Theorem 0.1 (Riesz–Thorin). *Let T be a linear map from $L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ to $L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$ satisfying*

$$\|Tf\|_{L^{q_j}} \leq M_j \|f\|_{L^{p_j}} \text{ for } j = 0, 1.$$

Then for $t \in (0,1)$ and for p_t and q_t defined by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad , \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|Tf\|_{L^{q_t}} \leq (M_0)^{1-t}(M_1)^t \|f\|_{L^{p_t}} \text{ for } f \in L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d).$$

Proof. First of all notice that if $f \in L^a \cap L^b$ with $a < b$ then $f \in L^c$ for any $c \in (a,b)$. To see this recall Hölder

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \text{ for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

Then, since $\frac{1}{c} = \frac{t}{a} + \frac{1-t}{b}$ for $t \in (0,1)$ from $|f| = |f|^t |f|^{1-t}$ we have

$$\|f\|_{L^c} = \||f|^t |f|^{1-t}\|_{L^c} \leq \||f|^t\|_{L^{\frac{a}{t}}} \||f|^{1-t}\|_{L^{\frac{b}{1-t}}} = \|f\|_{L^a}^t \|f\|_{L^b}^{1-t}.$$

For $p_t = p_0 = p_1 = \infty$ (in fact we can repeat a similar argument for $p_t = p_0 = p_1$ any fixed value in $[1,\infty]$) we then have

$$\|Tf\|_{L^{q_t}} \leq \|Tf\|_{L^{q_1}}^t \|Tf\|_{L^{q_0}}^{1-t} \leq (M_0)^{1-t}(M_1)^t \|f\|_{L^\infty}.$$

So let us suppose $p_t < \infty$. Then it is enough to prove

$$|\int Tfg dx| \leq (M_0)^{1-t}(M_1)^t \|f\|_{L^{p_t}} \|g\|_{L^{q_t'}} = (M_0)^{1-t}(M_1)^t$$

considering only $\|f\|_{L^{p_t}} = \|g\|_{L^{q_t'}} = 1$ for simple functions $f = \sum_{j=1}^m a_j \chi_{E_j}$ where we can take the E_j to be finite measure sets mutually disjoint. If $q_t' < \infty$ we can also reduce to simple functions $g = \sum_{k=1}^N b_k \chi_{F_k}$ where the F_j are finite measure sets mutually disjoint. The case $q_t' = \infty$ reduces to the case $p_t = \infty$ by duality. In fact, see Remark 16 p. 44 [2],

$$\|T\|_{\mathcal{L}(L^{p_t}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p_t'})}.$$

Notice that if both $p_0 < \infty$ and $p_1 < \infty$ and since we are treating $q_0 = q_1 = 1$, then $\|T\|_{\mathcal{L}(L^{p_j}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p_j'})} \leq M_j$ and so one reduces to the case $p_t = \infty$. If, say, $p_0 = \infty$, then $\|T\|_{\mathcal{L}(L^{p_1}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p_1'})} \leq M_1$ since $p_1 < \infty$, but $\|T\|_{\mathcal{L}(L^{p_0}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, (L^\infty)')} \leq M_0$, so in other words, we don't get a Lebesgue space. However, the issue is to bound for $f \in L^{p_0} \cap L^\infty$ a $T^* f \in L^1 \cap (L^\infty)' = L^1$ where $\|T^* f\|_{(L^\infty)'} = \|T^* f\|_{L^1}$, so that one can still apply the above argument used for $p_t = \infty$.

Let us turn to the case $p_t < \infty$ and $q_t' < \infty$. For $a_j = e^{\mathrm{i}\theta_j}|a_j|$ and $b_k = e^{\mathrm{i}\psi_k}|b_k|$ the polar representations, set

$$f_z := \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{\mathrm{i}\theta_j}\chi_{E_j} \text{ with } \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$$

$$g_z := \sum_{k=1}^{N} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{\mathrm{i}\psi_k}\chi_{F_k} \text{ with } \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Notice that since we are assuming $q_t' < \infty$, then $q_t > 1$ and so $\beta(t) = \frac{1}{q_t} < 1$, so that g_z is well defined. Similarly, since $p_t < \infty$ we have $\alpha(t) = \frac{1}{p_t} > 0$, so also f_z is well defined.
We consider now the function

$$F(z) = \int T f_z g_z dx.$$

Our goal is to prove $|F(t)| \leq M_0^{1-t} M_1^t$.
$F(z)$ is holomorphic in $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \leq \operatorname{Re} z \leq 1$. Boundedness follows from estimates like

$$||a_j|^{\frac{\alpha(z)}{\alpha(t)}}| = |a_j|^{\frac{\alpha(\operatorname{Re} z)}{\alpha(t)}} \text{ which is bounded for } 0 \leq \operatorname{Re} z \leq 1.$$

We have $F(t) = \int T f g dx$ since $f_t = f$ and $g_t = g$.
By the 3 lines lemma, see below, which yields $|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ if the two estimates below are true, our theorem is a consequence of the following two inequalities

$$|F(z)| \leq M_0 \text{ for } \operatorname{Re} z = 0 \;;$$
$$|F(z)| \leq M_1 \text{ for } \operatorname{Re} z = 1 \;.$$

For $z = \mathrm{i}y$ we have for $p_0 < \infty$

$$|f_{\mathrm{i}y}|^{p_0} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\alpha(\mathrm{i}y)}{\alpha(t)}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\frac{1}{p_0} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}} \right|^{p_0} \chi_{E_j}$$

$$= \sum_{j=1}^{m} \left| |a_j|^{\mathrm{i}y p_t\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} |a_j|^{\frac{p_t}{p_0}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}.$$

This implies

$$\|f_{\mathrm{i}y}\|_{p_0} = \left(\int_{\mathbb{R}^d} |f_{\mathrm{i}y}|^{p_0} dx\right)^{\frac{1}{p_0}} = \left(\int_{\mathbb{R}^d} |f|^{p_t} dx\right)^{\frac{1}{p_0}} = 1. \tag{0.1}$$

Notice that we have also $\|f_{\mathrm{i}y}\|_\infty = 1$ when $p_0 = \infty$.
Proceeding similarly, using $1 - \beta(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}$, for $z = \mathrm{i}y$ and $q_0' < \infty$ we have

$$|g_{\mathrm{i}y}|^{q_0'} = \sum_{k=1}^{N} \left| |b_k|^{\frac{1-\beta(\mathrm{i}y)}{1-\beta(t)}} \right|^{q_0'} \chi_{F_k} = \sum_{k=1}^{N} \left| |b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'} - \frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_0'}}{\frac{1}{q_t'}}} \right|^{q_0'} \chi_{F_k} = \sum_{j=1}^{N} |b_k|^{q_t'} \chi_{F_k} = |g|^{q_t'}.$$

This implies

$$\|g_{\mathrm{i}y}\|_{q_0'} = \left(\int_{\mathbb{R}^d} |g_{\mathrm{i}y}|^{q_0'} dx\right)^{\frac{1}{q_0'}} = \left(\int_{\mathbb{R}^d} |g|^{q_t'} dx\right)^{\frac{1}{q_0'}} = 1. \tag{0.2}$$

Notice that we have also $\|g_{\mathrm{i}y}\|_\infty = 1$ when $q_0' = \infty$.
Then

$$|F(\mathrm{i}y)| \leq \|Tf_{\mathrm{i}y}\|_{q_0}\|g_{\mathrm{i}y}\|_{q_0'} \leq M_0\|f_{\mathrm{i}y}\|_{p_0}\|g_{\mathrm{i}y}\|_{q_0'} = M_0.$$

By a similar argument

$$|f_{1+\mathrm{i}y}|^{p_1} = |f|^{p_t}$$
$$|g_{1+\mathrm{i}y}|^{q_1'} = |g|^{q_t'}.$$

Indeed by $\alpha(1+\mathrm{i}y) = \frac{1+\mathrm{i}y}{p_1} - \frac{\mathrm{i}y}{p_0}$

$$|f_{1+\mathrm{i}y}|^{p_1} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\alpha(1+\mathrm{i}y)}{\alpha(t)}} \right|^{p_1} \chi_{E_j} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\frac{1}{p_1} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}} \right|^{p_1} \chi_{E_j}$$

$$= \sum_{j=1}^{m} \left| |a_j|^{\frac{p_t}{p_1}} \right|^{p_1} \chi_{E_j} = \sum_{j=1}^{m} |a_j|^{p_t}\chi_{E_j} = |f|^{p_t}$$

and by $1 - \beta(1+\mathrm{i}y) = \frac{1+\mathrm{i}y}{q_1'} - \frac{\mathrm{i}y}{q_0'}$

$$|g_{1+\mathrm{i}y}|^{q_1'} = \sum_{k=1}^{N} \left| |b_k|^{\frac{1-\beta(1+\mathrm{i}y)}{1-\beta(t)}} \right|^{q_1'} \chi_{F_k} = \sum_{k=1}^{N} \left| |b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'} - \frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_1'}}{\frac{1}{q_t'}}} \right|^{q_1'} \chi_{F_k} = \sum_{j=1}^{N} |b_k|^{q_t'}\chi_{F_k} = |g|^{q_t'}.$$

Finally

$$|F(1+\mathrm{i}y)| \leq \|Tf_{1+\mathrm{i}y}\|_{q_1}\|g_{1+\mathrm{i}y}\|_{q_1'} \leq M_1\|f_{1+\mathrm{i}y}\|_{p_1}\|g_{1+\mathrm{i}y}\|_{q_1'} = M_1\|f\|_{p_t}^{\frac{p_t}{p_1}}\|g\|_{q_t'}^{\frac{q_t'}{q_1'}} = M_1.$$

□

Here we have used the following lemma.

Lemma 0.2 (Three Lines Lemma)**.** *Let $F(z)$ be holomorphic in the strip $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \leq \operatorname{Re} z \leq 1$ and such that*

$$|F(z)| \leq M_0 \text{ for } \operatorname{Re} z = 0\ ,$$
$$|F(z)| \leq M_1 \text{ for } \operatorname{Re} z = 1\ .$$

Then we have $|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $0 < \operatorname{Re} z < 1$.

Proof. Let us start with the special case $M_0 = M_1 = 1$ and set $B := \|F\|_{L^\infty}$. Set $h_\epsilon(z) := (1+\epsilon z)^{-1}$ with $\epsilon > 0$. Since $\operatorname{Re}(1+\epsilon z) = 1+\epsilon x \geq 1$ it follows $|h_\epsilon(z)| \leq 1$ in the strip. Furthermore $\operatorname{Im}(1+\epsilon z) = \epsilon y$ implies also $|h_\epsilon(z)| \leq |\epsilon y|^{-1}$. Consider now the two horizontal lines $y = \pm B/\epsilon$ and let R be the rectangle $0 \leq x \leq 1$ and $|y| \leq B/\epsilon$. In $|y| \geq B/\epsilon$ we have

$$|F(z)h_\epsilon(z)| \leq \frac{B}{|\epsilon y|} \leq \frac{B}{|\epsilon B/\epsilon|} = 1.$$

On the other hand by the maximum modulus principle

$$\sup_R |F(z)h_\epsilon(z)| = \sup_{\partial R} |F(z)h_\epsilon(z)| \leq 1,$$

where on the horizontal sides the last inequality follows from the previous inequality and on the vertical sides follows from $|F(z)| \leq 1$ for $\operatorname{Re} z = 0, 1$ and from $|h_\epsilon(z)| \leq 1$.
Hence in the whole strip $0 \leq x \leq 1$ we have $|F(z)h_\epsilon(z)| \leq 1$ for any $\epsilon > 0$. This implies

$$\lim_{\epsilon \searrow 0} |F(z)h_\epsilon(z)| = |F(z)| \leq 1$$

in the whole strip $0 \leq x \leq 1$.
In the general case $(M_0, M_1) \neq (1, 1)$ set $g(z) := M_0^{1-z} M_1^z$. Notice that

$$\begin{aligned}&g(z) = e^{(1-z)\log M_0} e^{z \log M_1} \Rightarrow |g(z)| = M_0^{1-x} M_1^x \Rightarrow \\ &\min(M_0, M_1) \leq |g(z)| \leq \max(M_0, M_1).\end{aligned}$$

So $F(z)g^{-1}(z)$ satisfies the hypotheses of the case $M_0 = M_1 = 1$ and so $|F(z)| \leq |g(z)| = M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$

□

One application of the Riesz–Thorin Theorem is the following.

Theorem 0.3 (Hausdorff–Young). *For $p \in [1, 2]$ and $f \in L^p(\mathbb{R}^n, \mathbb{C})$ then (**??**) defines a function $\mathcal{F}f \in L^{p'}(\mathbb{R}^n, \mathbb{C})$ where $p' = \frac{p}{p-1}$ and an operator remains defined which satisfies*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n,\mathbb{C})} \leq (2\pi)^{-n\left(\frac{1}{2}-\frac{1}{p'}\right)} \|f\|_{L^p(\mathbb{R}^n,\mathbb{C})}. \tag{0.3}$$

Another example of application of M. Riesz's Theorem is the following useful tool.

Lemma 0.4 (Young's Inequality). *Let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

where

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy < C, \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dx < C. \tag{0.4}$$

Then

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \text{ for all } p \in [1, \infty]$$

Proof. The case $p = 1, \infty$ follow immediately from (0.4). The intermediate cases from Riesz's Theorem. $\square$

Recall the formula

$$e^{-\varepsilon \frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}} dx \text{ for any } \varepsilon > 0. \tag{0.5}$$

1 Schrödinger equations

For $u_0 \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the linear homogeneous Schrödinger equation is

$$\mathrm{i}u_t + \triangle u = 0\,, u(0,x) = u_0(x). \tag{1.1}$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + \mathrm{i}|\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi)$. We have $e^{-\mathrm{i}t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t\mathrm{i})^{-\frac{d}{2}} e^{\frac{\mathrm{i}|x|^2}{4t}}$. This follows from the following generalization of (0.5) for $\operatorname{Re} z > 0$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2z}} dx.$$

This formula follows from the fact that both sides are holomorphic in $\operatorname{Re} z > 0$ and coincide for $z \in \mathbb{R}_+$. Then taking the limit $z \to 2\mathrm{i}$ for $\operatorname{Re} z > 0$ and using the continuity of $\mathcal{F}$ in $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ we get

$$e^{-\mathrm{i}|\xi|^2} = (4\pi\mathrm{i})^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{\frac{\mathrm{i}|x|^2}{4}} dx.$$

Then $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * u_0(x)$. In particular, for $u_0 \in L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and by Reisz's interpolation defines for any $t > 0$ an operator which we denote by

$$e^{\mathrm{i}\triangle t} u_0(x) = (4\pi\mathrm{i}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} u_0(y) dy \tag{1.2}$$

which is s.t. $e^{\mathrm{i}\triangle t} : L^p(\mathbb{R}^d, \mathbb{C}) \to L^{p'}(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and $p' = \frac{p}{p-1}$ with $\|e^{\mathrm{i}\triangle t} u_0\|_{L^{p'}} \le (4\pi t)^{-d(\frac{1}{2} - \frac{1}{p'})} \|u_0\|_{L^p}$ by Riesz interpolation.

Remark 1.1. Notice that for no $p \neq 2$ and $t > 0$ we have that $e^{\mathrm{i}\triangle t}$ defines a bounded operator $L^p(\mathbb{R}^d, \mathbb{C}) \to L^p(\mathbb{R}^d, \mathbb{C})$, see [9].

Remark 1.2. Notice that $e^{\triangle t} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator for all $1 \le p \le q \le \infty$.

Notice that (1.1) is time reversible. and if $u(t,x) = e^{\mathrm{i}\triangle t} u_0(x)$, then $v(t,x) = \overline{u(-t,x)} = e^{\mathrm{i}\triangle t}\overline{u_0}(x)$ is a solution.

