27 marzo

$$\begin{cases} \partial_t B(u,v) - B(u,v) = Q(u,v) \\ B(u,v)\big|_{t=0} = 0 \end{cases}$$

$$u = e^{t\Delta} u_0 + B(u,u) \qquad *$$

$$X = L^4([0,T), \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d))$$

Teor Sia $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$.
Allora $\exists$ $T > 0$ ed una soluzione di $*$, è unica

$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d))$

$\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d))$

Sia T_{u_0} la life span.

1) $\exists$ $c > 0$ t.c. $\qquad \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

$$|u_0|_{\dot{H}^{\frac{d}{2}-1}} \le c \implies$$

$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

$T_{u_0} = +\infty$

2) se $T_{u_0} < +\infty$ allora

$$\int_0^{T_{u_0}} |u(t)|^4_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d)} dt = +\infty$$

3) se u e v sono soluzioni allora

$$|u(t)-v(t)|^2_{\dot H^{\frac{d}{2}-1}} + \int_0^t |\nabla(u-v)|^2_{\dot H^{\frac{d}{2}-1}} dt$$

$$\leq |u_0 - v_0|^2_{\dot H^{\frac{d}{2}-1}} \, e^{C_d \int_0^t \left(|u|^4_{\dot H^{\frac{d-1}{2}}} + |v|^4_{\dot H^{\frac{d-1}{2}}}\right) dt'}$$

$u(t,x)$

$u_\lambda(t,x) = \lambda \, u(\lambda^2 t, \lambda x)$

$u_\lambda(0) = \lambda \, u_0(\lambda \cdot)$ $\quad \dot H^{\frac{d}{2}-1}(\mathbb{R}^d)$

$\|u_0\|_{\dot H^{\frac{d}{2}-1}} \leq M$

$T(M)$

se u, u_0 $T_{u_0} = T^* < +\infty$

$u_\lambda(t,x) = \lambda\, u(\lambda^2 t, \lambda x)$

$u_\lambda(t)$ ha tempo di esistenza

$\lambda^2 T^*_\lambda = T^*$

$$T^*_\lambda = \frac{T^*}{\lambda^2} \quad \text{contro}$$

e parità di norma del dato iniziale $\|u_\lambda(0)\|_{\dot H^{\frac{d}{2}-1}} = \|u_0\|_{\dot H^{\frac{d}{2}-1}}$

se $T_{u_0} < +\infty$

$$\|u\|_{L^\infty((0,T_{u_0}),\, \dot H^{\frac12}(\mathbb{R}^3))} = +\infty$$

<u>Dim</u>

Lemma $d = 2,3$ $\quad \exists\, \varepsilon_d > 0$ tc

$$\| Q(u,v) \|_{\dot H^{\frac{d}{2}-2}(\mathbb{R}^d,\mathbb{R}^d)}$$

$$= \left\| \frac{1}{2} \mathbb{P}\left[\operatorname{div}(u \otimes v) + \operatorname{div}(v \otimes u) \right] \right\|_{\dot H^{\frac{d}{2}-2}}$$

(applicheremo l'equazione del calore con $s = \frac{d}{2} - 1$ in modo che

$u_0 \in \dot H^s(\mathbb{R}^d)$

$\partial_t u - \Delta u = f$

$$f \in L^2([0,T], \dot H^{s-1}(\mathbb{R}^d))$$

$$\| Q(u,v) \|_{\dot H^{\frac{d}{2}-2}(\mathbb{R}^d)} \le C_d \| u \|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)} \| v \|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)}$$

<u>Dim</u> $d = 3$ $\qquad \frac{d-1}{2} = 1$

$$\| Q(u,v) \|_{\dot H^{-\frac{1}{2}}(\mathbb{R}^3)} \le$$

$$\| \mathbb{P} \operatorname{div}(u \otimes v) \|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$$

$$\leq \| \operatorname{div}(u \otimes v) \|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$$

$$= \sum_k \| \partial_j (u_j v_k) \|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$$

$$\leq \sum_{k,j} \| u_j \partial_j v_k \|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$$

