Riassunto :

Q : SPAZIO DELLE CONFIGURAZIONI (n-dimensionale)

$p \in Q \rightsquigarrow$ determina la configurazione del sistema,
cioè (per un sist. meccanico discreto)
la posizione di tutti gli N pti
$\bar{r}_i(q,t) \qquad i=1,\dots,N$

coord. $(q_1,\dots,q_n)$

TQ : SPAZIO DEGLI STATI

$(p,u) \in TQ \rightsquigarrow$ determina lo stato del sistema,
cioè (per un sist. meccanico discreto)
la posizione e la velocità
di tutti gli N pti

coord. $(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_n)$

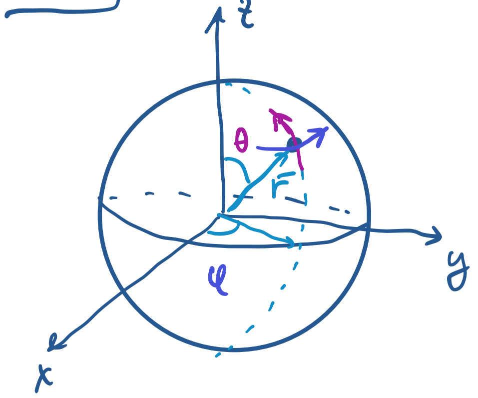
$$\bar{r}_i(q,t)\ , \quad \bar{v}_i(q,\dot{q},t) = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k}(q,t)\dot{q}_k + \frac{\partial \bar{r}}{\partial t}(q,t)$$
$i=1,\dots,N$

MOTO : funzioni $q_1(t),\dots,q_n(t) \rightsquigarrow$ permettono di descrivere il moto di tutti i pti
$$\bar{r}_i(t) = \bar{r}_i(q(t),t) \qquad i=1,\dots,N$$

EQ. del MOTO : dinamica del sistema data dalle
Eq. di Lagrange $\qquad L(q,\dot{q},t)$ è detta "Lagrangiana"
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_h}(q(t),\dot{q}(t),t) - \frac{\partial L}{\partial q_h}(q(t),\dot{q}(t),t) = 0 \qquad h=1,\dots,n$$

$\rightsquigarrow$ n eq. diff. (del secondo ordine) in n incognite

ES. PENDOLO SFERICO (pto materiale vincolato a una sfera di raggio R, soggetto alla forza di gravità)



$$x = R\,\text{sen}\theta\cos\varphi$$
$$y = R\,\text{sen}\theta\,\text{sen}\varphi \quad \leftarrow \quad \bar{r}(\varphi, \theta)$$
$$z = R\cos\theta$$

$$\frac{\partial \bar{r}}{\partial \varphi} \qquad \frac{\partial \bar{r}}{\partial \theta}$$

En. cinetica

$$\dot{x} = -R\,\text{sen}\varphi\,\text{sen}\theta\,\dot{\varphi} + R\cos\varphi\cos\theta\,\dot{\theta}$$
$$\dot{y} = R\cos\varphi\,\text{sen}\theta\,\dot{\varphi} + R\,\text{sen}\varphi\cos\theta\,\dot{\theta} \quad \leftarrow \quad \bar{v}(\bar{q}, \dot{\bar{q}})$$
$$\dot{z} = -R\,\text{sen}\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) =$$

$$= \frac{1}{2}m\Big[\big(R^2\text{sen}^2\varphi\,\text{sen}^2\theta\,\dot{\varphi}^2 + R^2\cos^2\varphi\cos^2\theta\,\dot{\theta}^2 - 2R^2\cancel{\cos\varphi\,\text{sen}\varphi}\cos\theta\,\text{sen}\theta\,\dot{\varphi}\dot{\theta}\big)$$
$$+ \big(R^2\cos^2\varphi\,\text{sen}^2\theta\,\dot{\varphi}^2 + R^2\text{sen}^2\varphi\cos^2\theta\,\dot{\theta}^2 + 2R^2\cancel{\cos\varphi\,\text{sen}\varphi}\cos\theta\,\text{sen}\theta\,\dot{\varphi}\dot{\theta}\big)$$
$$+ R^2\text{sen}^2\theta\,\dot{\theta}^2\Big] =$$