Let now $u(t,x) = e^{\mathrm{i}\triangle t}u_0(x)$, and for $\mathbf{v}, D \in \mathbb{R}^d$ consider $v_0(x) = e^{\mathrm{i}\frac{\mathbf{v}}{2}\cdot x}u_0(x-D)$. Then

$$v(t,x) := e^{\mathrm{i}\triangle t}v_0(x) = e^{\frac{\mathrm{i}}{2}\mathbf{v}\cdot x - \mathrm{i}\frac{\mathbf{v}^2}{4}t}u(t, x - t\mathbf{v} - D).$$

In the sequel, given $v, w \in L^2(\mathbb{R}^d, \mathbb{C})$ we will use the notation

$$\langle v, w\rangle = \mathrm{Re}\int_{\mathbb{R}^d} v(x)\overline{w}(x)dx. \tag{1.3}$$

In the sequel we will reinterpret the equation

$$\mathrm{i}u_t + \triangle u = f\,, \quad u(0) = u_0 \in H^1(\mathbb{R}^d) \tag{1.4}$$

in the integral form

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\int_0^t e^{\mathrm{i}(t-t')\triangle}f(t')dt'. \tag{1.5}$$

To understand this formula we will need Strichartz's inequalities.
We say that a pair (q,r) is *admissible* when

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \tag{1.6}$$

$$2 \le r \le \frac{2d}{d-2}\ (2 \le r \le \infty \text{ if } d = 1,\ 2 \le r < \infty \text{ if } d = 2). \tag{1.7}$$

Remark 1.3. The pair $(\infty, 2)$ is always admissible. The *endpoint* $(2, \frac{2d}{d-2})$ is admissible for $d \ge 3$ but the point $(2,\infty)$ is not for $d = 2$. The equality (1.6) needs to be true by the parabolic scaling $u(t,x) \rightsquigarrow u(\lambda^2 t, \lambda x)$, which preserves the set of solutions to (1.1).

We have the following important result.

Theorem 1.4 (Strichartz's estimates)**.** *The following facts hold.*

(1) For every $u_0 \in L^2(\mathbb{R}^d)$ we have $e^{\mathrm{i}\triangle t}u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$ for every admissible (q,r). Furthermore, there exists a C s.t.

$$\|e^{\mathrm{i}\triangle t}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \le C\|u_0\|_{L^2}. \tag{1.8}$$

(2) Let I be an interval and let $t_0 \in \overline{I}$. If (γ,ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then for any admissible pair (q,r) the function

$$\mathcal{T}f(t) = \int_{t_0}^t e^{\mathrm{i}\triangle(t-s)}f(s)ds \tag{1.9}$$

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\overline{I}, L^2(\mathbb{R}^d))$ and there exists a constant C independent of I and f s.t.

$$\|\mathcal{T}f\|_{L^q(I,L^r(\mathbb{R}^d))} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{R}^d))}. \tag{1.10}$$

2 Keel and Tao's proof of Strichartz estimates

We will follow the argument by Keel and Tao [8]. We will assume that (X, dx) is a measurable space and that H is a Hilbert space. We consider a family of operators $U(t) : H \to L^2(X)$. We assume the following two hypotheses.

(1) There exists a $C > 0$ s.t.

$$\|U(t)f\|_{L^2} \leq C\|f\|_H \text{ for all } f \in H;$$

(2) there exist a $\sigma > 0$ and a $C > 0$ s.t. for all $t \neq s$ and all $g \in L^1(X)$ we have

$$\|U(t)(U(s))^* g\|_{L^\infty} \leq C|t-s|^{-\sigma}\|g\|_{L^1}.$$

We say that a pair (q, r) is σ–*admissible* when

$$\begin{aligned} &\frac{2}{q} + \frac{2\sigma}{r} = \sigma \\ &r, q \geq 2 \text{ and } (q, r, \sigma) \neq (2, \infty, 1). \end{aligned} \tag{2.1}$$

Particularly important, for $\sigma > 1$, is the point $P = \left(2, \dfrac{2\sigma}{\sigma - 1}\right)$.

Notice that (1) implies $\|U^*(t)F\|_{L^2} \leq C\|F\|_{L^2}$ by duality and that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, (U(t))^* f\rangle_H$ [1]

Theorem 2.1 (Keel and Tao's Strichartz estimates)**.** *If $U(t)$ satisfies (1) and (2), and if furthermore there exists an appropriate scaling operator in X and H, then we have*

(3)

$$\|U(t)u_0\|_{L^q(\mathbb{R}, L^r(X))} \leq C_{q,r}\|u_0\|_H.$$

(4)

$$\|\int_{\mathbb{R}} (U(s))^* F(s) ds\|_H \leq C\|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

(5)

$$\|\int_{t>s} U(t)(U(s))^* F(s) ds\|_{L^q(\mathbb{R}, L^r(X))} \leq C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{\widetilde{q}'}(\mathbb{R}, L^{\widetilde{r}'}(X))}.$$

for all admissible pairs (q, r) and $(\widetilde{q}, \widetilde{r})$.

(3) is called the homogeneous estimate and (5) the non–homogeneous estimate or also the retarded estimate. (3) and (4) are equivalent by duality. The scaling operators are used only in Sect. 2.2.

[1] Notice that since $h \to \langle U(t)h, f\rangle_{L^2(X)}$ is continuous, an element $f^* \in H$ remains defined such that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, f^*\rangle_H$. The map $f \to f^*$ is linear, bounded and $(U(t))^* f := f^*$.

2.1 Proof of the nonendpoint homogeneous estimate

We consider the case $(q,r) \neq P$. The proof of this case predates the paper by Keel and Tao.
It is elementary that (4) is by duality and hypothesis (1) equivalent

$$\left| \int_{\mathbb{R}^2} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds \right| \leq C \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

So we have to prove the above estimate. Furthermore, it is enough to prove the above bound for

$$T(F,G) := \int_{t>s} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds. \tag{2.2}$$

By (1) we know that (3) holds for $q = \infty$ and $r = 2$. So pointwise

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = \left| \langle U(t)(U(s))^* F(s), G(t) \rangle_{L^2(X)} \right| \\ &\leq \|U(t)(U(s))^* F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)} \leq C^2 \|F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)}. \end{aligned}$$

Furthermore

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = \left| \langle U(t)(U(s))^* F(s), G(t) \rangle_{L^2(X)} \right| \leq \|U(t)(U(s))^* F(s)\|_{L^\infty(X)} \|G(t)\|_{L^1(X)} \\ &\leq C|t-s|^{-\sigma} \|F(s)\|_{L^1(X)} \|G(t)\|_{L^1(X)}. \end{aligned}$$

From the Riesz–Thorin Interpolation Theorem, see Theorem 0.1, we have (omitting the constant) for any $r \in [2, \infty]$

$$\begin{aligned} &\|U(t)(U(s))^* F(s)\|_{L^r(X)} \lesssim |t-s|^{-\sigma\left(1-\frac{2}{r}\right)} \|F(s)\|_{L^{r'}(X)} = |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} \\ &\text{where } \beta(r,\widetilde{r}) := \sigma - 1 - \frac{\sigma}{r} - \frac{\sigma}{\widetilde{r}}. \end{aligned}$$

Then we conclude

$$|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| \lesssim |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} \|G(t)\|_{L^{r'}(X)}.$$

For $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$, using the Hardy,Littlewood Sobolev inequality, see Theorem **??**, which requires $q > q'$,

$$|T(F,G)| \lesssim \| \int_{\mathbb{R}} |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} ds \|_{L^q(\mathbb{R})} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \lesssim \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

Notice that $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$ means

$$1 - \frac{2}{q} = -\sigma + 1 + 2\frac{\sigma}{r} \Leftrightarrow \frac{2}{q} + \frac{2\sigma}{r} = \sigma$$

and $-\beta(r,r) > 0$ means

$$r < \frac{2\sigma}{\sigma - 1}.$$

2.2 Proof of the endpoint homogeneous estimate

Here we consider the endpoint case $(q,r) = P = (2, \frac{2\sigma}{\sigma-1})$, when $\sigma > 1$.

The introduction of a scaling operator will simplify considerably the discussion. We will denote it by D_λ for $\lambda > 0$. We assume the following:

1. there exist operators $D_\lambda : H \to H$ s.t. $\langle D_\lambda f, D_\lambda g\rangle_H = \lambda^{-\sigma} \langle f, g\rangle_H$
2. there exist operators $D_\lambda : L^r(X) \to L^r(X)$ s.t $\|D_\lambda f\|_{L^r(X)} = \lambda^{-\frac{\sigma}{r}} \|f\|_{L^r(X)}$
3. in all cases $D_\lambda^{-1} = D_{\lambda^{-1}}$ and $D_\lambda^* = \lambda^{-\sigma} D_{\lambda^{-1}}$.

Notice that for $\sigma = \frac{d}{2}$, $H = L^2(\mathbb{R}^d)$ and $X = \mathbb{R}^d$ with $L^r(X)$ the standard Lebesgue spaces, then $D_\lambda f(x) := f(\lambda^{\frac{1}{2}} x)$ satisfies the desired requirements. Notice that we used the same notation for dilation operators in H and $L^r(X)$, but they are distinct operators.

Lemma 2.2. *Let the function $t \to U(t)$ satisfy (1) and (2) in Sect. 2. Then $t \to D_\lambda U(\lambda t) D_{\lambda^{-1}}$ satisfies (1) and (2) in Sect. 2 with exactly the same constants C.*

Proof. Indeed

$$\|D_\lambda U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} = \lambda^{-\frac{\sigma}{2}} \|U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} \le C \lambda^{-\frac{\sigma}{2}} \|D_{\lambda^{-1}} f\|_H = C\|f\|_H$$

and from $(D_\lambda U(\lambda s) D_{\lambda^{-1}})^* = D_\lambda (U(\lambda s))^* D_{\lambda^{-1}}$,

$$\begin{aligned}&\|D_\lambda U(\lambda t) D_{\lambda^{-1}} (D_\lambda U(\lambda s) D_{\lambda^{-1}})^* f\|_{L^\infty} \|D_\lambda U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \\ &= \|U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \le C\lambda^{-\sigma} |t-s|^{-\sigma} \|D_{\lambda^{-1}} f\|_{L^1} = C|t-s|^{-\sigma} \|f\|_{L^1}.\end{aligned}$$

□

After the above preliminary on scaling operators, expand

$$T(F,G) = \sum_{j\in\mathbb{Z}} T_j(F,G) \text{ where } T_j(F,G) := \int_{t-2^j > s > t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds. \tag{2.3}$$

We will prove

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^2 L^{r'}} \|G\|_{L^2 L^{r'}}. \tag{2.4}$$

We will prove the following.

Lemma 2.3. *For a fixed constant C dependent only on the constants in (1) –(2) Sect. 2. we have*

$$|T_j(F,G)| \le C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}. \tag{2.5}$$

with $(1/a, 1/b)$ in a sufficiently small, but fixed neighborhood of $(1/r, 1/r)$, dependent only on σ.

Proof. Notice that

$$T_j(F,G) = \int_{t-2^j>s>t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds$$
$$= 2^{2j} 2^{j\sigma} \int_{t-1>s>t-2} \left\langle D_{2^j}(U(2^j s))^* D_{2^{-j}} D_{2^j} F(2^j s), D_{2^j}(U(2^j t))^* D_{2^{-j}} D_{2^j} G(2^j t)\right\rangle_H dtds.$$

Suppose now that we have (2.4) in the particular case $j=0$. Then we have

$$|T_j(F,G)| \le C 2^{2j} 2^{j\sigma} \|D_{2^j} F(2^j s)\|_{L^2 L^{a'}} \|D_{2^j} G(2^j t)\|_{L^2 L^{b'}} = C 2^{2j} 2^{j\sigma} 2^{-j\left(1+\frac{\sigma}{a'}+\frac{\sigma}{b'}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}$$
$$= C 2^{j\left(2+\sigma-1-2\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{j\left(1-\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}$$

where we recall $\beta(a,b) = \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}$.

So we have reduced to the case $j=0$. Next we do another reduction. We claim that to prove the case $j=0$ it is enough to assume that F and G are supported in time intervals of length 1. Indeed, assuming this case, then we have

$$|T_0(F,G)| \le \sum_{n\in\mathbb{Z}} \left| \int_{n+1>t>n} dt \int_{t-1>s>t-2} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, ds \right|$$
$$\le C \sum_{n\in\mathbb{Z}} \|F\|_{L^2((n,n+1),L^{a'})} \|G\|_{L^2((n-2,n),L^{b'})} \le C \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-2,n),L^{a'})} \right)^{\frac{1}{2}}$$
$$= C\sqrt{2} \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-1,n),L^{b'})} \right)^{\frac{1}{2}} = C\sqrt{2} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}.$$

Hence, in the rest of the proof we will assume that F and G are supported in time intervals of length 1. To prove (2.5) for $j=0$ we consider three cases:

(i) $a = b = \infty$;

(ii) $2 \le a < r$ and $b = 2$;

(iii) $a = 2$ and $2 \le b < r$.

Then the desired result follows by interpolation.

Let us start with (i). The proof is elementary and straightforward, because we have

$$|T_0(F,G)| \le \int dt \int_{t-1>s>t-2} |\langle U(t)(U(s))^* F(s), G(t)\rangle_{L^2(X)}| ds$$
$$\le C \int dt \int_{t-1>s>t-2} |t-s|^{-\sigma} \|F(s)\|_{L^1} \|G(t)\|_{L^1} \le C \int dt \int_{t-1>s>t-2} \|F(s)\|_{L^1} \|G(t)\|_{L^1}$$
$$\le C \|F\|_{L^1 L^1} \|G\|_{L^1 L^1} \le C \|F\|_{L^2 L^1} \|G\|_{L^2 L^1}.$$

Let us now consider (ii). Here we will use the Strichartz estimates in Sect. 2.1. We have

$$\begin{aligned}
|T_0(F,G)| &\le \int |\left\langle \int_{t-1>s>t-2} (U(s))^* F(s)ds, (U(t))^* G(t) \right\rangle_H |dt \\
&\le \int \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \|(U(t))^* G(t)\|_H dt \\
&\le \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \int \|(U(t))^* G(t)\|_H dt \\
&\le C\|G\|_{L^1L^2} \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H ,
\end{aligned}$$

where we used (1) in Sect. 2. Now, using the non endpoint Strichartz estimates in Sect. 2.1 (notice here $2 \le a < r$) we have, for $(q(a), a)$ admissible,

$$\sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \le C\|F\|_{L^{q(a)'}L^{a'}} \le C\|F\|_{L^2L^{a'}}.$$

This proves (ii) and by symmetry yields also (iii). □

Now we need to show that (2.5) implies (2.4). Obviously, we cannot just take $a = b = r$ and sum up, since $\beta(r,r) = 0$. To give an intuition on how to overcome this problem, Keel and Tao consider functions of the form

$$F(t) = 2^{-\frac{k}{r'}} f(t)\chi_{E(t)}(x) \text{ and } G(s) = 2^{-\frac{\widetilde{k}}{r'}} g(s)\chi_{\widetilde{E}(s)}(x), \tag{2.6}$$

with scalar functions $f(t), g(s)$ and $E(t)$ resp. $\widetilde{E}(s)$ sets of size 2^k resp. $2^{\widetilde{k}}$. Applying (2.5) we obtain

$$\begin{aligned}
|T_j(F,G)| &\le C2^{-j\left(\sigma-1-\frac{\sigma}{a}-\frac{\sigma}{b}\right)} 2^{-\frac{k}{r'}} 2^{\frac{k}{a'}} 2^{-\frac{\widetilde{k}}{r'}} 2^{\frac{\widetilde{k}}{b'}} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)} 2^{-(k+\widetilde{k})\left(\cancel{1}-\frac{1}{r}\right)+\cancel{(k+\widetilde{k})}-\frac{k}{a}-\frac{\widetilde{k}}{b}} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)+k\left(\frac{1}{r}-\frac{1}{a}\right)+\widetilde{k}\left(\frac{1}{r}-\frac{1}{b}\right)} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{(k-j\sigma)\left(\frac{1}{r}-\frac{1}{a}\right)+\left(\widetilde{k}-j\sigma\right)\left(\frac{1}{r}-\frac{1}{b}\right)} \|f\|_{L^2}\|g\|_{L^2}.
\end{aligned} \tag{2.7}$$

Notice now that we can adjust (a,b) s.t. for a fixed small $\varepsilon > 0$ the last term equals

$$C2^{-\varepsilon|k-j\sigma|-\varepsilon\left|\widetilde{k}-j\sigma\right|} \|f\|_{L^2}\|g\|_{L^2} \tag{2.8}$$

whose sum for $j \in \mathbb{Z}$ is finite.

