24 marzo

$\{u_n\}$

$$\begin{cases} \partial_t u_n - \Delta u_n = -\mathbb{P}\,\mathrm{div}(\rho_{\varepsilon_n} * u_n \otimes u_n) \\ u_n|_{t=0} = \rho_{\varepsilon_n} * u_0 \end{cases}$$

$u_n \rightharpoonup u$

in $L^{\frac{10}{3}}(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R}^3)$

$\forall\, T > 0$ e $\forall\, K \subset\subset \mathbb{R}^3$

$u_n \to u$ in $L^2([0,T] \times \mathbb{R}^3, \mathbb{R}^3)$

$\phi \in L^2(\mathbb{R}^3, \mathbb{R}^3)$

$\langle u_n(t), \phi \rangle \longrightarrow \langle u(t), \phi \rangle$

in $C^0([0,T])$

$\langle \chi_K P_{n_0} u_n(t), \phi \rangle \longrightarrow \langle \chi_K P_{n_0} u(t), \phi \rangle$

$$u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3, \mathbb{R}^3))$$

$$\nabla u \in L^2(\mathbb{R}_+, L^2(\mathbb{R}^3, \mathbb{R}^3))$$

$$|u(t)|_{L^2}^2 + 2\int_0^t |\nabla u|_{L^2}^2 \, dt'$$

$$\leq |u_0|_{L^2}^2$$

$$< u_n(t), \psi(t) > =$$

$$= \int_0^t \Big(< u_n, \Delta \psi > + < u_n, \partial_t \psi >$$

$$< \mathbb{P} \operatorname{div}(\varsigma_{\varepsilon_n} * u_n \otimes u_n), \psi \Big) \, dt'$$

$$+ < \varsigma_{\varepsilon_n} * u_0, \psi(0) >$$

$$\forall \; \psi \in C^\infty_{c\sigma}([0, +\infty) \times \mathbb{R}^3, \mathbb{R}^3)$$

$$< u(t), \Psi(t) > =$$

$$= \int_0^t \left(< u , \Delta \Psi > + < u , \partial_t \Psi > \right) dt'$$

$$+ \lim_{n \to +\infty} \int_0^t < \mathbb{P} \mathrm{div} (\rho_{\varepsilon_n} * u_n \otimes u_n), \Psi) \, dt'$$

$$+ < u_0, \Psi(0) >$$

Resto da dimostrare che

$$\lim_{n \to +\infty} \int_0^t < (\rho_{\varepsilon_n} * u_n \otimes u_n), \nabla \Psi > dt'$$

$$= \int_0^t < u \otimes u , \nabla \Psi > dt'$$

$t \in [0,T]$ $\Psi \in C^0([0,T], H^1)$

$\forall \varepsilon > 0$ so che $\exists$ $K \subset\subset \mathbb{R}^d$

t.c. $\sup_{0 \le t \le T} \| \chi_{\mathbb{R}^d \setminus K} \nabla \Psi \|_{L^2} < \varepsilon$

$$\int_0^t dt' \left| \int_{\mathbb{R}^d \setminus K} \rho_{\varepsilon_n} * u_n \otimes u_n : \nabla \psi \, dx \right|$$

$$\leq \int_0^t dt' \quad |\rho_{\varepsilon_n} * u_n \otimes u_n|_{L^2(\mathbb{R}^d)} \quad |\nabla \psi|_{L^2(\mathbb{R}^d \setminus K)}$$

$$\leq \int_0^t dt' |\rho_{\varepsilon_n} * u_n \otimes u_n|_{L^2(\mathbb{R}^d)} \quad \varepsilon$$

$$\leq T^{\frac{4-d}{4}} \left| \, |\rho_{\varepsilon_n} * u_n \otimes u_n|_{L^2(\mathbb{R}^d)} \right|_{L^{\frac{4}{d}}(0,T)} \varepsilon$$

$$\leq T^{\frac{4-d}{4}} \varepsilon \, \left| |\cancel{\rho_{\varepsilon_n}} * u_n|_{L^4(\mathbb{R}^d)} \, |u_n|_{L^4(\mathbb{R}^d)} \right|_{L^{\frac{4}{d}}(0,T)}$$

$$\leq T^{\frac{4-d}{4}} \varepsilon \quad \left| \quad |u_n|^2_{L^4(\mathbb{R}^d)} \right|_{L^{\frac{4}{d}}(0,T)}$$

$$|u_n|_{L^4(\mathbb{R}^d)} \leq |u_n|^{1-\frac{d}{4}}_{L^2(\mathbb{R}^d)} \quad |\nabla u_n|^{\frac{d}{4}}_{L^2(\mathbb{R}^d)}$$

$$\lesssim T^{\frac{4-d}{4}} \varepsilon \left| \, |u_n|^{2(1-\frac{d}{4})}_{L^2(\mathbb{R}^d)} \quad |\nabla u_n|^{\frac{d}{2}}_{L^2(\mathbb{R}^d)} \right|_{L^{\cancel{4/d}}(0,T)}$$