$$\leq \| u \nabla v \|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$$

$L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ e' il duale di

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

$$\frac{1}{3} = \frac{1}{2} - \frac{\frac{1}{2}}{3} = \frac{1}{2} - \frac{1}{6}$$

$$\leq \| u \nabla v \|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\leq |u|_{L^6} |\nabla v|_{L^2}$$

$$\leq |\nabla u|_{L^2(\mathbb{R}^3)} |\nabla v|_{L^2(\mathbb{R}^3)}$$

$$\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$$

$$\cancel{\exists\ C_d \quad t.c.}$$

<u>Lemma</u> $$|Q(u,v)|_{L^2([0,T], \dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d))}$$

$$\leq C_d |u|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} \cdot$$

$$\cdot |v|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}$$

Dim

$$\left| |Q(u,v)|_{\dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d)} \right|_{L^2([0,T])}$$

$$\leq C_d \left| |u|_{\dot{H}^{\frac{d-1}{2}}} |v|_{\dot{H}^{\frac{d-1}{2}}} \right|_{L^2([0,T])}$$

$$\leq C_d |u|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} |v|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}$$

$$\begin{cases} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \\ B(u,v)\big|_{t=0} = 0 \end{cases} \quad Q(u,v) \in$$

$$\in L^2([0,T], \dot H^{\frac{d}{2}-2}) = L^2([0,T], \dot H^{s-1})$$

$1 - \mathbb{P}$

$$\begin{cases} (\partial_t - \Delta)(1-\mathbb{P}) B(u,v) = 0 \\ (1-\mathbb{P}) B(u,v)\big|_{t=0} = 0 \end{cases} \quad s = \frac{d}{2} - 1$$

$$u, v \in X = L^4([0,T], \dot H^{\frac{d-1}{2}})$$

$s = \frac{d}{2} - 1, \; p = 4$

$$|B(u,v)|_{L^4([0,T], \dot H^{\frac{d-1}{2}})}$$

$$= \Big| \, |B(u,v)|_{\dot H^{s+\frac{2}{p}}} \Big|_{L^p(0,T)} \qquad s + \frac{2}{p} = \frac{d}{2} - 1 + \frac{1}{2} = \frac{d-1}{2}$$

$$\leq |Q(u,v)|_{L^2([0,T], \dot H^{s-1})}$$

$$= |Q(u,v)|_{L^2([0,T], \dot H^{\frac{d}{2}-2}(\mathbb{R}^d))}$$

$$\leq C_d \, |u|_X \, |v|_X$$

$$|B(u,v)|_X \leq C_d \, |u|_X \, |v|_X$$

$$\|B\| \leq C_d \qquad X = L^2\left((0,T), \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d)\right)$$

$$\begin{cases} \partial_t u - \Delta u = Q(u,u) \\ u_{|t=0} = u_0 \end{cases}$$

$$u = e^{t\Delta} u_0 + B(u,u)$$

Applicheremo

Lemma Dato $B : X \times X \to X$ e dato $\alpha < \frac{1}{4\|B\|}$

se $x_0 \in D_X(0,\alpha)$

allora $\exists$ una unica soluzione $x \in \overline{D_X(0,2\alpha)}$ di

$$x = x_0 + B(x,x)$$

$$u = e^{t\Delta} u_0 + B(u,u) \quad .$$

$$\| e^{t\Delta} u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}$$

$$\leq \| e^{t\Delta} u_0 \|_{L^4([0,+\infty), \dot{H}^{\frac{d-1}{2}})}$$

$$\leq \| u_0 \|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)}$$

$$\|B\| \leq C_d$$

$$\frac{1}{4\|B\|} \geq \frac{1}{4C_d}$$

Se $\quad \| u_0 \|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} < \frac{1}{4C_d}$

$$\Rightarrow \quad \| e^{t\Delta} u_0 \|_{L^4(\mathbb{R}_+, \dot{H}^{\frac{d-1}{2}})} \leq \| u_0 \|_{\dot{H}^{\frac{d}{2}-1}}$$

$$< \frac{1}{4C_d}$$

$\Rightarrow \exists$ una soluzione

$$u \in L^4(\mathbb{R}_+, \dot{H}^{\frac{d-1}{2}})$$

$$u = e^{t\Delta} u_0 + B(u,u)$$

Se ora $|u_0|_{\dot{H}^{\frac{d}{2}-1}} \geq \frac{1}{4C_d}$

Vogliamo T t.c.