To convert the above intuition in a proof we consider the following preliminary lemma.

Lemma 2.4. *Let $p \in (0,\infty)$. Then any $f \in L^p_x$ can be written as*

$$f = \sum_{k\in\mathbb{Z}} c_k\chi_k$$

where meas(suppχ_k) $\le 2\ 2^k$, $|\chi_k| \le 2^{-\frac{k}{p}}$ and $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.

Proof. Consider the distribution function $\lambda(\alpha) = \text{meas}(\{|f(x)| > \alpha\})$. Then for each k consider

$$\alpha_k := \inf_{\lambda(\alpha)<2^k} \alpha\,, \quad c_k := 2^{\frac{k}{p}}\alpha_k\,, \quad \chi_k := \frac{1}{c_k}\chi_{(\alpha_{k+1},\alpha_k]}(|f|)f.$$

Notice that $\{\alpha_k\}_{k\in\mathbb{Z}}$ is decreasing in k (since, the larger k, the larger is the set $\{\alpha : \lambda(\alpha) < 2^k\}$).
We show the desired properties. We have

$$\text{supp}\chi_k \subseteq \{x : \alpha_{k+1} < |f(x)| \le \alpha_k\} \subseteq \{x : |f(x)| > \alpha_{k+1}\}.$$

Then we get the 1st inequality:

$$\text{meas}(\text{supp}\chi_k) \le \text{meas}(\{x : |f(x)| > \alpha_{k+1}\}) = \lim_{\alpha\to\alpha_{k+1}^+} \lambda(\alpha) = \sup\{\lambda(\alpha) : \alpha > \alpha_{k+1}\}) \le 2^{k+1}.$$

Next, by $|f(x)| \le \alpha_k$ in $\text{supp}\chi_k$, we have

$$|\chi_k(x)| \le 2^{-\frac{k}{p}}\frac{|f(x)|}{\alpha_k} \le 2^{-\frac{k}{p}}.$$

Let now $\lim_{k\to+\infty} \alpha_k = \inf_{k\in\mathbb{Z}} \alpha_k = \underline{\alpha}$ and $\lim_{k\to-\infty} \alpha_k = \sup_{k\in\mathbb{Z}} \alpha_k = \overline{\alpha}$. Then we claim that $\underline{\alpha} = 0$ and that $|f(x)| \le \overline{\alpha}$ a.e. Indeed, suppose that $|f(x)| > \overline{\alpha}$ on a set of positive measure. There there is $\alpha > \overline{\alpha}$ with $\lambda(\alpha) > 2^k$ for some $k \in \mathbb{Z}$. Then $\alpha_k \ge \alpha > \overline{\alpha}$, which is a contradiction. On the other hand, suppose we have $0 < \alpha < \underline{\alpha}$. Then $\lambda(\alpha) = \infty$, since otherwise $\lambda(\alpha) < 2^k$ for a k, and then $\alpha \ge \alpha_k \ge \underline{\alpha} > \alpha$, getting a contradiction. But by Chebyshev's inequality,

$$\infty > \|f\|_{L^p}^p \ge \alpha^p\lambda(\alpha),$$

hence getting a contradiction. The above claim and the obvious fact that for any x we have $|f(x)| \in (\alpha_{k+1}, \alpha_k]$ for at most one k, prove $f = \sum_{k\in\mathbb{Z}} c_k\chi_k$ (the claim guarantees the existence of one such k).
We have $\|f\|_{L^p} \le 2^{\frac{1}{p}}\|c_k\|_{\ell^p}$ by

$$\begin{aligned}\|f\|_{L^p}^p &= \int |f|^p dx = \int \sum_{k\in\mathbb{Z}} |c_k|^p|\chi_k|^p dx = \sum_{k\in\mathbb{Z}} |c_k|^p \int |\chi_k|^p dx \le \sum_{k\in\mathbb{Z}} |c_k|^p 2^{-k}\text{meas}(\text{supp}\chi_k) \\ &\le 2\sum_{k\in\mathbb{Z}} |c_k|^p\end{aligned}$$

Next we have

$$\sum_{k\in\mathbb{Z}} |c_k|^p = \sum_{k\in\mathbb{Z}} 2^k\alpha_k^p = \int_{\mathbb{R}_+} \alpha^p \left(\sum 2^k\delta(\alpha - \alpha_k)\right) d\alpha = \int_{\mathbb{R}_+} \alpha^p(-F'(\alpha))d\alpha$$

where

$$F(\alpha) := \sum_{k\in\mathbb{Z}} 2^k H(\alpha_k - \alpha) = \sum_{\alpha_k>\alpha} 2^k \le \sum_{2^k\le\lambda(\alpha)} 2^k \le 2\lambda(\alpha).$$

Then, integrating by parts and using (**??**),

$$\sum_{k\in\mathbb{Z}} |c_k|^p = p\int_{\mathbb{R}_+} \alpha^{p-1}F(\alpha)d\alpha \le 2p\int_{\mathbb{R}_+} \alpha^{p-1}\lambda(\alpha)d\alpha = 2\|f\|_{L^p}^p,$$

so that $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.

□

Furthermore we have the following.

Lemma 2.5. *Let $1 \le q, r < \infty$ and let $f \in L^q(I, L^r_x)$ with I an interval. Then we can write the expansion of Lemma 2.4*

$$f = \sum_{k\in\mathbb{Z}} c_k(t)\chi_k(t) \tag{2.9}$$

with $t \to \{c_k(t)\}$ a map in $L^q(I, \ell^r)$.

Proof. Formally this follows immediately from

$$\|\{c_k(t)\}\|_{L^q(I,\ell^r)} = \|\|\{c_k(t)\}\|_{\ell^r}\|_{L^q(I)} \le 2^{\frac{1}{p}}\|\|f\|_{L^r_x}\|_{L^q(I)}.$$

However one needs to argue that the function $t \to \{c_k(t)\}$ is measurable. By a density argument it is enough to consider the case of simple functions $f = \sum_{j=1,\dots,n} \chi_{E_j}(t)g_j(x)$ with E_j mutually disjoints sets. Then $\lambda(t,\alpha) = \text{meas}(\{|f(t,x)| > \alpha\}) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\lambda_j(\alpha)$ with λ_j the distribution function of g_j. Then $\alpha_k(t) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\alpha_k^{(j)}$ with $\alpha_k^{(j)}$ defined like in Lemma 2.4 for each g_j. Then

$$\{c_k(t)\} = \sum_{j=1,\dots,n} \chi_{E_j}(t)\{c_k^{(j)}\} \text{ for } c_k^{(j)} = 2^{\frac{k}{p}}\alpha_k^{(j)}.$$

This is measurable in t.

□

Consider now the

$$F(t) = \sum_{k\in\mathbb{Z}} f_k(t)\chi_k(t)\,, \quad G(s) = \sum_{k\in\mathbb{Z}} g_k(s)\widetilde{\chi}_k(s). \tag{2.10}$$

By (2.6)–(2.8) e have

$$\sum_j |T_j(F,G)| \le \sum_{j,k,\widetilde{k}} |T_j(f_k\chi_k, g_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})| \le C \sum_{j,k,\widetilde{k}} 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}$$
$$= C\sum_{k,\widetilde{k}}\left(\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\right)\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}.$$

We claim that for a fixed $C = C(\sigma,\varepsilon)$

$$\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \le C2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|). \tag{2.11}$$

To prove this inequality, it is not restrictive to assume $k \le \widetilde{k}$. Then the summation on the left can be rewritten as

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} + \sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} + \sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma}.$$

Then (here $[t]\in\mathbb{Z}$ is the integer part of $t\in\mathbb{R}$, defined by $[t]\le t<[t]+1$)

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} = 2^{-\varepsilon(k+\widetilde{k})}\sum_{j\le\left[\frac{k}{\sigma}\right]} 2^{2\varepsilon j\sigma} = 2^{-\varepsilon(k+\widetilde{k})}\sum_{j=0}^{\infty} 2^{2\varepsilon\sigma\left(\left[\frac{k}{\sigma}\right]-j\right)} = C_{\varepsilon\sigma}\ 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\left[\frac{k}{\sigma}\right]}$$
$$\le C_{\varepsilon\sigma}2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\frac{k}{\sigma}} = C_{\varepsilon\sigma}2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma}2^{-\varepsilon|k-\widetilde{k}|} \text{ where } C_{\varepsilon\sigma} = \frac{1}{1-2^{-2\varepsilon\sigma}}.$$

We have

$$\sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma} \le 2^{\varepsilon(k+\widetilde{k})}\sum_{j\ge\left[\frac{\widetilde{k}}{\sigma}\right]+1} 2^{-2\varepsilon j\sigma} = 2^{\varepsilon(k+\widetilde{k})}\sum_{j=0}^{\infty} 2^{-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1+j\right)} = C_{\varepsilon\sigma}2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1\right)}$$
$$\le C_{\varepsilon\sigma}2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\frac{\widetilde{k}}{\sigma}} = C_{\varepsilon\sigma}2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma}2^{-\varepsilon|k-\widetilde{k}|}.$$

Finally

$$\sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} = 2^{-\varepsilon(\widetilde{k}-k)}\sum_{\left[\frac{k}{\sigma}\right]+1\le j\sigma\le\left[\frac{\widetilde{k}}{\sigma}\right]} 1 = 2^{-\varepsilon(\widetilde{k}-k)}\left(\left[\frac{\widetilde{k}}{\sigma}\right]-\left[\frac{k}{\sigma}\right]-1\right) \le \sigma^{-1}2^{-\varepsilon(\widetilde{k}-k)}(\widetilde{k}-k)$$

Hence (2.11) is proved. From this we conclude that for a fixed C

$$\sum_j |T_j(F,G)| \leq C \sum_{k,\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|f_k\|_{L_t^2}\|g_{\widetilde{k}}\|_{L_t^2}$$

$$\leq C\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\left\|\{\sum_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|g_{\widetilde{k}}\|_{L_t^2}\}\right\|_{\ell^2(\mathbb{Z})}$$

$$\leq C\left(\sum_k 2^{-\varepsilon|k|}(1+|k|)\right)\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}$$

where we used Lemma 0.4. So, using $r' \leq 2$,

$$\sum_j |T_j(F,G)| \leq C'\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} = C'\|\|\{f_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}$$

$$\leq C''\|\|\{f_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2} \leq C'''\|\|F\|_{L_x^{r'}}\|_{L_t^2}\|\|\|G\|_{L_x^{r'}}\|_{L_t^2}$$

which completes the proof of (2.4).

2.3 Proof of the non homogeneous estimate

We need to prove that for all admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$ we have

$$|T(F,G)| \leq C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}\|G\|_{L^{\widetilde{q}'}(\mathbb{R},L^{\widetilde{r}'}(X))}. \tag{2.12}$$

We have already proved that this is true for $(q,r)=(\widetilde{q},\widetilde{r})$. Furthermore, proceeding like in Lemma 2.3

$$|T(F,G)| \leq \int\left|\left\langle\int_{t>s}(U(s))^*F(s)ds,(U(t))^*G(t)\right\rangle_H\right|dt$$

$$\leq \int \|\int_{t>s}(U(s))^*F(s)ds\|_H\|(U(t))^*G(t)\|_H dt \leq \sup_t \|\int_{t>s}(U(s))^*F(s)ds\|_H \int \|(U(t))^*G(t)\|_H dt$$

$$\leq C\|G\|_{L^1L^2}\sup_t\|\int_{t>s}(U(s))^*F(s)ds\|_H,$$

Then, by (4) in Theorem 2.1 (that is the dual homogenous estimates, which are already proved) for any admissible pair (q,r)

$$\sup_t\|\int_{t>s}(U(s))^*F(s)ds\|_H = \sup_t\|\int_{\mathbb{R}}(U(s))^*F(s)\chi_{(-\infty,t)}(s)ds\|_H \leq C\|F\chi_{(-\infty,t)}\|_{L^{q'}(\mathbb{R},L^{r'})} \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'})}.$$

So (2.12) holds for $(\widetilde{q},\widetilde{r})=(\infty,2)$ and any admissible pair (q,r). Obviously, symmetrically (2.12) holds for $(q,r)=(\infty,2)$ and any admissible pair $(\widetilde{q},\widetilde{r})$. Finally, let us consider (q,r)

and $(\widetilde{q}, \widetilde{r})$ not in one of the cases already covered. Then it is not restrictive to assume that $(\widetilde{q}, \widetilde{r}) = (a_{t_0}, b_{t_0})$ for $t_0 \in (0,1)$ where

$$\left(\frac{1}{a_t}, \frac{1}{b_t}\right) = t\left(\frac{1}{q}, \frac{1}{r}\right) + (1-t)\left(\frac{1}{\infty}, \frac{1}{2}\right).$$

In the cases $t = 0, 1$ the inequality holds, because these are cases considered above. By a generalization of Riesz–Thorin, Theorem 0.1, the inequality holds also for the intermediate t's. □

2.4 Improved non–homogeneous Strichartz estimates

While the homogeneous Strichartz estimates (1.8) are optimal, the non–homogeneous Strichartz estimates (1.10) as described in Claim 2 of Theorem 1.4 are not optimal.
We say that a pair (q, r) is *acceptable* when

$$\frac{1}{q} < \frac{d}{2} - \frac{d}{r} \tag{2.13}$$

$$2 \leq r \leq \infty \text{ and } 2 \leq r < \infty. \tag{2.14}$$

Remark 2.6. Admissible pairs are acceptable, but the viceversa is not necessarily true.

We state without proof the following theorem from [7]

Theorem 2.7 (Inhomogeneous Strichartz estimates)**.** *Statement 2 in Theorem 1.4 is true for any pairs (q, r) and (γ, ρ) which are acceptable, satisfy*

$$\frac{1}{q} + \frac{1}{\gamma} = \frac{d}{2}\left(1 - \frac{1}{r} - \frac{1}{\rho}\right) \tag{2.15}$$

and the following conditions:

- *if $d = 1$ no further conditions;*
- *if $d = 2$, $r < \infty$ and $\rho < \infty$*
- *if $d \geq 3$ we distinguish two cases.*

  1. *The non–sharp case*

$$\frac{1}{q} + \frac{1}{\gamma} < 1, \tag{2.16}$$

$$\frac{d-2}{d}\frac{1}{r} \leq \frac{1}{\rho} \text{ and } \frac{d-2}{d}\frac{1}{\rho} \leq \frac{1}{r} \tag{2.17}$$

2. The sharp case

$$\frac{1}{q} + \frac{1}{\gamma} = 1, \tag{2.18}$$

$$\frac{d-2}{d}\ \frac{1}{r} < \frac{1}{\rho} \textit{ and } \frac{d-2}{d}\ \frac{1}{\rho} < \frac{1}{r} \textit{ and} \tag{2.19}$$

$$\frac{1}{r} \leq \frac{1}{q} \textit{ and } \frac{1}{\rho} \leq \frac{1}{\gamma}. \tag{2.20}$$

3 The semilinear Schrödinger equation

There is a vast literature on semilinear Schrödinger equations. For a survey, with a concise discussion of some physical motivations, we refer to [14]. Here though, we consider only the mathematical formalism and only the pure power semilinear Schrödinger equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + \lambda |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ \qquad\qquad u(0,x) = u_0(x) \end{cases} \tag{3.1}$$

for $\lambda \in \{1, -1\}$ and $p > 1$. Here $p < d^*$ with $d^* = \infty$ for $d = 1, 2$ and $d^* = \frac{d+2}{d-2}$ for $d \geq 3$. We collect here a number of facts needed later.