$$= \frac{1}{2}m\left(R^2\text{sen}^2\theta\,\dot{\varphi}^2 + R^2\cos^2\theta\,\dot{\theta}^2 + R^2\text{sen}^2\theta\,\dot{\theta}^2\right)$$

$$= \frac{1}{2}m\left(R^2\text{sen}^2\theta\,\dot{\varphi}^2 + R^2\dot{\theta}^2\right) \leftarrow a = \begin{pmatrix} mR^2\text{sen}^2\theta & 0 \\ 0 & mR^2 \end{pmatrix}$$

dip. da q

$$= \frac{1}{2}\sum_{h,k} a_{hk}\,\dot{q}_h\dot{q}_k$$

$$V = mgz = mgR\cos\theta$$

$$L = \frac{mR^2}{2}\left(\text{sen}^2\theta\,\dot{\varphi}^2 + \dot{\theta}^2\right) - mgR\cos\theta$$

$L(\varphi, \theta, \dot{\varphi}, \dot{\theta})$

$n = 2$ gradi di libertà (ci aspettiamo 2 eq. di Lagrange)

Eq. di Lagrange

$\theta(t)$ $\quad$ $\dot\varphi(t)$

$$\frac{\partial L}{\partial \dot\varphi} = mR^2 \operatorname{sen}^2\theta\,\dot\varphi \quad\rightarrow\quad \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = 2mR^2 \operatorname{sen}\theta\cos\theta\,\dot\theta\dot\varphi + mR^2\operatorname{sen}^2\theta\,\ddot\varphi$$

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \quad\rightarrow\quad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = mR^2\ddot\theta$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \theta} = mR^2 \operatorname{sen}\theta\cos\theta\,\dot\varphi^2 + mgR\operatorname{sen}\theta$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 2mR^2\operatorname{sen}\theta\cos\theta\,\dot\theta\dot\varphi + mR^2\operatorname{sen}^2\theta\,\ddot\varphi$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = mR^2\ddot\theta - mR^2\operatorname{sen}\theta\cos\theta\,\dot\varphi^2 - mgR\operatorname{sen}\theta$$

$$\ddot\theta = \operatorname{sen}\theta\cos\theta\,\dot\varphi^2 + \frac{g}{R}\operatorname{sen}\theta$$

INVARIANZA PER CAMBIAMENTO DI COORDINATE

Iniziamo con un esempio.

ES. Punto vincolato sul piano non soggetto a forze

1 – coord cartesiane $\bar{r}(q_1, q_2) = \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}$

legate da trasf. di coord

2 – coord polari $\bar{r}(\tilde{q}_1, \tilde{q}_2) = \begin{pmatrix} \tilde{q}_1 \cos\tilde{q}_2 \\ \tilde{q}_1 \operatorname{sen}\tilde{q}_2 \end{pmatrix}$

Trasformazione di coordinate

$$\begin{cases} q_1 = \tilde{q}_1 \cos\tilde{q}_2 \\ q_2 = \tilde{q}_1 \operatorname{sen}\tilde{q}_2 \end{cases} \quad (*) \quad \leadsto \quad \begin{aligned} \dot{q}_1 &= \dot{\tilde{q}}_1 \cos\tilde{q}_2 - \tilde{q}_1 \dot{\tilde{q}}_2 \operatorname{sen}\tilde{q}_2 \\ \dot{q}_2 &= \dot{\tilde{q}}_1 \operatorname{sen}\tilde{q}_2 + \tilde{q}_1 \dot{\tilde{q}}_2 \cos\tilde{q}_2 \end{aligned}$$

$$L(q_1, q_2) = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) \xrightarrow{L \circ (*)} \tilde{L}(\tilde{q}_1 + \tilde{q}_2) = \frac{m}{2}\left(\dot{\tilde{q}}_1^2 + \tilde{q}_1^2 \dot{\tilde{q}}_2^2\right)$$