$$|e^{t\Delta} u_0|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} < \frac{1}{4C_d}$$

$$\rho > 0$$

$$u_0 = P_\rho u_0 + (1-P_\rho) u_0$$

$$\widehat{P_\rho u_0}(\xi) = \chi_{\substack{B(0,\rho) \\ \mathbb{R}^d}}(\xi) \, \hat{u}_0(\xi)$$

Per $\rho >> 1$

$$|(1-P_\rho) u_0|_{\dot{H}^{\frac{d-1}{2}}} < \frac{1}{8C_d}$$

$$|e^{t\Delta}u_0|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} =$$

$$\le |P_\varrho e^{t\Delta}u_0|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} + \frac{1}{8C_d}$$

$$|P_\varrho e^{t\Delta}u_0|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} =$$

$$= |\chi_{[0,\varrho)}(\sqrt{-\Delta})\, e^{t\Delta}u_0|_{L^4((0,T),\dot H^{\frac{d-1}{2}})}$$

$$= |\sqrt{\varrho}\, e^{t\Delta}\frac{(-\Delta)^{\frac14}}{\sqrt{\varrho}}\chi_{[0,\varrho]}(\sqrt{-\Delta})\, u_0|_{L^4((0,T),\dot H^{\frac d2-1})}$$

$$\le \sqrt{\varrho}\, |\underbrace{\chi_{[0,\varrho)}(\sqrt{-\Delta})}_{P_\varrho}\, e^{t\Delta}u_0|_{L^4((0,T),\dot H^{\frac d2-1})}$$

$$= \sqrt{\varrho}\, | e^{t\Delta}P_\varrho u_0|_{L^4((0,T),\dot H^{\frac d2-1})}$$

$$\le \sqrt{\varrho}\, T^{\frac14} |e^{t\Delta}\cancel{P_\varrho} u_0|_{L^\infty((0,T),\dot H^{\frac d2-1})}$$

$$\le \sqrt{\varrho}\, T^{\frac14} |u_0|_{\dot H^{\frac d2-1}} \le \frac{1}{8C_d}$$

$$T = \left(\frac{1}{8 \rho^{\frac{1}{2}} C_d |u_0|_{\dot{H}^{\frac{d}{2}-1}}} \right)^4$$

$$|e^{t\Delta} u_0|_{L^4((0,T),\dot{H}^{\frac{d-1}{2}})}$$

$$\leq |P_\rho e^{t\Delta} u_0|_{L^4((0,T),\dot{H}^{\frac{d-1}{2}})} + \frac{1}{8C_d}$$

$$< \frac{1}{8C_d}$$

$$T_{u_0} > T(u_0) =$$

$$= \left(\frac{1}{8 \rho^{\frac{1}{2}} C_d |u_0|_{\dot{H}^{\frac{d}{2}-1}}} \right)^4$$

$$\partial_t u - \Delta u = Q(u,u)$$

$$\partial_t v - \Delta v = Q(v,v)$$

$$w = u - v \qquad \overbrace{Q(u,u) - Q(v,v)}^{Q(u-v,\,u+v)}$$

$$\begin{cases} \partial_t w - \Delta w = Q(u,u) - Q(v,v) \\ w(0) = u_0 - v_0 \end{cases}$$

$$\begin{cases} \partial_t w - \Delta w = Q(w, u+v) \\ w(0) = u_0 - v_0 \end{cases}$$

$$\Delta_w = |w(t)|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac{d}{2}-1}} \, dt'$$

$$= |w(0)|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t \langle Q(w, u+v), w\rangle_{\dot H^{\frac{d}{2}-1}} \, dt'$$

<u>Lemma</u>

$$\langle Q(a,b), c\rangle_{\dot H^{\frac{d}{2}-1}} \le C_d |a|_{\dot H^{\frac{d-1}{2}}} |b|_{\dot H^{\frac{d-1}{2}}} |c|_{\dot H^{\frac{d}{2}}}$$