Lemma 3.1. *We have the following facts.*

(1) For $1 < p < d^*$ we have the Gagliardo–Nirenberg inequality:

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \leq C_p \|\nabla u\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \textit{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}. \tag{3.2}$$

(2) The map $u \to |u|^{p-1}u$ is a locally Lipschitz from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.

(3) For $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have $\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla \overline{u}$ and belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$.

Proof. For (1) see Theorem **??**.

We turn (2). By (3.2) we know that $u \to |u|^{p-1}u$ maps $H^1(\mathbb{R}^d) \to L^{p+1}(\mathbb{R}^d) \to L^{\frac{p+1}{p}}(\mathbb{R}^d)$. Furthermore this map is locally Lipschitz:

$$\begin{aligned} &\| |u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}} \leq C\|(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{\frac{p+1}{p}}} \\ &\leq C'(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}} \end{aligned}$$

where we have used, for $w = v - u$,

$$|u|^{p-1}u - |v|^{p-1}v = \int_0^1 \frac{d}{dt}\left(|u+tw|^{p-1}(u+tw)\right) dt =$$
$$\int_0^1 |u+tw|^{p-1} dtw + \int_0^1 (u+tw)\frac{d}{dt}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-1}{2}} dt = \int_0^1 |u+tw|^{p-1}dtw+$$
$$\sum_{j=1}^{2}\int_0^1 (u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)dtw_j$$

which from $|u+tw| \le |u|+|v|$ for $t \in [0,1]$ and

$$\left|(u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)w_j\right| \le (p-1)|u+tw|^{p-1}|w|$$

yields

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \le p(|u|+|v|)^{p-1}|u-v| \le p2^{p-1}(|u|^{p-1}+|v|^{p-1})|u-v|,$$

where in the last step we used, for $|u| \ge |v|$,

$$(|u|+|v|)^{p-1} \le 2^{p-1}|u|^{p-1} \le 2^{p-1}(|u|^{p-1}+|v|^{p-1}).$$

Next, we show that we have an embedding $L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$. Indeed, this is equivalent to $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ with in turn is a consequence of (3.2).

We turn (3). First of all we claim that if $G \in C^1(\mathbb{C},\mathbb{C})$ with $G(0)=0$ and $|\nabla G| \le M < \infty$, then $\nabla(G(u)) = \partial_u G(u)\nabla u + \partial_{\overline{u}} G(u)\nabla\overline{u}$ in the sense of distributions. This claim can be proved like Proposition 9.5 in [2] and we skip the proof here.
Let us now consider an increasing function $g \in C^\infty(\mathbb{R}_+,\mathbb{R})$ s.t.

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} & \text{for } 0 \le s \le 1 \\ 2^{\frac{p-1}{2}} & \text{for } s \ge 2 \end{cases}$$

and let us define $G_m(u) = m^{p-1} g\left(\frac{|u|^2}{m^2}\right) u$ for $m \in \mathbb{N}$. Then, by the claim, for all $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int G_m(u)\ \partial_j\varphi = \int \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u}\right)\ \varphi. \tag{3.3}$$

Let us take now the limit for $m \to \infty$. We have

$$\int G_m(u)\ \partial_j\varphi = \int |u|^{p-1}u\ \partial_j\varphi - \int_{|u|\ge m} |u|^{p-1}u\ \partial_j\varphi + \int_{|u|\ge m} G_m(u)\ \partial_j\varphi.$$

Now we have

$$\int_{|u|\geq m} |u|^{p-1}u \ \partial_j \varphi \xrightarrow{m\to\infty} 0 \text{ by Dominated Convergence}$$

since $\chi_{\{|u|\geq m\}}(x) \xrightarrow{m\to\infty} 0$ a.e. by Chebyshev's inequality. Similarly

$$\begin{aligned}&\left|\int_{|u|\geq m} G_m(u)\ \partial_j\varphi\right| \leq \int_{|u|\geq m} |G_m(u)\ \partial_j\varphi| \leq 2^{p-1}\int_{|u|\geq m} m^{p-1}|u|\ \partial_j\varphi| \\ &\leq 2^{p-1}\int_{|u|\geq m} |u|^p|\partial_j\varphi| \xrightarrow{m\to\infty} 0\end{aligned}$$

Next, we consider the limit of the r.h.s. of (3.3). For $G(u) = |u|^{p-1}u$ we have

$$\begin{aligned}&\int (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j \overline{u})\ \varphi = \int (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j \overline{u}) \\ &- \int_{|u|\geq m} (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j \overline{u})\ \varphi + \int_{|u|\geq m} (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j \overline{u})\ \varphi.\end{aligned}$$

Then, like before, the terms of the 2nd line converge to 0 as $m \to \infty$ and so we conclude that all $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have

$$-\int |u|^{p-1}u\ \partial_j\varphi = \int \left(p|u|^{p-1}\partial_j u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \partial_j\overline{u}\right)\ \varphi.$$

The fact of belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$ follows immediately from Hölder inequality.

□

Important are the following quantities:

$$\begin{aligned}E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx \\ P_j(u) &= \frac{1}{2}\,\mathrm{Im}\int_{\mathbb{R}^d} \partial_j u\, \overline{u} dx \\ Q(u) &= \int_{\mathbb{R}^d} |u|^2 dx.\end{aligned} \tag{3.4}$$

Here $E(u)$ is the energy, $P_j(u)$ for $j = 1, ..., d$ are the linear momenta and $Q(u)$ is the mass or charge.

Remark 3.2. Notice that $Q, P_j \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$ while $E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$. We will show that the above quantities are conserved for solutions in $H^1(\mathbb{R}^d, \mathbb{C})$. Here E is the hamiltonian. The system is invariant under the transformation $u \to e^{i\vartheta}u$ for $\vartheta \in \mathbb{R}$ and the transformations $u(x_1, ...x_{j-1}, x_j, x_{j+1}, ..., x_d) \to u(x_1, ...x_{j-1}, x_j - \tau, x_{j+1}, ..., x_d)$ for $\tau \in \mathbb{R}$. The related Noether invariants are Q and P_j.

Remark 3.3. Notice that if $u(t,x)$ solves the equation (3.1) then also $\tau^{\frac{2}{p}}u(\tau^2 t, \tau x)$ solves the equation (3.1), with initial value $\tau^{\frac{2}{p}}u_0(\tau\cdot)$. Notice that

$$\|\tau^{\frac{2}{p}}u_0(,\tau\cdot)\|_{\dot{H}^{s_p}(\mathbb{R}^d)} = \|u_0\|_{\dot{H}^{s_p}(\mathbb{R}^d)} \text{ for } s_p = \frac{d}{2} - \frac{2}{p}.$$

$\dot{H}^{s_p}(\mathbb{R}^d)$ is the critical space for the equation (3.1) and equation (3.1) is critical for $\dot{H}^{s_p}(\mathbb{R}^d)$. Equation (3.1) is supercritical for $\dot{H}^{s}(\mathbb{R}^d)$ with $s < s_p$. In practice when an equation is critical or supercritical the well–posedness is either hard to prove or not true.

3.1 The local existence

We will consider the following integral formulation of (3.1):

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{3.5}$$

Proposition 3.4 (Local well posedness in $L^2(\mathbb{R}^d)$)**.** *For any $p \in (1, 1+4/d)$ and any $u_0 \in L^2(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of* (3.5) *with*

$$u \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.6}$$

Furthermore, there exists a (decreasing) function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{L^2}) > 0$.
Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'], L^2(\mathbb{R}^d)) \cap L^q([-T',T'], L^{p+1}(\mathbb{R}^d)) \tag{3.7}$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], L^b(\mathbb{R}^d))$ for all admissible pairs (a,b).

Remark 3.5. We will prove later that for $p \in (1, 1+2/d)$ that we can take $T = \infty$ always.

Proof. The proof is a fixed point argument. We set for an $a > 0$ to be fixed below

$$E(T,a) = \Big\{ v \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) : \\ \|v\|_T := \|v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))} \leq a \Big\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Our first aim is to show that for $T = T(\|u_0\|_{L^2})$ sufficiently small, then $\Phi : E(T,a) \to E(T,a)$ is a contraction.
By Strichartz's estimates

$$\|\Phi(u)\|_T \leq c_0\|u_0\|_{L^2} + c_0\||u|^{p-1}u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ = c_0\|u_0\|_{L^2} + c_0\|u\|^p_{L^{pq'}([-T,T],L^{p+1})}$$

We will see in a moment that

$$p \in (1, 1+4/d) \Longleftrightarrow pq' < q. \tag{3.8}$$

Assuming this for a moment, by Hölder we conclude that for a $\theta > 0$

$$\|\Phi(u)\|_T \le c_0\|u_0\|_{L^2} + c_0(2T)^\theta \|u\|^p_{L^q([-T,T],L^{p+1})} \le c_0\|u_0\|_{L^2} + c_0(2T)^\theta a^p.$$

So for $c_0(2T)^\theta a^{p-1} < 1/2$, which can be obtained by picking T small enough, we have

$$\|\Phi(u)\|_T \le c_0\|u_0\|_{L^2} + \frac{a}{2} \le a$$

if $a \ge 2c_0\|u_0\|_{L^2}$. Hence $\Phi\left(E(T,a)\right) \subseteq E(T,a)$. Let us fix here an $a > 2c_0\|u_0\|_{L^2}$. Now let us show that Φ is a contraction for T small enough. We have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_T \le c_0 \||u|^{p-1}u - |v|^{p-1}v\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\
&\le c_0 C \|(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}}\|_{L^{q'}(-T,T)} \\
&\le c_0 C (\|u\|^{p-1}_{L^q([-T,T],L^{p+1}} + \|v\|^{p-1}_{L^q([-T,T],L^{p+1})})\|u-v\|_{L^\rho([-T,T],L^{p+1})}
\end{aligned}$$

where $\frac{p-1}{q} + \frac{1}{\rho} = \frac{1}{q'}$. Since we are still assuming (3.8), we must have $\rho < q$, for $\rho \ge q$ would imply $pq' \ge q$, contrary to (3.8). Then by Hölder and for an appropriate $\theta > 0$

$$\|\Phi(u) - \Phi(v)\|_T \le c_0 C 2a^{p-1}T^\theta \|u-v\|_{L^q([-T,T],L^{p+1})} \le c_0 C 2a^{p-1}T^\theta \|u-v\|_T.$$

So, for $c_0 C 2a^{p-1}T^\theta < 1$, where $a > 2c_0\|u_0\|_{L^2}$, we obtain that Φ is a contraction and we obtain the existence and uniqueness of the solution.

Next, let us prove (3.8). Obviously $pq' < q$ is equivalent to $p/q < 1 - 1/q$, in turn to $(p+1)/q < 1$, that is to $1/q < 1/(p+1)$. But $1/q = d/4 - d/(2p+2)$, so the last inequality is equivalent to

$$d/4 < \left(\frac{d}{2} + 1\right)/(p+1) \Leftrightarrow p+1 < \frac{2d+4}{d} = 2 + \frac{4}{d}$$

and this yields the desired result.

We have proved the existence of a $T = T(\|u_0\|_{L^2})$ with the desired properties. Then there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ such that for any $v_0 \in V$ we have $a > 2c_0\|u_0\|_{L^2}$. Then there is a corresponding solution $v(t)$ in $C([-T,T],L^2(\mathbb{R}^d)) \cap L^q([-T,T],L^{p+1}(\mathbb{R}^d))$. Let now $T' \in (0,T)$ to be fixed. Using the equation and proceeding like above,

$$\begin{aligned}
&\|u-v\|_{T'} \le c_0\|u_0 - v_0\|_{L^2} + c_0 C(2T')^\theta \left(\|u\|^{p-1}_{T'} + \|v\|^{p-1}_{T'}\right)\|u-v\|_{T'} \\
&\le c_0\|u_0 - v_0\|_{L^2} + c_0 C(2T')^\theta 2\left((2c_0\|v_0\|_{L^2})^{p-1} + (2c_0\|u_0\|_{L^2})^{p-1}\right)\|u-v\|_{T'}.
\end{aligned}$$

Adjusting T', we can assume that it satisfies (recall $a > 2c_0 \max\{\|v_0\|_{L^2}, \|u_0\|_{L^2}\}$)

$$4c_0 C(2T')^\theta a^{p-1} < 1/2.$$

Notice that here $T' = T'(\|u_0\|_{L^2})$. Renaming $T = T'$, from the above we get

$$\|u - v\|_T \leq 2c_0\|u_0 - v_0\|_{L^2}$$

and this gives the desired Lipschitz continuity.

Finally, the last statement follows from (3.5) and the Strichartz Estimates.

□

Proposition 3.6 (Local well posedness in $H^1(\mathbb{R}^d)$). *For any $p \in (1, d^*)$ and any $u_0 \in H^1(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of (3.5) with*

$$u \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.9}$$

Furthermore, there exists a (decreasing) function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{H^1}) > 0$.
Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $H^1(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T', T'], L^2(\mathbb{R}^d)) \cap L^q([-T', T'], W^{1,p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a, b).

Proof. The proof is similar to that of Proposition 3.4. The proof is a fixed point argument.This time we set

$$E^1(T,a) = \Big\{ v \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) : \\ \|v\|_T^{(1)} := \|v\|_{L^\infty([-T,T],H^1(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],W^{1,p+1}(\mathbb{R}^d))} \leq a \Big\}$$

and, as before, use $\Phi(u)$ for the r.h.s. of (3.5). We need to show that by taking T sufficiently small then $\Phi : E^1(T,a) \to E^1(T,a)$ and is a contraction. The argument is similar to the one in Proposition 3.4 and is based on the Strichartz estimates. We will only consider some of the estimates. By Lemma 3.1 and Strichartz's estimates, we have

$$\begin{aligned} \|\nabla\Phi(u)\|_T &\leq c_0\|u_0\|_{H^1} + c_0\||u|^{p-1}\nabla u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ &= c_0\|u_0\|_{L^2} + c_0\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})}\|\nabla u\|_{L^q([-T,T],L^{p+1})}. \end{aligned} \tag{3.10}$$

where $\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$. Notice that if $\beta < q$, we can proceed exactly like in Proposition 3.4. However this works only for $p \in (1, 1 + 4/d)$, which is not necessarily true here. Instead, using the Sobolev Embedding we bound

$$\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})} \lesssim \|u\|^{p-1}_{L^\beta([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}\|u\|^{p-1}_{L^\infty([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^{p-1}.$$

So, inserting this in the previous inequality we get

$$\|\nabla\Phi(u)\|_T \leq c_0\|u_0\|_{H^1} + c_0(2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^p. \tag{3.11}$$

Here it is important to remark that the admissible pair $(q, p+1)$ is s.t. $q > 2$. Indeed, for $d = 1, 2$ it is always true that, if $p + 1 < \infty$, then the q in (3.38) is $q > 2$. On the other hand, for $d \geq 3$ recall that

$$p + 1 < d^* + 1 = \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}.$$

And so again, since $(q, p+1)$ differs from the endpoint admissible pair $(2, \dfrac{2d}{d-2})$, we necessarily have $q > 2$ also if $d \geq 3$.
In turn, the fact that $q > 2$ implies that the β in the above formulas is $\beta < \infty$. This implies that we can pick T small enough s.t. $(2T)^{\frac{p-1}{\beta}} a^{p-1} < 1/2$, which from (3.11) yields $\|\Phi(u)\|_T^{(1)} \leq c_1\|u_0\|_{H^1} + a/2 \leq a$ for $a > 2c_1\|u_0\|_{H^1}$. From these arguments, it is easy to conclude that there exists a $T(\|u_0\|_{H^1})$ s.t. for $T \in (0, T(\|u_0\|_{H^1}))$ we have $\Phi\left(E^1(T,a)\right) \subseteq E^1(T,a)$. Proceeding similarly and like in Proposition 3.4, it can be shown that there exists a $T_1(\|u_0\|_{H^1})$ s.t. for $T \in (0, T_1(\|u_0\|_{H^1}))$ the map Φ is a contraction inside $E^1(T,a)$. The Lipschitz continuity in terms of the initial data can be shown like in Proposition 3.4 and the last statement follows from the Strichartz estimates.