$$\quad\quad = L(q(\tilde{q}), \dot{q}(\tilde{q}, \dot{\tilde{q}}))$$

Eq. Lagrange

$$\begin{cases} \ddot{q}_1 = 0 \\ \ddot{q}_2 = 0 \end{cases}$$

soluz. con dato iniziale

$q_1^0 = q_2^0 = 1$

$\dot{q}_1^0 = \dot{q}_2^0 = v/\sqrt{2}$

$$q_1(t) = 1 + \frac{v}{\sqrt{2}} t$$

$$q_2(t) = 1 + \frac{v}{\sqrt{2}} t$$

Eq. Lagrange

$$\begin{cases} \ddot{\tilde{q}}_1 = \tilde{q}_1 \dot{\tilde{q}}_2^2 \\ 2\dot{\tilde{q}}_1 \tilde{q}_1 \dot{\tilde{q}}_2 + \tilde{q}_1^2 \ddot{\tilde{q}}_2 = 0 \end{cases}$$

soluz. con dato iniziale

$\tilde{q}_1^0 = \sqrt{2}$ $\quad \tilde{q}_2^0 = \pi/4$

$\dot{\tilde{q}}_1^0 = v$ $\quad \dot{\tilde{q}}_2^0 = 0$

$$\tilde{q}_1(t) = vt$$

$$\tilde{q}_2(t) = \pi/4$$

$$\leadsto \quad q(t) = q(\tilde{q}(t))$$

Trasf. di coord.

$$q_h = q_h(\tilde{q}_1, \dots, \tilde{q}_m, t) \qquad \text{t.c.} \qquad \det\left(\frac{\partial q_h}{\partial \tilde{q}_k}\right) \neq 0 \qquad (\star)$$

$$\dot{q}_h = \dot{q}_h(\tilde{\bar{q}}, \dot{\tilde{\bar{q}}}, t) = \sum_{k=1}^{m} \frac{\partial q_h}{\partial \tilde{q}_k} \dot{\tilde{q}}_k + \frac{\partial q_h}{\partial t}$$

Prop. Dato un sist. Lagrangiano con Lagrangiana $L(\bar{q}, \dot{\bar{q}}, t)$, si consideri il camb. di coord. (regolare e invertibile) $(\star)$ e sia $\tilde{L}(\tilde{\bar{q}}, \dot{\tilde{\bar{q}}}, t)$ la Lagrangiana ottenuta da L per SOSTITUZIONE di $(\star)$, cioè

$$\tilde{L}(\tilde{\bar{q}}, \dot{\tilde{\bar{q}}}, t) \equiv L\left(\bar{q}(\tilde{\bar{q}}, t), \dot{\bar{q}}(\tilde{\bar{q}}, \dot{\tilde{\bar{q}}}, t), t\right) \qquad (\bullet)$$

Allora $\tilde{\bar{q}}(t)$ è soluzione delle eq. di Lagrange con Lagrangiana $\tilde{L}$ SE E SOLO SE $\bar{q}(t) = \bar{q}(\tilde{\bar{q}}(t), t)$ è soluzione delle eq. di Lagrange con Lagrangiana L

Dim.

$$\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} \overset{(\bullet)}{=} \sum_{l=1}^{m} \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \dot{\tilde{q}}_h} \overset{(\star)}{=} \sum_{l=1}^{m} \frac{\partial L}{\partial \dot{q}_l} \cancel{\sum_{k=1}^{m}} \frac{\partial q_l}{\partial \tilde{q}_k} \underbrace{\frac{\partial \dot{\tilde{q}}_k}{\partial \dot{\tilde{q}}_h}}_{\delta_{kh}}$$

$$= \sum_{l=1}^{m} \frac{\partial L}{\partial \dot{q}_l} \frac{\partial q_l}{\partial \tilde{q}_h}$$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} = \sum_{l=1}^{m} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}\right) \frac{\partial q_l}{\partial \tilde{q}_h} + \sum_{l=1}^{m} \frac{\partial L}{\partial \dot{q}_l} \frac{d}{dt} \frac{\partial q_l}{\partial \tilde{q}_h}(\tilde{\bar{q}}(t), t)$$

