\qquad$ ρ density matrix

$H = H_S + H_B + V$

$|J_x\rangle$ eigenstate of some observable for the total system

$|j_x\rangle$ " " for the sys. alone

$\Pi_x = |J_x\rangle\langle J_x|$ $\qquad$ $\pi_x = |j_x\rangle\langle j_x|$

$$P(J_f, t \mid J_0, t=0) = \mathrm{Tr}\{\Pi_f U_t \Pi_0 U_t^\dagger \Pi_f\}$$

$\rightarrow$ time ordering

$$U_t = T \exp\left[-i\int_0^t dt' \left(H_B + H_S(t) + V\right)\right]$$

let's introduce the time reversed Hamiltonian

$H_B + H(t-t') + V$

$$U_{\tilde t} = T_{op}\left[-i\int_0^t dt' \left[H_B + H_S(t-t') + V\right]\right]$$

Time reversal operator Θ : $[\Theta, H] = 0$

$-iH\Theta \cdot = \Theta\, iH \cdot$ $\qquad$ antilinearity $\qquad$ $\tilde A = \Theta A \Theta^{-1}$

$|\tilde J_f\rangle = \Theta |J_f\rangle$ $\qquad$ $\tilde\Pi_x = \Theta |J_x\rangle\langle J_x| \Theta^{-1}$

$$P(\tilde J_0, t \mid \tilde J_f, t=0) = \mathrm{Tr}\{\tilde\Pi_0 U_{\tilde t} \tilde\Pi_f U_{\tilde t}^\dagger \tilde\Pi_0\}$$
$$= \mathrm{Tr}\{\Pi_0 U_t^\dagger \Pi_f U_t \Pi_0\} = P(J_f, t \mid J_0, t=0)$$

$$P(j_f, t \mid j_0) = \mathrm{Tr}\{\pi_f \otimes \mathbb{1}_B\, U_t\, \pi_0 \otimes \rho_B(0)\, U_t^\dagger\, \pi_f \otimes \mathbb{1}_B\}$$
$$= \mathrm{Tr}_S\{\pi_f\, \rho_S(t)\, \pi_f\}$$

$\tilde\rho_{tot}(0) = \Theta\, \pi_f \otimes \rho_B\, \Theta^{-1}$

$$P(\tilde j_0, t \mid \tilde j_f) = \mathrm{Tr}\{\tilde\pi_0 \otimes \mathbb{1}_B\, U_{\tilde t}\, \tilde\pi_f \otimes \tilde\rho_B(0)\, U_{\tilde t}^\dagger\, \tilde\pi_0 \otimes \mathbb{1}_B\}$$
$$= \mathrm{Tr}\{\pi_0 \otimes \mathbb{1}_B\, U_t^\dagger\, \pi_f \otimes \rho_B(0)\, U_t\, \pi_0 \otimes \mathbb{1}_B\}$$
$$\neq P(j_f, t, j_0)$$