□

Proposition 3.7 (Conservation laws)**.** *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.6. Then all the three quantities in* (3.4) *are constant in t.*

Proof. For $u \in C((-T_2, T_1), H^1(\mathbb{R}^d))$ a maximal solution of (3.5) we will show that there exists $[-T, T] \subset (-T_2, T_1)$ where $E(u(t)) = E(u(0))$, $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$. In fact this shows that $E(u(t))$, $Q(u(t))$ and $P_j(u(t))$ are locally constant in t. Since these functions are continuous in t, the set of $t \in (-T_2, T_1)$ where $E(u(t)) = E(u(0))$ is closed in $(-T_2, T_1)$; on the other hand, it is also open in $(-T_2, T_1)$ since $E(u(t))$ is locally constant, and hence we have $E(u(t)) = E(u(0))$ for all $t \in (-T_2, T_1)$. Similarly $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$ for all $t \in (-T_2, T_1)$.

Step 1: truncations of the NLS. For $\varphi \in C_c^\infty(\mathbb{R}, [0,1])$ a function with $\varphi = 1$ near 0 and with support contained in the ball $B_{\mathbb{R}^d}(0, r_0)$, consider [2] the operators $\mathbf{Q}_n = \varphi(\sqrt{-\triangle}/n)$. The truncations $\mathbf{Q}_n(|u|^{p-1}u)$ are locally Lipschitz functions from $H^1(\mathbb{R}^d)$ into itself as they are compositions $H^1(\mathbb{R}^d) \overset{|u|^{p-1}u}{\to} H^{-1}(\mathbb{R}^d) \overset{\mathbf{Q}_n}{\to} H^1(\mathbb{R}^d))$ of a locally Lipschitz function, Lemma 3.1, and of bounded linear maps.

[2] Notice that using everywhere the projections $\mathbf{P}_n = \chi_{[0,n]}(\sqrt{-\triangle})$ would be a bad choice for this proof. Difficulties would arise from the fact proved by C.Feffermann [6] that $\mathbf{P}_n$ for $d \geq 2$ is bounded from $L^p(\mathbb{R}^d)$ into itself only if $p = 2$. On the other hand it is elementary that the $\mathbf{Q}_n$ are of the form $\rho_{\frac{1}{n}} *$ for a $\rho \in \mathcal{S}(\mathbb{R}^d)$ and so are uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for all p and form a sequence converging strongly to the identity operator.

We consider the following truncations of the NLS

$$\begin{cases} \mathrm{i}u_{nt} = -\mathbf{P}_{nr_0}\triangle u_n + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) \text{ for } (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u_n(0) = \mathbf{Q}_n u_0. \end{cases} \tag{3.12}$$

By the theory of ODE's, there exists a maximal solution $u_n(t) \in C^1(-T_1(n), T_2(n)), H^1(\mathbb{R}^d))$ of (3.12) . Furthermore, if $T_2(n) < \infty$ then we must have blow up

$$\lim_{t\nearrow T_2(n)} \|u_n(t)\|_{H^1} = +\infty \text{ if } T_2(n) < \infty \tag{3.13}$$

with a similar blow up phenomenon if $T_1(n) < \infty$.
To get bounds on this sequence of functions we consider invariants of motion. The following will be proved later.

Claim 3.8. The following functions are invariants of motion of (3.12):

$$\begin{aligned} &E_n(v) := \frac{1}{2}\|P_{nr_0}\nabla v\|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n v|^{p+1} dx \\ &P_j(v) \text{ with } j = 1, ..., d, \\ &Q(v). \end{aligned} \tag{3.14}$$

We assume Claim 3.8 and proceed. It is easy to check that $u_n = \mathbf{P}_{nr_0}u_n$. We claim that $T_1(n) = T_2(n) = \infty$. Indeed by $Q(u_n(t)) = Q(\mathbf{Q}_n u_0) \le Q(u_0)$ we have

$$\|u_n(t)\|_{H^1} = \|\mathbf{P}_{nr_0}u_n(t)\|_{H^1} \le nr_0\|u_n(t)\|_{L^2} = nr_0\|\mathbf{Q}_n u_0\|_{L^2} \le nr_0\|u_0\|_{L^2}. \tag{3.15}$$

Let us now fix M such that $\|u_0\|_{H^1} < M$ and let us set

$$\theta_n := \sup\{\tau > 0 : \|u_n(t)\|_{H^1} < 2M \text{ for } |t| < \tau.\} \tag{3.16}$$

Our main focus is now to prove that there exists a fixed $T(M) > 0$ s.t. $\theta_n \ge T(M)$ for all n.
First of all we prove that $u_n \in C^{0,\frac{1}{2}}((-\theta_n,\theta_n), L^2)$ with a fixed Hölder constant $C(M)$. By interpolation

$$\begin{aligned} &\|u_n(t) - u_n(s)\|_{L^2} \lesssim \|u_n(t) - u_n(s)\|^{\frac{1}{2}}_{H^1}\|u_n(t) - u_n(s)\|^{\frac{1}{2}}_{H^{-1}} \\ &\le \sqrt{2}\|u_n\|^{\frac{1}{2}}_{L^\infty((-\theta_n,\theta_n),H^1)}\|u_{nt}\|^{\frac{1}{2}}_{L^\infty((-\theta_n,\theta_n),H^{-1})}\sqrt{|t-s|} \\ &\le C(M)\sqrt{|t-s|} \text{ for } t, s \in (-\theta_n, \theta_n) \end{aligned} \tag{3.17}$$

Now we want to prove

$$\|u_n(t)\|^2_{H^1} \le \|u_0\|^2_{H^1} + C(M)t^b \text{ for some fixed } b > 0 \text{ and for } t \in (-\theta_n, \theta_n). \tag{3.18}$$

From $E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0)$ and $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ we get

$$\|u_n(t)\|^2_{H^1} + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = \|\mathbf{Q}_n u_0\|^2_{H^1} + \frac{2\lambda}{p+1}\int_{\mathbb{R}^n} |\mathbf{Q}^2_n u_0|^{p+1} dx.$$

Hence using Hölder and Gagliardo–Nirenberg

$$
\begin{aligned}
&\|u_n(t)\|_{H^1}^2 \le \|u_0\|_{H^1}^2 + \frac{2}{p+1}\int_{\mathbb{R}^d} \left| \ |\mathbf{Q}_n u_n(t)|^{p+1} - |\mathbf{Q}_n^2 u_0|^{p+1} \right| dx \\
&\le \|u_0\|_{H^1}^2 + C\int_{\mathbb{R}^d} (|\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p)|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0| dx \\
&\le \|u_0\|_{H^1}^2 + C\| |\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p \|_{L^{\frac{p+1}{p}}} \|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0\|_{L^{p+1}} \\
&\le \|u_0\|_{H^1}^2 + C_1 \left(\|\mathbf{Q}_n u_n(t)\|_{L^{p+1}}^p + \|\mathbf{Q}_n^2 u_0\|_{L^{p+1}}^p \right) \|u_n(t) - \mathbf{Q}_n u_0\|_{\dot H^1}^{\alpha} \|u_n(t) - \mathbf{Q}_n u_0\|_{L^2}^{1-\alpha}
\end{aligned}
$$

Then by (3.17) with $s=0$, the Sobolev Embedding Theorem and (3.16) we get (3.18). Now for $T(M)$ defined s.t. $C(M)T(M)^b = 2M^2$ (for the $C(M)$ in (3.18)) from (3.18) we get

$$\|u_n(t)\|_{L^\infty([-T(M),T(M)],H^1)} \le \sqrt{3}M. \tag{3.19}$$

Since $\sqrt{3}M < 2M$ this obviously means that $T(M) < \theta_n$ since, if we had $\theta_n \le T(M)$ then, by the fact that $u_n \in C^1(\mathbb{R}, H^1)$, the definition of θ_n in (3.16) would be contradicted. Hence we have

$$\|u_n\|_{L^\infty([-T(M),T(M)],H^1)} < 2M \tag{3.20}$$

This completes step 1, up to Claim 3.8.

The proof of Claim 3.8 is rather elementary and involves applying to (3.12) $\langle \ , u_{nt}\rangle$, $\langle \ , \mathrm{i}u_n\rangle$ and $\left\langle \ , \partial_{x_j} u_n \right\rangle$ and integration by parts. We will do this now, but then we will discuss also the fact that Claim 3.8 is just a consequence of the fact that (3.12) is a hamiltonian system with hamiltonian E_n and that the invariance of Q resp. P_j just due to Nöther principle and the invariance with respect to multiplication by $e^{\mathrm{i}\vartheta}$ resp. translation.

Indeed, applying $\langle \cdot, u_{nt}\rangle$ to (3.12)

$$
\begin{aligned}
0 &= -\langle \mathbf{P}_{nr_0} \triangle u_n, u_{nt}\rangle + \lambda \langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), u_{nt}\rangle \\
&= -\langle \triangle u_n, u_{nt}\rangle + \lambda\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathbf{Q}_n u_{nt}\rangle = \frac{d}{dt} E_n(u_n).
\end{aligned}
$$

Notice furthermore that, by $u_n = \mathbf{P}_{nr_0} u_n$, we have

$$E_n(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx.$$

Similarly when we apply $\langle \cdot, \mathrm{i}u_n\rangle$ to (3.12) we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2} = -\langle \mathbf{P}_{nr_0}\triangle u_n, \mathrm{i}u_n\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle. \tag{3.21}$$

We have to show that r.h.s. are equal to 0. We observe that the the 1st term is 0 because the bounded operator $\mathrm{i}\mathbf{P}_{nr_0}\triangle$ of $L^2(\mathbb{R}^d)$ into itself is antisymmetric: $(\mathrm{i}\mathbf{P}_{nr_0}\triangle)^* = -\mathrm{i}\mathbf{P}_{nr_0}\triangle$. For the 2nd term we use

$$\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle = \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle = \lambda \,\mathrm{Re}\,\mathrm{i}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = 0.$$

This yields $\frac{d}{dt}Q(u_n(t)) = 0$. In a similar fashion we can prove $\frac{d}{dt}P_j(u_n(t)) = 0$.

These computations obscure somewhat the following simple facts. First of all, (3.12) and, in a somewhat formal sense also (3.1), is a hamiltonian system. First of all, the symplectic form is

$$\Omega(X,Y) := \langle \mathrm{i}X, Y\rangle \tag{3.22}$$

where

$$\langle f, g\rangle = \mathrm{Re} \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx. \tag{3.23}$$

Notice that Ω satisfies the following definition for $X = L^2(\mathbb{R}^d, \mathbb{C})$ or $X = H^1(\mathbb{R}^d, \mathbb{C})$.

Definition 3.9. Let X be a Banach space on $\mathbb{R}$ and let X' be its dual. A strong symplectic form is a 2-form ω on X s.t. $d\omega = 0$ (i.e. ω is closed) and s.t. the map $X \ni x \to \omega(x, \cdot) \in X'$ is an isomorphism.

Definition 3.10 (Gradient)**.** Let $F \in C^1(L^2(\mathbb{R}^d, \mathbb{C}), \mathbb{R})$. Then the gradient $\nabla F \in C^0(L^2(\mathbb{R}^d, \mathbb{C}), L^2(\mathbb{R}^d, \mathbb{C}))$ is defined by

$$\langle \nabla F(u), Y\rangle = dF(u)Y \text{ for all } u, Y \in L^2(\mathbb{R}^d, \mathbb{C}).$$

Notice that

$$\begin{aligned}\langle \nabla E_n(u), Y\rangle &= \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{P}_{nr_0}\nabla(u+tY)\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n(u+tY)|^{p+1}dx\right)\Bigg|_{t=0} \\ &= \left\langle -\mathbf{P}_{nr_0}\triangle u + \lambda\mathbf{Q}_n(|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u), Y\right\rangle.\end{aligned} \tag{3.24}$$

We are interested in hamiltonian vector fields.

Definition 3.11 (Hamiltonian vector field)**.** Let ω be a strong symplectic form on the Banach space X and $F \in C^1(X, \mathbb{R})$. We define the Hamiltonian vector field X_F with respect to ω by

$$\omega(X_F(u), Y) := dF(u)Y \text{ for all } u, Y \in X.$$

From $\Omega(X_F, Y) = \langle \mathrm{i}X_F, Y\rangle = \langle \nabla F, Y\rangle$ we conclude $X_F = -\mathrm{i}\nabla F$. Then from (3.24) it is straightforward to conclude that (3.12) is a hamiltonian system with hamiltonian E_n.

Definition 3.12 (Poisson bracket)**.** Let ω be a strong symplectic form in a Banach space X and let $F, G \in C^1(X, \mathbb{R})$. Then the Poisson bracket $\{F, G\}$ is given by

$$\{F, G\}(u) := \omega(X_F(u), X_G(u)) = dF(u)X_G(u).$$

So, for Ω we have $\{F, G\} = \langle \nabla F, -\mathrm{i}\nabla G\rangle = \langle \mathrm{i}\nabla F, \nabla G\rangle$. Now notice that if $F \in C^1(X, \mathbb{R})$ then

$$\frac{d}{dt}\left(F(u_n(t))\right) = \langle \nabla F(u_n(t)), \dot{u}_n(t)\rangle = \langle \nabla F(u_n(t)), -\mathrm{i}\nabla E_n(u_n(t))\rangle = \{F, E_n\}|_{u_n(t)} \tag{3.25}$$

Notice now that the map $u \to e^{\mathrm{i}\vartheta} u$ leaves E_n invariant. In particular the last assertion implies that

$$\begin{aligned} 0 &= \frac{d}{d\vartheta} E_n(u)\Big|_{\vartheta=0} = \frac{d}{d\vartheta} E_n(e^{\mathrm{i}\vartheta}u)\Big|_{\vartheta=0} \\ &= \langle \nabla E_n(u), \mathrm{i}u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla Q(u)\rangle = \langle \mathrm{i}\nabla Q(u), \nabla E_n(u)\rangle = \{Q, E_n\}|_u \end{aligned}$$

But then, since $\{Q, E_n\} = 0$, by (3.25) we obviously have $\frac{d}{dt}(Q(u_n(t))) = 0$.

Let us consider now, for $\{\overrightarrow{e}_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, the transformation $(\tau_{\lambda \overrightarrow{e}_j} F)(x) := F(x - \lambda \overrightarrow{e}_j)$. Obviously E_n is invariant by this transformation and

$$\begin{aligned} 0 &= \frac{d}{d\lambda} E_n(u)\Big|_{\lambda=0} = \frac{d}{d\lambda} E_n(\tau_{\lambda \overrightarrow{e}_j} u)\Big|_{\lambda=0} \\ &= -\langle \nabla E_n(u), \partial_j u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla P_j(u)\rangle = \langle \mathrm{i}\nabla P_j(u), \nabla E_n(u)\rangle = \{P_j, E_n\}|_u \end{aligned}$$

But then, since $\{P_j, E_n\} = 0$, by (3.25) we obviously have $\frac{d}{dt}(P_j(u_n(t))) = 0$.
The above argument gives a link between group actions and invariants.