$$= \sum_{m=1}^{m} \frac{\partial^2 q_l}{\partial \tilde{q}_h \partial \tilde{q}_m} \dot{\tilde{q}}_m + \frac{\partial^2 q_l}{\partial \tilde{q}_h \partial t} = \quad f(\tilde{q}, \dot{\tilde{q}}, t) \text{ valutata in } \tilde{q}(t)$$

$$= \frac{\partial}{\partial \tilde{q}_h}\left(\sum_{m=1}^{m} \frac{\partial q_l}{\partial \tilde{q}_m} \dot{\tilde{q}}_m + \frac{\partial q_l}{\partial t}\right) = \frac{\partial}{\partial \tilde{q}_h} \dot{q}_l$$

$$= \sum_{l=1}^{m} \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}\right) \frac{\partial q_l}{\partial \tilde{q}_h} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \tilde{q}_h}\right]$$

$$\frac{\partial \tilde{L}}{\partial \tilde{q}_h} = \sum_{l=1}^{m} \left[\frac{\partial L}{\partial q_l} \frac{\partial q_l}{\partial \tilde{q}_h} + \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial \tilde{q}_h} \right]$$

$$\Rightarrow \underbrace{\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} - \frac{\partial \tilde{L}}{\partial \tilde{q}_h}}_{\text{eq. di Lagr. di } \tilde{L}\ (\tilde{*})} = \overbrace{\sum_{l=1}^{m} \Big(\underbrace{\frac{\partial q_l}{\partial \tilde{q}_h}}_{\equiv J_{hl} \text{ invertibile}} \Big) \Big[\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l}}_{\text{eq. di Lagr. di } L\ (*)} \Big]}^{\sum_l J_{hl} V_l} \qquad h=1,\dots,m$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} \equiv V_l$$

Se $(*)$ sono sodd. $\forall l \Longrightarrow$ $(\tilde{*})$ sono soddisf. $\forall h$ (manifesto)

Se $(\tilde{*})$ sono sodd. $\forall h \Longrightarrow \sum_{l=1}^{m} J_{hl} V_l = 0 \iff J \cdot \overline{V} = 0$

$\underset{J \text{ invertibile}}{\Longrightarrow}$ $(*)$ sono sodd. $\forall l$

(moltiplichiamo a destra e sinistra per J^{-1})

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Diverse Lagrangiane possono portare alle stesse eq. del moto.

<u>Prop.</u> Per ogni scelta della funzione $F(\bar{q},t)$ e della cost. $c \neq 0$, la Lagrangiana $L(\bar{q},\dot{\bar{q}},t)$ e la Lagrangiana

$$L'(\bar{q},\dot{\bar{q}},t) = c\,L(\bar{q},\dot{\bar{q}},t) + \varphi(\bar{q},\dot{\bar{q}},t),$$

dove $\varphi(\bar{q},\dot{\bar{q}},t) = \sum_{k=1}^{m} \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}$,

"derivata totale" $\varphi(\bar{q}(t),\dot{\bar{q}}(t),t) = \frac{d}{dt}F(\bar{q}(t),t)$

conducono alle stesse eq. di Lagrange.

"Lagrangiane che differiscono per una DERIVATA TOTALE (in t) sono (classicam.) EQUIVALENTI"

Dim.
$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot q_h} = c\,\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_h} + \underbrace{\frac{d}{dt}\frac{\partial \varphi}{\partial \dot q_h}}$$

$$= \frac{d}{dt}\frac{\partial F}{\partial q_h} = \sum_{l=1}^{m}\frac{\partial^2 F}{\partial q_h \partial q_l}\dot q_l + \cancel{\frac{\partial^2 F}{\partial q_h \partial t}}$$

$$\frac{\partial \mathcal{L}'}{\partial q_h} = c\,\frac{\partial \mathcal{L}}{\partial q_h} + \underbrace{\frac{\partial \varphi}{\partial q_h}}$$

$$= \sum_{k=1}^{m}\frac{\partial^2 F}{\partial q_h \partial q_k}\dot q_k + \cancel{\frac{\partial^2 F}{\partial q_h \partial t}}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot q_h} - \frac{\partial \mathcal{L}'}{\partial q_h} = c\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_h} - \frac{\partial \mathcal{L}}{\partial q_h}\right) \qquad //$$

$c \neq 0$