<u>Dim</u>

$$|\langle Q(a,b), c\rangle_{\dot H^{\frac{d}{2}-1}}| \le$$

$$\leq |Q(a,b)|_{\dot H^{\frac{d}{2}-2}} \; |c|_{\dot H^{\frac{d}{2}}}$$

$$\leq C_d \; |a|_{\dot H^{\frac{d-1}{2}}} |b|_{\dot H^{\frac{d-1}{2}}} \; |c|_{\dot H^{\frac{d}{2}}}$$

$$\Delta_w = |w(t)|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac{d}{2}-1}} \, dt'$$

$$= |w(0)|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t \langle Q(w, u+v), w\rangle_{\dot H^{\frac{d}{2}-1}} \, dt'$$

$$\leq |w(0)|^2_{\dot H^{\frac{d}{2}-1}} +$$

$$+ 2C_d \int_0^t |w(t')|_{\dot H^{\frac{d-1}{2}}} \overbrace{\left(|u(t')|_{\dot H^{\frac{d-1}{2}}} + |v(t')|_{\dot H^{\frac{d-1}{2}}}\right)}^{N(t')}$$

$$|\nabla w(t')|_{\dot H^{\frac{d}{2}-1}} \quad dt'$$

$$|w(t')|_{\dot H^{\frac{d-1}{2}}} \leq |w(t')|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} \underbrace{|\nabla w(t')|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}}}_{|w(t')|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}}}}$$

$$\frac{d-1}{2} \overset{\checkmark}{=} \frac{1}{2}\left(\frac{d}{2}-1\right) + \frac{1}{2}\;\frac{d}{2} =$$

$\Delta_w \leq$

$$\leq |w(0)|^2_{\dot H^{\frac{d}{2}-1}} +$$

$$+ 2C_d \int_0^t |w(t')|_{\dot H^{\frac{d-1}{2}}} \quad N(t')$$

$$|\nabla w(t')|_{\dot H^{\frac{d}{2}-1}} \quad dt'$$

$$\Delta_w \leq |w(0)|^2_{\dot H^{\frac{d}{2}}}$$

$$+ \int_0^t N(t') [w(t')]^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} \; |\nabla w(t')|^{\frac{3}{2}}_{\dot H^{\frac{d}{2}-1}}$$

$$ab \leq \frac{1}{4} a^4 + \frac{3}{4} b^{\frac{4}{3}}$$

$$N(t') [w(t')]^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} \; |\nabla w(t')|^{\frac{3}{2}}_{\dot H^{\frac{d}{2}-1}} =$$

$$= \left(|w(t')|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} N(t') \left(\frac{3}{4}\right)^{\frac{3}{4}} \right) \left(\frac{4}{3} |\nabla w(t')|^2_{\dot H^{\frac{d}{2}-1}} \right)^{\frac{3}{4}}$$

$$\leq \left(\frac{3}{4}\right)^3 |w(t')|^2_{\dot H^{\frac{d}{2}-1}} N^4(t')$$

$$+ \; |\nabla w(t')|^2_{\dot H^{\frac{d}{2}-1}}$$

$$\Delta_w = |w(t)|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac d2-1}}\, dt'$$

$$\le |w(0)|^2_{\dot H^{\frac d2-1}} +$$

$$+ C\int_0^t |w(t')|^2_{\dot H^{\frac d2 -1}} N^4(t')\, dt'$$

$$+ \int_0^t |\nabla w|^2_{\dot H^{\frac d2-1}}\, dt$$

$$Y(t) := |w(t)|^2_{\dot H^{\frac d2-1}} + \int_0^t |\nabla w|^2_{\dot H^{\frac d2-1}}\, dt'$$

$$\le |w(0)|^2_{\dot H^{\frac d2-1}} + C\int_0^t N^4(t')\, |\underline{w(t')}|^2_{\dot H^{\frac d2-1}}\, dt'$$

$$Y(t) \le |w(0)|^2_{\dot H^{\frac d2-1}} + C\int_0^t N^4(t')\, Y(t')\, dt'$$

$$Y(t) \le |w(0)|^2_{\dot H^{\frac d2-1}}\, e^{C\int_0^t N^4(t')\, dt}$$