Step 2: Convergence $u_n \to u$. Let us consider $I := [-T, T] \subseteq [-T(M), T(M)] \cap (-T_2, T_1)$. Obviously we have

$$u_n(t) = e^{\mathrm{i}t\triangle}\mathbf{Q}_n u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n(|\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s))ds.$$

Taking the difference with (3.5) we obtain

$$\begin{aligned} u(t) - u_n(t) &= e^{\mathrm{i}t\triangle}(1 - \mathbf{Q}_n)u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}(1-\mathbf{Q}_n)|u(s)|^{p-1}u(s)ds \\ &- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n \left(|u(s)|^{p-1}u(s) - |\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s)\right) ds \\ &- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n \left(|\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s) - |\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s)\right) ds. \end{aligned}$$

Then we have

$$\begin{aligned} &\|u - u_n\|_{L^q(I,W^{1,p+1})} + \|u - u_n\|_{L^\infty(I,H^1)} \le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\ &+ c_0\||u|^{p-1}u - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\ &+ c_0\||\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}. \end{aligned}$$

and so, for a fixed $\vartheta > 0$

$$\|u-u_n\|_{L^q(I,W^{1,p+1})}+\|u-u_n\|_{L^\infty(I,H^1)}\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+c_0C|I|^{\vartheta}\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+\|\mathbf{Q}_nu\|^{p-1}_{L^\infty(,H^1)}\right)\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+c_0C|I|^{\vartheta}\left(\|\mathbf{Q}_nu\|^{p-1}_{L^\infty(I,H^1)}+\|\mathbf{Q}_nu_n\|^{p-1}_{L^\infty(I,H^1)}\right)\|\mathbf{Q}_n(u-u_n)\|_{L^q(I,W^{1,p+1})}$$
$$\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+c_0C|I|^{\vartheta}2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+c_0C|2T|^{\vartheta}\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+(C(M))^{p-1}\right)\|u-u_n\|_{L^q(I,W^{1,p+1})}.$$

Then, taking T small so that $c_0C|2T|^{\vartheta}\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+(C(M))^{p-1}\right)<1/2$ we conclude

$$\|u-u_n\|_{L^q(I,W^{1,p+1})}+\|u-u_n\|_{L^\infty(I,H^1)}\le 2c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+$$
$$2c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}+2c_0C|I|^{\vartheta}2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}.$$

But now we have r.h.s.$\overset{n\to\infty}{\to}0$. Hence we have proved that there exist $T>0$ s.t.

$$\lim_{n\to+\infty}\|u-u_n\|_{L^\infty([-T,T],H^1)}=0. \tag{3.26}$$

Now, taking the limit for $n\to+\infty$ in $Q(u_n(t))=Q(\mathbf{Q}_nu_0)$ and $P_j(u_n(t))=P_j(\mathbf{Q}_nu_0)$ we obtain $Q(u(t))=Q(u_0)$ and $P_j(u(t))=P_j(u_0)$ for all $t\in[-T,T]$. Similarly, taking the limit for $n\to+\infty$ in $E_n(u_n)=E_n(\mathbf{Q}_nu_0)$ and with a little bit of work, we obtain $E(u(t))=E(u_0)$ for all $t\in[-T,T]$.

□

Corollary 3.13. *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.4. Then $Q(u(t))=Q(u_0)$. In particular, the solutions in in Proposition 3.4 are globally defined.*

Proof. As above it is enough to show that $Q(u(t))=Q(u_0)$ for $t\in[-T,T]$ for some $T>0$. So let us take the T in the statement of Proposition 3.4 and let us take $T'\in(0,T)$. There exists a sequence $u_0^{(n)}\in H^1(\mathbb{R}^d,\mathbb{C})$ with $u_0^{(n)}\overset{n\to\infty}{\to}u_0$ in $L^2(\mathbb{R}^d,\mathbb{C})$. So for $n\gg1$ we have $u_0^{(n)}\in V$, the V in (3.7). In particular, for the corresponding solutions u_n we have $u^{(n)}\overset{n\to\infty}{\to}u$ in $C([-T',T'],L^2(\mathbb{R}^d))$. Then, since $Q(u^{(n)}(t))=Q(u_0^{(n)})$ for $t\in([-T',T']$, taking the limit we obtain $Q(u(t))=Q(u_0)$ for $t\in([-T',T']$. Since $T'\in(0,T)$ is arbitrary and $t\to Q(u(t))$ is continuous, we have $Q(u(t))=Q(u_0)$ for $t\in([-T,T]$. This implies that $t\to Q(u(t))$ is locally constant, and hence it is constant.

□

Remark 3.14. It can be shown that under the hypotheses of Proposition 3.6 there are unique maximal solutions to (3.5) of the type $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ with $T > 0$ and $S > 0$ and with

$$\lim_{t\to T^-} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty \text{ if } T < +\infty \text{ and}$$
$$\lim_{t\to -S^+} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty \text{ if } S < +\infty.$$

3.2 Conservation of regularity

In this subsection we will prove the following.

Theorem 3.15 (Conservation or regularity)**.** *Let $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ be a maximal solution of (3.5). Suppose that the initial value satisfies $u_0 \in H^2(\mathbb{R}^d)$. Then*

$$u \in C^0((-S,T), H^2(\mathbb{R}^d)). \tag{3.27}$$

Proposition 3.16 (Local well posedness in $H^2(\mathbb{R}^d)$)**.** *For any $p \in (1, d^*)$ and any $u_0 \in H^2(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of (3.5) with*

$$u \in L^\infty([-T,T], H^2(\mathbb{R}^d)) \cap W^{1,\infty}([-T,T], L^2(\mathbb{R}^d)) \cap W^{1,q}([-T,T], W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.28}$$

Furthermore, there exists a (decreasing) function $T(\cdot) : [0,+\infty) \to (0,+\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{H^2}) > 0$.
Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'], H^2(\mathbb{R}^d)) \cap W^{1,q}([-T',T'], L^{p+1}(\mathbb{R}^d)) \cap L^q([-T',T'], W^{2,p+1}(\mathbb{R}^d)) \tag{3.29}$$

and is Lipschitz.

Proof (sketch). As before we use $\Phi(u)$ for the r.h.s. of (3.5). Recall that for a an appropriate multiple of $\|u_0\|_{H^1}$ we have that Φ is a contraction of $E^1(T,a)$ into itself.

Let now set for an M to be defined below,

$$E^2(T,M) = \Big\{ v \in L^\infty([-T,T], \dot{H}^2(\mathbb{R}^d)) \cap \dot{W}^{1,q}([-T,T], L^{p+1}(\mathbb{R}^d)) \cap E^1(T,a) :$$
$$\|v\|_T^{(2)} := \|v\|_{L^\infty([-T,T],\dot{H}^2(\mathbb{R}^d))} + \|\partial_t v\|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))} \leq M \Big\}.$$

We need to show that by taking T sufficiently small then $\Phi : E^2(T,M) \to E^2(T,M)$ and is a contraction. First of all, T will be smaller than the T in Proposition 3.6. Here we consider estimates not done already. Starting from (3.5) we write

$$\Phi(u)(t) = e^{\mathrm{i}t\triangle} u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}s\triangle} |u(t-s)|^{p-1} u(t-s) ds.$$

Now we have

$$\partial_t \Phi(u)(t) = \mathrm{i}e^{\mathrm{i}t\triangle}\triangle u_0 - \lambda \mathrm{i}e^{\mathrm{i}t\triangle}|u(0)|^{p-1}u(0) - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}s\triangle}\partial_t \left(|u(t-s)|^{p-1}u(t-s)\right) ds.$$

Then for any admissible pair (γ, r)

$$\|\mathrm{i}e^{\mathrm{i}t\triangle}\triangle u_0\|_{L^r(\mathbb{R},L^\gamma(\mathbb{R}^d))} \leq C_{Strichartz}\|u_0\|_{H^2(\mathbb{R}^d)}$$

and

$$\|\mathrm{i}e^{\mathrm{i}t\triangle}|u(0)|^{p-1}u(0)\|_{L^r(\mathbb{R},L^\gamma(\mathbb{R}^d))} \leq C_{Strichartz}\||u(0)|^{p-1}u(0)\|_{L^2(\mathbb{R}^d)} = C_{Strichartz}\|u(0)\|^p_{L^{2p}(\mathbb{R}^d)}$$
$$\leq C_{Strichartz}C_{Sobolev}\|u(0)\|^p_{H^s(\mathbb{R}^d)}$$

where we used $H^2(\mathbb{R}^d) \hookrightarrow L^{2p}(\mathbb{R}^d)$ which follows by Sobolev's embedding by $\frac{1}{2p} = \frac{1}{2} - \frac{s}{d}$ for a $s \in (0, 2)$. Notice that for $s \leq 1$ we can bound the above by $\|u(0)\|^p_{H^1(\mathbb{R}^d)}$. If $s > 1$, it is elementary to check that

$$s = (1-\alpha) + 2\alpha \text{ for a } \alpha \in (0,1) \text{ with } \alpha p < 1.$$

Indeed,

$$\frac{1}{2p} = \frac{1}{2} - \frac{s}{d} \Rightarrow s = \frac{d}{2}\left(1 - \frac{1}{p}\right) = \frac{d}{2}\,\frac{p-1}{p} = \frac{d}{2}\,\frac{1}{p'}.$$

Notice that incidentally that

$$\frac{1}{2d^*} = \frac{d-2}{2d+4} = \frac{1}{2} - \frac{\sigma}{d} \Rightarrow \frac{\sigma}{d} = \frac{1}{2}\left(1 - \frac{d-2}{d+2}\right) = \frac{2}{d+2}$$

from which we derive $\sigma = \frac{2d}{d+2} \leq 2$ and we have $s < \sigma$ (confirming that $s < 2$). Now

$$s = (1-\alpha) + 2\alpha = 1 + \alpha \Rightarrow \alpha p = sp - p = \frac{d}{2}(p-1) - p = \left(\frac{d}{2} - 1\right)p - \frac{d}{2}$$
$$< \frac{d-2}{2}d^* - \frac{d}{2} = \frac{d-2}{2}\,\frac{d+2}{d-1} - \frac{d}{2} = 1.$$

Then using interpolation in all cases we conclude

$$\begin{aligned}
&\|e^{\mathrm{i}t\triangle}|u(0)|^{p-1}u(0)\|_{L^r(\mathbb{R},L^\gamma(\mathbb{R}^d))} \leq C_{Strichartz}\||u(0)|^{p-1}u(0)\|_{L^2(\mathbb{R}^d)} \\
&\leq C_{Strichartz}C_{Sobolev}\|u(0)\|^{p(1-\alpha)}_{H^1(\mathbb{R}^d)}\|u(0)\|^{p\alpha}_{\dot{H}^2(\mathbb{R}^d)} \\
&\leq C_{Strichartz}C_{Sobolev}a^{p(1-\alpha)}M^{p\alpha} \text{ for } 0 \leq \alpha p < 1.
\end{aligned} \tag{3.30}$$

We have

$$
\begin{aligned}
&\| \int_0^t e^{\mathrm{i}s\triangle}\partial_t \left(|u(t-s)|^{p-1} u(t-s) \right) ds \|_{L^r([-T,T],L^\gamma(\mathbb{R}^d))} \\
&= \| \int_0^t e^{\mathrm{i}(t-s)\triangle}\partial_s \left(|u(s)|^{p-1} u(s) \right) ds \|_{L^r([-T,T],L^\gamma(\mathbb{R}^d))} \\
&\le C_{Strichartz} \| |u|^{p-1}\partial_t u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \le C_{Strichartz} \|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})} \|\partial_t u\|_{L^q([-T,T],L^{p+1})} \\
&\le C_{Strichartz} (2T)^{\frac{p-1}{\beta}} \|u\|^{p-1}_{L^\infty([-T,T],H^1)} \|\partial_t u\|_{L^q([-T,T],L^{p+1})} \\
&\le C_{Strichartz} (2T)^{\frac{p-1}{\beta}} a^{p-1} M.
\end{aligned}
$$

From the above we see that for a constant $C(d,a)$ and $T = T(d,a,M)$ such that

$$\|\partial_t \Phi(u)\|_{L^r([-T,T],L^\gamma(\mathbb{R}^d))} \le C(d,a) \left(\|u_0\|_{H^2(\mathbb{R}^d)} + a^{p(1-\alpha)} M^{p\alpha} \right).$$

Choosing M s.t.

$$\frac{M}{8} \ge C(d,a)\|u_0\|_{H^2(\mathbb{R}^d)} \text{ and } \frac{M}{8} \ge \max\{1, C(d,a)\} a^{p(1-\alpha)} M^{p\alpha} \tag{3.31}$$

we obtain

$$\|\partial_t \Phi(u)\|_{L^r([-T,T],L^\gamma(\mathbb{R}^d))} \le \frac{M}{4} \text{ for all admissible pairs } (r,\gamma). \tag{3.32}$$

Next, we have

$$\triangle \Phi(u) = -\mathrm{i}\partial_t \Phi(u) + \lambda |u|^{p-1} u \tag{3.33}$$

Then like in (3.30)

$$
\begin{aligned}
&\|\triangle \Phi(u)\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} \le \|\partial_t \Phi(u)\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} + \| |u|^{p-1} u\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} \\
&\le \|\partial_t \Phi(u)\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} + \|u\|^{p(1-\alpha)}_{L^\infty([-T,T],H^1(\mathbb{R}^d))} \|u\|^{p\alpha}_{L^\infty([-T,T],\dot{H}^2(\mathbb{R}^d))}
\end{aligned}
$$

Hence we conclude

$$\|\triangle \Phi(u)\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} \le \frac{M}{4} + a^{p(1-\alpha)} M^{p\alpha} \le \frac{M}{4} + \frac{M}{8} < \frac{M}{2}. \tag{3.34}$$

So we have found that Φ maps $E^2(T,M)$ into itself. We skip the proof that, by taking T appropriately small, Φ is a contraction. Notice also that by Theorem 1.4 we have $\Phi(u) \in C^0([-T,T], L^2(\mathbb{R}^d))$ for all $u \in E^2(T,M)$. □

Proof of Theorem 3.15. By standard arguments Proposition 3.16 implies that if $T < +\infty$ in (3.27) then

$$\lim_{t\to T^-} \|u(t)\|_{H^2(\mathbb{R}^d)} = +\infty. \tag{3.35}$$

Now we claim that if $u \in C^0([0,T], H^1(\mathbb{R}^d))$, then $\triangle u \in C^0([0,T], L^2(\mathbb{R}^d))$. Let $\|u\|_{L^\infty([0,T_1],H^1(\mathbb{R}^d))} \leq M < \infty$. Then there exists a constant C_M dependent only on M such that From $u = \Phi(u)$ and (3.32) for $T_1 \in (0,T)$ we have

$$\begin{aligned}
\|\triangle u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} &\leq \|\partial_t \Phi(u)\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} + \||u|^{p-1}u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} \\
&\leq C_M \left(1 + \|\triangle u\|^{p\alpha}_{L^\infty([0,T_1],L^2(\mathbb{R}^d))}\right).
\end{aligned}$$

Since $p\alpha < 1$ if

$$\|\triangle u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} \xrightarrow{T_1 \to T^-} +\infty \tag{3.36}$$

then for T_1 close enough to T we have

$$\|\triangle u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} \left(1 - C_M \|\triangle u\|^{p\alpha-1}_{L^\infty([0,T_1],L^2(\mathbb{R}^d))}\right) < \frac{1}{2}\|\triangle u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} \leq C_M$$

which implies the following, which contradicts (3.36) and completes the proof,

$$\lim_{T_1 \to T^-} \|\triangle u\|_{L^\infty([0,T_1],L^2(\mathbb{R}^d))} \leq 2C_M.$$

□

3.3 The global existence

We start with the following observation.

Lemma 3.17. *Let $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ be a maximal solution as of Proposition 3.6. Then if $T < \infty$ we have*

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty. \tag{3.37}$$

Analogously, $\lim_{t \searrow -S} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$ *if $S < \infty$.*

Remark 3.18. Notice that it is very important for this lemma that $p < d^*$. Indeed, in the *energy* critical case $p = d^*$, the above statement is false.

Proof. Suppose by contradiction that there exists a solution with $T < \infty$ for which there is a sequence $t_j \nearrow T$ s.t. $\|u(t_j)\|_{H^1(\mathbb{R}^d)} \leq M < \infty$. Then by Proposition 3.6 one can extend $u(t)$ beyond $t_j + T(M) > T$ and get a contradiction.

□

Corollary 3.19. *If $\lambda > 0$ the solutions of Proposition 3.6 are globally defined.*

Proof. Indeed if a solution has maximal interval of existence $(-S,T)$ with $T < \infty$, we must have (3.37). But for $\lambda > 0$ we have $\|\nabla u(t)\|_{L^2} \leq 2E(u(t)) = 2E(u_0)$.