$$\leq T^{\frac{k-d}{k}} \varepsilon \quad |u_0|^{2(1-\frac{d}{4})}_{L^2(\mathbb{R}^d)} \quad \Big| \, |\nabla u_n|^{\frac{d}{2}}_{L^2(\mathbb{R}^d)} \Big|_{L^{\frac{4}{d}}(0,T)}$$

$$= T^{\frac{k-d}{k}} \varepsilon \quad |u_0|^{2(1-\frac{d}{4})}_{L^2(\mathbb{R}^d)} \quad \underbrace{\Big| \, |\nabla u_n|_{L^2(\mathbb{R}^d)} \Big|^{\frac{d}{2}}_{L^2(0,T)}}_{\leq |u_0|^{\frac{d}{2}}_{L^2(\mathbb{R}^d)}}$$

$$\int_0^t dt' \left| \int_{\mathbb{R}^d \setminus K} \rho_{\varepsilon_n} * u_n \otimes u_n : \nabla \varphi \, dx \right|$$

$$\leq C_T \, \varepsilon \, |u_0|^2_{L^2(\mathbb{R}^d)}$$

$$\lim_{n\to+\infty}\int_0^t \langle (\rho_{\varepsilon_n} * u_n \otimes u_n), \nabla\psi \rangle_{L^2(K)} dt'$$

$$= \int_0^t \langle u \otimes u , \nabla\psi \rangle_{L^2(K)} dt'$$

Verifichemo che

$$* \lim_{n\to+\infty} \rho_{\varepsilon_n} * u_n \otimes u_n = u \otimes u$$

in $L^1([0,T], L^2(K))$

$*$ e' una conseguenza di

$$\lim_{n\to+\infty} u_n = u \text{ in } L^2([0,T], L^4(K))$$

$$\| \rho_{\varepsilon_n} * u_n \otimes u_n - u \otimes u \|_{L^1([0,T], L^2(K))}$$

$$|\rho_{\varepsilon_n} * u_n \otimes u_n - u \otimes u_n + u \otimes u_n - u \otimes u|_{L^1([0,T], L^2(K))}$$

$$|u \otimes u_n - u \otimes u|_{L^1([0,T], L^2(K))}$$

$$= |u \otimes (u_n - u)|_{L^1([0,T], L^2(K))}$$

$$\leq |u|_{L^2([0,T], L^4(K))} |u_n - u|_{L^2([0,T], L^4(K))}$$

$$|\nabla u|_{L^2([0,T], L^2(\mathbb{R}^4))}$$

$$\gtrsim |u|_{L^2([0,T], L^6(\mathbb{R}^4))}$$

$$\leq C_T |u_0|_{L^2} |u_n - u|_{L^2([0,T], L^4(K))}$$

$$\downarrow$$
$$0$$

$$|\rho_{\varepsilon_n} * u_n \otimes u_n - u \otimes u_n|_{\underbrace{L^1([0,T],L^2(K))}_{X}}$$

$$= |\rho_{\varepsilon_n} * u_n \otimes u_n - \rho_{\varepsilon_n} * u \otimes u_n$$

$$\rho_{\varepsilon_n} * u \otimes u_n - u \otimes u_n|_X$$

$$n$$

$$|\rho_{\varepsilon_n} * u \otimes u_n - u \otimes u_n|_X \to 0$$

$$\overbrace{|\rho_{\varepsilon_n} * u - u|_{L^2([0,T],L^4(\mathbb{R}^d))}}^{\to 0}$$

$$|u_n|_{L^2([0,T],L^4(K))}$$

$$\downarrow$$

$$|u|_{L^2([0,T],L^4(K)))}$$

$$\rho_{\varepsilon_n}^* - 1 \xrightarrow{n\to+\infty} 0$$

$$(\rho_\varepsilon * - id) f \longrightarrow 0 \quad \text{in } L^4$$

$$\forall f \in L^4(\mathbb{R}^d)$$

$u \in L^2$

$$|\rho_{\varepsilon_n} * u - u|_{L^2([0,T], L^4(\mathbb{R}^d))}$$

$$\downarrow n \to +\infty$$

$$0$$

si dimostra prima supponendo

$$u \in C^0([0,T], L^4(\mathbb{R}^d))$$

e poi $u \in L^2([0,T], L^4(\mathbb{R}^d))$

$$\lim_{n\to+\infty} u_n = u \quad \text{in } L^2([0,T], L^4(K))$$

$$K \subset\subset \mathbb{R}^d$$

$$\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$$

t.c. $\chi_{|K} \equiv 1$

pongo $\Omega = \mathrm{supp}\,\chi$

e assumo $|\nabla \chi|_{L^\infty(\mathbb{R}^d)} \le 1$

$$|f|_{L^4(K)} \le |\chi f|_{L^2(\mathbb{R}^d)} \le$$

$$\le |\chi f|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{4}} \; |\nabla(\chi f)|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}$$

$$\le |f|_{L^2(\Omega)}^{1-\frac{d}{4}} \left(|\nabla f|_{L^2(\mathbb{R}^d)} + |f|_{L^2(\Omega)} \right)^{\frac{d}{4}}$$

$$\leq |f|_{L^2(\Omega)}^{1-d/4} \quad |f|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}}$$