□

Corollary 3.20. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.6 are globally defined.*

Proof. We have

$$2E(u(t)) \geq \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}.$$

Notice that

$$\alpha(p+1) = \frac{d}{2}(p+1) - d < 2 \Longleftrightarrow (p+1) - 2 < \frac{4}{d} \Longleftrightarrow p < 1 + \frac{4}{d}.$$

But then, if (3.37) happens, we have

$$2E(u_0) = \lim_{t \nearrow T} 2E(u(t)) \geq \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \left(1 - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)-2}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)}\right)$$
$$= \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = +\infty,$$

which is absurd. □

Corollary 3.21. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.6 are globally defined.*

3.4 Local existence for the L^2 critical case

We consider now equation (3.5) for $p = 1 + \frac{4}{d}$. Notice that in this case $(p+1, p+1)$ is an admissible pair.

Theorem 3.22. *For any $u_0 \in L^2(\mathbb{R}^d)$ there exists a unique maximal solution of (3.5) with $p = 1 + \frac{4}{d}$ with*

$$u \in C([0,T^*), L^2(\mathbb{R}^d)) \cap L^{p+1}_{loc}([0,T^*), L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.38}$$

Furthermore, the mass is preserved, we have $u \in L^a([0,T], L^b(\mathbb{R}^d))$ for any admissible pair, if $T \in (0,T^*)$.
There is continuity with respect to the initial data. And finally, if $T^* < \infty$, then

$$\lim_{T \to T^*} \|u\|_{L^a([0,T],L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } b \geq p+1. \tag{3.39}$$

Proposition 3.23. *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^{p+1}([0,T), L^{p+1}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0,T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T), L^{p+1}(\mathbb{R}^d)).$$

The mass is constant. Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0,T'], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T'], L^{p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([0,T], L^b(\mathbb{R}^d))$ for all admissible pairs (a,b).

Proof. The proof is a fixed point argument. We set like before

$$E(T,\delta) = \left\{ v \in L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d)) : \|v\|_{L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d))} \le 2\delta \right\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5).
By Strichartz's estimates

$$\begin{aligned} \|\Phi(u)\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} &< \delta + c_0 \||u|^{p-1}u\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &= \delta + c_0 \|u\|^p_{L^{p+1}([0,T]\times\mathbb{R}^d))} \le \delta + c_0 2^p \delta^p < 2\delta, \end{aligned}$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T,\delta)$. Now we show that Φ is a contraction in $E(T,\delta)$. We have

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} \le c_0 \||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &\le c_0 C \|(|u|^{p-1} + |v|^{p-1})|u-v|\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &\le c_0 C \left(\|u\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))} + \|v\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))} \right) \|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} \\ &\le c_0 C 2^{p-1} \delta^{p-1} \|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))}, \end{aligned}$$

which is a contraction for $\delta > 0$ small enough. The remaining part is also similar to that in Proposition 3.4. In particular, let us now discuss the conservation of mass. The first observation is that if $u_0 \in H^1(\mathbb{R}^d)$ then we have $u \in C([0,T], H^1(\mathbb{R}^d))$. Ti orove this we observe that $u \in C([0,\tau], H^1(\mathbb{R}^d))$ by Proposition 3.6 and if it is not possible to take $\tau \ge T$, then we will have a maximal interval of existence $u \in C([0,\tau), H^1(\mathbb{R}^d))$ with $\tau \in (0,T)$ and blow up $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. But, for $s < \tau_1 < \tau$,

$$\|\nabla u\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} < \|\nabla e^{\mathrm{it}\triangle} u_0\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} + c_0 \|u\|^{p-1}_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} \|\nabla u\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))}.$$

Now, for s close to τ we will have

$$c_0 \|u\|^{p-1}_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} < 1/2$$

and so, taking $\tau_1 \xrightarrow{\tau}$

$$\|\nabla u\|_{L^{p+1}([s,\tau]\times\mathbb{R}^d))} \le 2 \|\nabla e^{\mathrm{it}\triangle} u_0\|_{L^{p+1}([s,\tau]\times\mathbb{R}^d))}.$$

and in particular

$$\|\nabla u\|_{L^{p+1}([0,\tau]\times\mathbb{R}^d))} < +\infty.$$

Feeding this back in Strichartz inequality, we have

$$\|\nabla u\|_{L^\infty([0,\tau],L^2(\mathbb{R}^d)))} < \|\nabla u_0\|_{L^2(\mathbb{R}^d)} + c_0\|u\|^{p-1}_{L^{p+1}([0,\tau]\times\mathbb{R}^d))}\|\nabla u\|_{L^{p+1}([0,\tau]\times\mathbb{R}^d))} < +\infty,$$

which excludes the blow up $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. So we conclude that $u \in C([0,T], H^1(\mathbb{R}^d))$ and that, energy, momenta and mass of $u(t)$ are constant in $[0,T]$. If now $u_0 \notin H^1(\mathbb{R}^d)$, we consider a sequence $u_{0n} \in H^1(\mathbb{R}^d)$ with $u_{0n} \xrightarrow{n\to\infty} u_0$ in $L^2(\mathbb{R}^d)$. For any $T' \in (0,T)$, we have by well posedness that for the corresponding solutions we have $u_n \xrightarrow{n\to\infty} u$ in $C([0,T'], L^2(\mathbb{R}^d))$. Then $Q(u_n) \xrightarrow{n\to\infty} Q(u)$ in $C([0,T'],\mathbb{R})$. Since $Q(u_n)$ are constant functions, also $Q(u)$ is constant in $[0,T']$ for all $T' < T$.

□

Proof of Theorem 3.22. Clearly we have $\|e^{it\triangle}u_0\|_{L^{p+1}([0,T),L^{p+1}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.23 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.39). Suppose that it is false, and that there is a maximal solution in $[0,T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),L^b(\mathbb{R}^d))} < +\infty \text{ for an admissible pair with } b \geq p+1. \tag{3.40}$$

Then if $b > p+1$, we have

$$\|u\|_{L^{p+1}([0,T^*),L^{p+1}(\mathbb{R}^d))} \leq \|u\|^{\mu}_{L^\infty([0,T^*),L^2(\mathbb{R}^d))}\|u\|^{1-\mu}_{L^a([0,T^*),L^b(\mathbb{R}^d))} \text{ for } \mu = \frac{\frac{1}{p+1}-\frac{1}{b}}{\frac{1}{2}-\frac{1}{b}}.$$

So (3.40) holds also for $b = p+1$. Now, for s close to T^* we have from (3.5)

$$e^{i(t-s)\triangle}u(s) = u(t) + i\lambda\int_s^t e^{i(t-t')\triangle}|u(t')|^{p-1}u(t')dt'.$$

This yields

$$\|e^{i(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \leq \|u\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} + C\|u\|^p_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0.$$

So

$$\sup_{s<T<T^*}\|e^{i(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} < \delta/2 \Longrightarrow \|e^{i(t-s)\triangle}u(s)\|_{L^{p+1}([s,T^*],L^{p+1}(\mathbb{R}^d))} \leq \delta/2$$

where we used the continuity in T of $T \to \|e^{i(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))}$. Therefore by the continuity there exists $\varepsilon > 0$ small enough so that $\|e^{i(t-s)\triangle}u(s)\|_{L^{p+1}([s,T_*+\varepsilon],L^{p+1}(\mathbb{R}^d))} < \delta$. Then the solution u can be extended beyond T^* also in the interval $[0, T_* + \varepsilon]$.

□

Example 3.24. In the case $\lambda = -1$ of the L^2- critical focusing NLS

$$\mathrm{i}u_t = -\triangle u - |u|^{\frac{4}{d}} u \text{ in } \mathbb{R} \times \mathbb{R}^d, \tag{3.41}$$

there are related solutions in $H^1\left(\mathbb{R}^d, [0, +\infty)\right)$to

$$-\Delta\phi + \phi - |\phi|^{p-1}\phi = 0. \tag{3.42}$$

In 1–d they are explicit,

$$\phi(x) = \frac{(\frac{p-1}{2} + 1)^{\frac{4}{p-1}}}{\cosh^{\frac{2}{p-1}}(\frac{p-1}{2}x)}. \tag{3.43}$$

For $d \geq 2$ there are many types of solitons. For example, the ones in (3.43) are *ground states*, and they are the only ones in $d = 1$. But in $d \geq 2$ there are also excited states. Notice that if $u(t, x)$ is a solution of (3.41), then also the following is a solution,

$$v(t, x) = t^{-\frac{d}{2}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{\mathrm{i}\frac{x^2}{4t}}.$$

Since now, given a solution $\phi(x)$ of (3.42), then $u(t, x) = e^{\mathrm{i}t + \frac{\mathrm{i}}{2}\mathbf{v}\cdot x - \mathrm{i}\frac{\mathbf{v}^2}{4}t}\phi(x - t\mathbf{v} - D)$ is a solution of (3.41), it follows, choosing $\mathbf{v} = D = 0$, that

$$S(t, x) := t^{-\frac{d}{2}}\phi\left(\frac{x}{t}\right) e^{\mathrm{i}\frac{x^2}{4t}} e^{-\frac{\mathrm{i}}{t}} \text{ so also } S(T - t, x) := (T - t)^{-\frac{d}{2}}\phi\left(\frac{x}{T - t}\right) e^{\mathrm{i}\frac{x^2}{4(T-t)}} e^{-\frac{\mathrm{i}}{T-t}}.$$

Obviously this for $T > 0$ has maximal positive lifespan T. Then, for any admissible pair (q, r) with $r > 2$, we have

$$\|S(T - t, x)\|_{L^r(\mathbb{R}^d)} = (T - t)^{-\frac{d}{2} + \frac{d}{r}} \|\phi\|_{L^r(\mathbb{R}^d)} = (T - t)^{-\frac{2}{q}} \|\phi\|_{L^r(\mathbb{R}^d)} \notin L^q(0, T).$$

3.5 The H^1 critical cases

We consider now equation (3.5) for $p = 1 + \frac{4}{d-2}$. We will consider the admissible pair (γ, ρ)

$$\text{admissible pair } (\gamma, \rho) \text{ given by} \rho = \frac{2d^2}{d^2 - 2d + 4}, \quad \gamma = \frac{2d}{d - 2}. \tag{3.44}$$

Notice that it is an admissible pair because

$$\frac{2}{\gamma} + \frac{d}{\rho} = \frac{d}{2}.$$

Indeed

$$\frac{\not{2}}{\frac{\not{2}d}{d-2}} + \frac{\not{d}}{\frac{2d^{\not{2}}}{d^2-2d+4}} = \frac{d - 2}{d} + \frac{d^2 - 2d + 4}{2d} = 1 - \frac{2}{d} + \frac{d}{2} - 1 + \frac{2}{d} = \frac{d}{2}$$

Theorem 3.25. *For any $u_0 \in H^1(\mathbb{R}^d)$ there exists a unique maximal solution of* (3.5) *with $p = 1 + \frac{4}{d-2}$ with*

$$u \in C([0,T^*), H^1(\mathbb{R}^d)) \cap C^1([0,T^*), H^{-1}(\mathbb{R}^d)). \tag{3.45}$$

Furthermore, the mass and energy are preserved, we have $u \in L^a([0,T], W^{1,b}(\mathbb{R}^d))$ for any admissible pair, if $T \in (0,T^*)$.
There is continuity with respect to the initial data in the following sense. If $0 < T' < T^*$ and if $u_{0n} \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^d)$ then for the corresponding solutions we have we have $u_n \xrightarrow{n\to\infty} u$ in $L^p([0,T'], H^1(\mathbb{R}^d))$ for any $p < \infty$.

And finally, if $T^* < \infty$, then

$$\lim_{T\to T^*} \|u\|_{L^a([0,T],L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } d > b > 2. \tag{3.46}$$

The proof of Theorem 3.25 is based on Proposition 3.26 below. In the course of the proof, we will consider admissible pairs (a,b) with $b \in (2,d)$ the number $\frac{1}{b^*} := \frac{d-b}{bd} = \frac{1}{b} - \frac{1}{d}$. Then there exists an admissible pair (α,β) such that

$$\begin{aligned} \frac{1}{\beta'} &= \frac{1}{\beta} + \frac{\frac{4}{d-2}}{b^*} \text{ which can be rewritten as} \\ 1 &= \frac{2}{\beta} + \frac{4}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right). \end{aligned} \tag{3.47}$$

Here notice that for $b^* = \infty$, that is when $b = d$, then $\beta = 2$, and if $b^* = \frac{2d}{d-2}$, that is in the case $b = 2$, we have $\beta = \frac{2d}{d-2}$, which is the endpoint. So for $b \in (2,d)$ we have the intermediate cases $2 < \beta < \frac{2d}{d-2}$. We claim that the α in (α,β) satisfies

$$\begin{aligned} \frac{1}{\alpha'} &= \frac{1}{\alpha} + \frac{\frac{4}{d-2}}{a} \text{ or, equivalently} \\ 1 &= \frac{2}{\alpha} + \frac{4}{d-2}\,\frac{d}{2}\,\left(\frac{1}{2} - \frac{1}{b}\right). \end{aligned} \tag{3.48}$$

So in other words, we need to show

$$\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) = \left(\frac{1}{2} - \frac{d}{d-2}\left(\frac{1}{2} - \frac{1}{b}\right), 1 - \frac{2}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right)\right) \text{ for any } b \in (2,d). \tag{3.49}$$

It is enough to check the endpoints, in fact recall that $\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ lays in a line, so it is enough to prove (3.49) just for two values of b, because then this will imply the equality for all values of b. If $b^* = \infty$, that is when $b = d$, then $\beta = 2$, which implies $\alpha = \infty$, and so (3.48) becomes

$$1 = \frac{\frac{4}{d-2}}{a} = \frac{\frac{4}{d-2}}{\frac{4}{d-2}},$$

which is obviously correct.
Looking at $b = 2$, then as we mentioned, we have the endpoint $(\alpha, \beta) = \left(2, \frac{2d}{d-2}\right)$, which makes (3.48) true because $\alpha' = 2$ and $a = 0$.
It is interesting to check when $(a, b) = (\alpha, \beta)$ we obtain exactly the admissible pair in (3.44). Indeed,

$$1 = \frac{2}{a} + \frac{\frac{4}{d-2}}{a} \Longleftrightarrow a = 2 + \frac{4}{d-2} = \frac{2d}{d-2} = \gamma.$$

Finally, since the map $\frac{1}{a} \to \frac{1}{\alpha}$ in (3.48) is affine and $\frac{1}{\gamma}$ is a fixed point, in any case when $a \neq \alpha$ it follows that γ is in between them, and so also ρ is in between b and β and that

$$\text{there exists a } \theta \in (0,1) \text{ with } \left(\frac{1}{\gamma}, \frac{1}{\rho}\right) = \theta \left(\frac{1}{a}, \frac{1}{b}\right) + (1-\theta) \left(\frac{1}{\alpha}, \frac{1}{\beta}\right). \tag{3.50}$$

Proposition 3.26. *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0,T], H^1(\mathbb{R}^d)) \cap L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d)).$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0,T'], L^2(\mathbb{R}^d)) \cap L^\gamma([0,T'), W^{1,\rho}(\mathbb{R}^d)$$

and is Lipschitz.

Finally, we have $u \in L^a([0,T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a,b) and mass and energy are preserved.

Proof (sketch). The proof is by a contraction argument. We set like before

$$E(T,\delta) = \left\{ v \in L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d)) : \|v\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} \leq 2\delta \right\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Let us open a small parenthesis now, and let us pick an admissible pair (a,b) with $b \in (2,d)$. Then, for $\frac{1}{b^*} = \frac{d-b}{bd} = \frac{1}{b} - \frac{1}{d}$ and (α, β) admissible like in (3.48), by Strichartz estimates, by the Chain Rule in Lemma 3.1 and by $p - 1 = \frac{4}{d-2}$, we have

$$\begin{aligned}
&\|\Phi(u)\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} \leq \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0 \|u^{p-1} \langle \nabla \rangle u\|_{L^{\alpha'}([0,T), W^{1,\beta'}(\mathbb{R}^d))} \\
&\leq \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0 \|u\|^{p-1}_{L^a([0,T], L^{b^*})} \|u\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} \\
&\leq \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0' \|u\|^{p-1}_{L^a([0,T], W^{1,b})} \|u\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))}
\end{aligned}$$

So, in the particular case $(a, b) = (\alpha, \beta) = (\rho, \gamma)$, we have

$$\|\Phi(u)\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} \le \|e^{\mathrm{i}t\triangle}u_0\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} + c_0'\|u\|^p_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))}$$

Hence in $E(T, \delta)$ we have

$$\|\Phi(u)\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} < \delta + c_0' 2^p \delta^p < 2\delta,$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T, \delta)$. In a similar fashion we prove that Φ is a contraction in $E(T, \delta)$. We skip the proof on the conservation of mass, energy and momenta.