$$|f|_{L^4(K)}$$

$$\leq |f|_{L^2(\Omega)}^{1-d/4} \quad |f|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}}$$

$$|u-u_n|_{L^2([0,T],L^4(K))} =$$

$$\Big| \ |u-u_n|_{L^4(K)} \ \Big|_{L^2([0,T])}$$

$$\leq \Big| \ |u-u_n|_{L^2(\Omega)}^{1-\frac{d}{4}} \ |u-u_n|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \ \Big|_{L^2([0,T])}$$

$$\frac{1}{2} = \frac{4-d}{8} + \frac{d}{8}$$

$$\leq \left\| \|u-u_n\|_{L^2(\Omega)}^{\frac{4-d}{4}} \right\|_{L^{\frac{8}{4-d}}(0,T)} \left\| \|u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \right\|_{L^{\frac{8}{d}}(0,T)}$$

$$= \underbrace{\left\| \|u-u_n\|_{L^2(\Omega)} \right\|_{L^2([0,T])}^{\frac{4-d}{4}}} \left\| \|u-u_n\|_{H^1(\mathbb{R}^d)} \right\|_{L^2(0,T)}^{\frac{d}{4}}$$

in $L^2([0,T]\times\Omega, \mathbb{R}^d)$

$u_n \longrightarrow u$

$n \to +\infty$

0

$$\|u\|_{L^2((0,T),H^1)} + \|u_n\|_{L^2((0,T),H^1)}$$

$$\leq C_T \|u_0\|_{L^2}$$

$$\partial_t u - \Delta u = -\mathbb{P}\,\mathrm{div}(u\otimes u) = Q(u,u)$$

$$Q(u,v) = -\frac{1}{2}\mathbb{P}\left(\mathrm{div}(u\otimes v) + \mathrm{div}(v\otimes u)\right)$$

$$\begin{cases} \partial_t B(u,v) - B(u,v) = Q(u,v) \\ B(u,v)\big|_{t=0} = 0 \end{cases}$$

$$u = e^{t\Delta} u_0 + B(u,u) \qquad (*)$$

$$X = L^4\left([0,T), \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d)\right)$$

Teor Sia $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$.
Allora $\exists\ T > 0$ ed una soluzione di $(*)$, è unica

$$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d))$$

$$\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d))$$

Sia T_{u_0} la life span.

1) $\exists\ C > 0$ t.c. $\qquad \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

$$|u_0|_{\dot{H}^{\frac{d}{2}-1}} \le C \implies$$

$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \qquad L^3(\mathbb{R}^3)$

$T_{u_0} = +\infty$

2) se $T_{u_0} < +\infty$ allora

$$\int_0^{T_{u_0}} \| u(t) \|^4_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)} dt = +\infty \qquad \dot{H}^{\frac{1}{2}}(\mathbb{R}^3,\mathbb{R}^3)$$

3) se u e v sono soluzioni allora

$$\| u(t) - v(t) \|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t \| \nabla (u-v) \|^2_{\dot{H}^{\frac{d}{2}-1}} dt$$

$$\leq \| u_0 - v_0 \|_{\dot{H}^{\frac{d}{2}-1}} \, e^{C_d \int_0^t \left(\|u\|^4_{\dot{H}^{\frac{d-1}{2}}} + \|v\|^4_{\dot{H}^{\frac{d-1}{2}}} \right) dt'}$$

$$\begin{cases} \partial_t u - \Delta u = - \mathbb{P} \operatorname{div}(u \otimes u) \\ u(0,\cdot) = u_0 \end{cases}$$

$u(t,x) \qquad \lambda > 0$

$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$

$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \qquad \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$

$$u_\lambda(0,x) = \lambda u_0(\lambda x) \qquad \lambda^{\frac{d}{q}} u_0(\lambda\,\cdot)$$

$$|\lambda u_0(\lambda\,\cdot)|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} = |u_0|_{\dot{H}^{\frac{d}{2}-1}}$$

$$|\lambda u_0(\lambda\,\cdot)|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} =$$

$$= \left|\lambda\, |\xi|^{\frac{1}{2}} \quad \lambda^{-3}\, \hat{u}_0\left(\frac{\cdot}{\lambda}\right)\right|_{L^2(\mathbb{R}^3)}$$

$$= \lambda \lambda^{-3} \lambda^{\frac{1}{2}} \left| \quad \left|\frac{\xi}{\lambda}\right|^{\frac{1}{2}} \hat{u}_0\left(\frac{\xi}{\lambda}\right)\right|_{L^2(\mathbb{R}^3)}$$

$$= \lambda \lambda^{-3+\frac{1}{2}} \quad \lambda^{\frac{3}{2}} \quad \underbrace{\left|\, |\xi|^{\frac{1}{2}}\, \hat{u}(\xi)\right|_{L^2(\mathbb{R}^2}}_{|u_0|_{\dot{H}^{\frac{1}{2}}}}$$

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

$$\frac{1}{3} = \frac{1}{2} - \frac{\frac{1}{2}}{3} = \frac{1}{2} - \frac{1}{6}$$

$$B^{\frac{1}{2}}_{r\,\infty}$$