Proof of Theorem 3.25. Clearly we have $\|e^{\mathrm{i}t\triangle}u_0\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.26 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.46). Suppose that it is false, and that there is a maximal solution in $[0, T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),W^{1,b}(\mathbb{R}^d))} < +\infty. \tag{3.51}$$

But then

$$\|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le \|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} + c_0'\|u\|^{p-1}_{L^a([0,T],W^{1,b})}\|u\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))}$$

and the fact that $\|u\|^p_{L^a([s,T],W^{1,b}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0$, implies

$$\|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le 2\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))}$$

for $s < T < T^*$ with s and T close to T^*. This implies in fact that also

$$\|u\|_{L^\alpha([0,T^*),W^{1,\beta}(\mathbb{R}^d))} < +\infty. \tag{3.52}$$

Then, by

$$e^{\mathrm{i}(t-s)\triangle}u(s) = u(t) + \mathrm{i}\lambda \int_s^t e^{\mathrm{i}(t-t')\triangle}|u(t')|^{p-1}u(t')dt',$$

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le \|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} + c_0'\|u\|^{p-1}_{L^a([s,T],W^{1,b})}\|u\|_{L^\alpha([s,T),W^{1,\beta}(\mathbb{R}^d))}$$
$$\xrightarrow{s<T\to T^{*-}} 0.$$

Since there exists a $\theta \in [0, 1]$ with the following, see (3.50),

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T_*+\varepsilon],W^{1,\rho}(\mathbb{R}^d))} \le \|e^{\mathrm{i}(t-s)\triangle}u(s)\|^\theta_{L^a([s,T],W^{1,b}(\mathbb{R}^d))}\|e^{\mathrm{i}(t-s)\triangle}u(s)\|^{1-\theta}_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))}$$

it follows that we can arrange $\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T_*+\varepsilon],W^{1,\rho}(\mathbb{R}^d))} < \delta$, for s close enough to T^* and for $\varepsilon > 0$ arbitrarily small. But then the solution u can be extended beyond T^*.

We skip here the discussion of the well posedness.

□

4 The dispersive equation

Here we will consider dispersive equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{4.1}$$

with $1 + 4/d < p < d^*$. In this §we will give a partial proof of the following classical result.

Theorem 4.1 (Scattering)**.** *Consider the unique solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$. Then*

$$u \in L^a(\mathbb{R}, W^{1,b}(\mathbb{R}^d)) \text{ for any admissible pair} \tag{4.2}$$

and there exist $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t\to\pm\infty} \|u(t) - e^{\mathrm{i}t\triangle}u_\pm\|_{H^1(\mathbb{R}^d)} = 0. \tag{4.3}$$

Remark 4.2. Scattering (the *completeness of scattering operators*) refers specifically to (4.3). Notice that for $1 < p \le 1 + 2/d$ Scattering (4.3) is false. For $1 + 2/d < p \le 1 + 4/d$ is an open problem.

Here the key deep statement is (4.2). In fact, (4.2) implies easily (4.3), as we show now in the case $+$. So, assume (4.2), and in particular let

$$u \in L^q(\mathbb{R}_+, W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{4.4}$$

From (3.5) with $\lambda = 1$, we have

$$e^{-\mathrm{i}t\triangle}u(t) = u_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds,$$

so that, for $t_1 < t_2$, we have

$$e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1) = -\mathrm{i}\int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds.$$

Then

$$\begin{aligned} &\|e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1)\|_{H^1} \le \left\|\int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds\right\|_{H^1} \\ &\le \|u\|^{p-1}_{L^\alpha([t_1,t_2],L^{p+1})}\|u\|_{L^q([t_1,t_2],W^{1,p+1})} \end{aligned} \tag{4.5}$$

where $\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'}$. We claim that $\alpha > q$. Otherwise $\alpha \le q$ and so

$$\frac{p}{q} \le \frac{1}{q'} \Leftrightarrow p + 1 \le q.$$

So, from $p > 1 + \frac{4}{d}$, $(q, p+1)$ is an admissible pair with both entries $> 2 + \frac{4}{d}$. But $\left(2+\frac{4}{d}, 2+\frac{4}{d}\right)$ is an admissible pair, so we get an absurd and we conclude $\alpha > q$.
So, let us consider the pair (α, β) which is admissible (notice that $\alpha > q > 2$ implies $\infty > \alpha > 2$ and so $2 < \beta < p+1$. We claim that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \text{ with } \tau \in [0,1]. \tag{4.6}$$

Assuming this, (4.5) can be majorized yielding

$$\|e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1)\|_{H^1} \le c_0 \|u\|^{p-1}_{L^\alpha([t_1,t_2],W^{1,\beta})} \|u\|_{L^q([t_1,t_2],W^{1,p+1})} \xrightarrow{t_1<t_2\to+\infty} 0.$$

This implies that there exists

$$u_+ = \lim_{t\to+\infty} e^{-\mathrm{i}t\triangle}u(t) \text{ in } H^1(\mathbb{R}^d).$$

Then we have

$$e^{\mathrm{i}t\triangle}u_+ - u(t) = -\mathrm{i}\int_t^\infty e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds.$$

As above,

$$\|e^{\mathrm{i}t\triangle}u_+ - u(t)\|_{H^1} \le \|u\|^{p-1}_{L^\alpha([t,\infty),W^{1,\beta})} \|u\|_{L^q([t,\infty),W^{1,p+1})} \xrightarrow{t\to+\infty} 0,$$

which proves the limit (4.3).
Turning to the proof of (4.6), obviously $\alpha > q$ implies $\beta < p+1$ so that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d}$$

with $\tau > 0$. Since $2 < \beta < p+1 < +\infty$, for $d = 1, 2$ we have $\tau < 1$. For $d \ge 3$ we have $2 < \beta < p+1 < \frac{2d}{d-2}$. Since $\frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$,

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

which implies $0 < \tau < 1$ by

$$\frac{1-\tau}{d} > \frac{1}{2} - \frac{1}{\beta}.$$

As we indicated above, in Theorem 4.1, the deep statement is (4.2). The proof is rather complicated. For this we will need the following which we will discuss only for dimension $d \ge 3$.

Theorem 4.3. *Let $d \ge 3$. Then given a solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$ we have*

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^r(\mathbb{R}^d)} = 0 \textit{ for all } 2 < r < \frac{2d}{d-2}. \tag{4.7}$$

Remark 4.4. Notice that it is enough to prove only case $r = p+1$. In fact, for $2 < r < p+1$ there is an exponent $\alpha \in (0,1)$ with

$$\|u(t)\|_{L^r(\mathbb{R}^d)} \le \|u_0\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{p+1}(\mathbb{R}^d)}$$

which yields (4.7) while for $p+1 < r < \frac{2d}{d-2}$ there is an exponent $\alpha \in (0,1)$ with

$$\begin{aligned}\|u(t)\|_{L^r(\mathbb{R}^d)} &\le \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \le \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \\ &\le \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} (2E(u_0))^{1-\alpha}\end{aligned}$$

which again yields (4.7).

Theorem 4.3 is deep result and implies (4.2) rather easily as we see now. We will use the following elementary lemma.

Lemma 4.5. *consider a function $f(x) = a - x + bx^{\alpha}$ for $x \ge 0$, $a, b > 0$, $\alpha > 1$. We assume that there are $0 < x_0 < x_1$ s.t. $f(x_0) = f(x_1) = 0$. Let now $\phi \in C(I, [0, +\infty))$ be such that $\phi(t) \le a + b\phi^{\alpha}(t)$ for all $t \in I$ and that there exists a point $t_0 \in I$ s.t. $\phi(t_0) \le x_0$. Then $\phi(t) \le x_0$ for all $t \in I$*

Proof. Since $f(\phi(t)) \ge 0$ for all t, and ϕ is continuous, the image of ϕ is either in $[0, x_0]$ or in $[x_1, +\infty)$. Obviously, the first case needs to occur. □

Proof that Theorem 4.3 implies (4.2) *(sketch).* Consider

$$u(t) = e^{\mathrm{i}(t-S)\triangle} u(S) - \mathrm{i} \int_S^t e^{\mathrm{i}(t-s)\triangle} |u(s)|^{p-1} u(s) ds,$$

Then by the Strichartz estimates

$$\begin{aligned}\|u\|_{L^q((S,t),W^{1,p+1})} &\le C\|u(S)\|_{H^1} + C \left\| \|u\|^{p-1}_{L^{p+1}_x} \|u\|_{W^{1,p+1}_x} \right\|_{L^{q'}(S,t)} \\ &= C\|u(S)\|_{H^1} + C \left(\int_S^t \|u\|^{(p-1)q'-(q-q')}_{L^{p+1}_x} \|u\|^{q-q'}_{L^{p+1}_x} \|u\|^{q'}_{W^{1,p+1}_x} ds \right)^{\frac{1}{q'}} \\ &\le C\|u(S)\|_{H^1} + C\|u\|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)} \|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.\end{aligned}$$

Here

$$p - \frac{q}{q'} = p + 1 - q > 0 \Leftrightarrow p > 1 + 4/d.$$

From Theorem 4.3, applied to $r = p+1$, we know $\|u\|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)} \xrightarrow{S\to+\infty} 0$. Furthermore, using conservation of mass and energy, there is a uniform upper bound for $\|u(S)\|_{H^1}$. There exists a constant $C_0 > 0$ s.t. for any $\epsilon > 0$ there is $S_0 > 0$ such that for any $S_0 < S < t$,

$$\|u\|_{L^q((S,t),W^{1,p+1})} \le C_0 + \epsilon \|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.$$

Picking $\epsilon > 0$ sufficiently small, by Lemma 4.5 we conclude that there exists a fixed constant X_0 s.t.

$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq X_0 \text{ for any } S_0 < S < t.$$

In particular we can take $t = \infty$. Since we know that $u \in L^q_{loc}(\mathbb{R}, W^{1,p+1})$, we conclude that $\|u\|_{L^q(\mathbb{R}_+,W^{1,p+1})} < +\infty$. Time reversibility of the NLS, yields the same result for negative times. The Strichartz estimates, yield $u \in L^a(\mathbb{R}, W^{1,b})$ for any admissible pair. Like in (3.10), we have

$$\|u\|_{L^a((S,t),W^{1,b})} \leq c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),L^{p+1})}\|u\|_{L^q((S,t),W^{1,p+1})}$$

where $\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'}$ with here $\alpha > q$ by the discussion in the proof of (3.8). So now let (α, β) be an admissible pair. We have $W^{1,\beta}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$, so, up to a change of constants, we get

$$\|u\|_{L^a((S,t),W^{1,b})} \leq c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),W^{1,\beta})}\|u\|_{L^q((S,t),W^{1,p+1})} \tag{4.8}$$

and in particular

$$\|u\|_{L^\alpha((S,t),W^{1,\beta})} \leq c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),W^{1,\beta})}\|u\|_{L^q((S,t),W^{1,p+1})}. \tag{4.9}$$

If in (4.9) we have $p \leq 2$, then since $\|u\|_{L^q((S,t),W^{1,p+1})} \xrightarrow{S\to+\infty} 0$ the factor $\|u\|_{L^\alpha((S,t),W^{1,\beta})}$ remains bounded for $t \to +\infty$ if $S \gg 1$. If instead $p - 1 > 1$ we can apply Lemma 4.5. So we conclude that in all cases $\|u\|_{L^\alpha((S,t),W^{1,\beta})}$ remains bounded for $t \to +\infty$ if $S \gg 1$. Inserting this information in (4.8), we get the same conclusion for $\|u\|_{L^a((S,t),W^{1,b})}$.

5 Proof of Theorem 4.3

Lemma 5.1. *Let $p \in [1, \infty)$ and $q < d$ with $0 \leq q \leq p$. Then we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \leq \left(\frac{p}{d-q}\right)^q \|u\|^{p-q}_{L^p(\mathbb{R}^d)} \|\nabla u\|^q_{L^p(\mathbb{R}^d)}. \tag{5.1}$$

Proof. The general case $u \in W^{1,p}(\mathbb{R}^d)$ reduces to the special case $u \in C^\infty_c(\mathbb{R}^d)$. In fact, if (5.1) is valid for all $u \in C^\infty_c(\mathbb{R}^d)$, then for a $u \in W^{1,p}(\mathbb{R}^d)$ with $u \notin C^\infty_c(\mathbb{R}^d)$, we can consider a sequence $C^\infty_c(\mathbb{R}^d) \ni u_n \xrightarrow{n\to+\infty} u$ in $W^{1,p}(\mathbb{R}^d)$. Then, up to subsequence, we have $u_n(x) \xrightarrow{n\to+\infty} u(x)$ for a.a. $x \in \mathbb{R}^d$, see p. 94 [2]. Then, by Fathou's Lemma

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{|u_n(x)|^p}{|x|^q} dx \\ &\leq \lim_{n\to\infty} \left(\frac{p}{d-q}\right)^q \|u_n\|^{p-q}_{L^p(\mathbb{R}^d)} \|\nabla u_n\|^q_{L^p(\mathbb{R}^d)} = \left(\frac{p}{d-q}\right)^q \|u\|^{p-q}_{L^p(\mathbb{R}^d)} \|\nabla u\|^q_{L^p(\mathbb{R}^d)}. \end{aligned}$$

So we will prove (5.1) for $u \in C_c^\infty(\mathbb{R}^d)$. Let $z(x) := |x|^{-q}x$. Then

$$\nabla \cdot z = \nabla(|x|^{-q}) \cdot x + |x|^{-q}\nabla \cdot x = -q|x|^{-q-1}\frac{x}{|x|} \cdot x + d|x|^{-q} = (d-q)|x|^{-q}.$$

Integrating the identity

$$|u|^p \nabla \cdot z = \nabla \cdot (|u|^p z) - p|u|^{p-1}\nabla|u| \cdot z,$$

we obtain for arbitrary $r > 0$

$$\begin{aligned}(d-q)\int_{|x|>r} \frac{|u(x)|^p}{|x|^q}dx &= \int_{|x|>r} \nabla \cdot (|u|^p z)\, dx - p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx \\ &\leq -p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx \leq p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}dx,\end{aligned}$$

where we used

$$\int_{|x|>r} \nabla \cdot (|u|^p z)\, dx = -\int_{|x|=r} |u|^p z \cdot \frac{x}{|x|}dS = -\int_{|x|=r} |u|^p |x|^{-q+1} dS \leq 0.$$

Using $1 - \frac{1}{q} + \frac{p-q}{pq} + \frac{1}{p} = 1$ and Hölder inequality, we have

$$\begin{aligned}p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}dx &= p\int_{|x|>r} \frac{|u|^{\frac{p(q-1)}{q}}}{|x|^{q-1}}|u|^{\frac{p-q}{q}}|\nabla u| dx \\ &\leq p\left(\int_{|x|>r} \frac{|u|^p}{|x|^q}dx\right)^{\frac{q-1}{q}} \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p-q}{q}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

This yields

$$\int_{|x|>r} \frac{|u(x)|^p}{|x|^q}dx \leq \left(\frac{p}{d-q}\right)^q \|u\|_{L^p(\mathbb{R}^d)}^{p-q} \|\nabla u\|_{L^p(\mathbb{R}^d)}^q$$

and, taking $r \to 0^+$, we obtain (5.1).

□

Lemma 5.2. *For $d \geq 4$ there exists a C_d s.t. we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^3}dx \leq C_d \|u\|_{H^2(\mathbb{R}^d)}^2. \tag{5.2}$$

Proof. We proceed as above for $q = 3$ and $p = 2$, to obtain

$$\begin{aligned}(d-3)\int_{|x|>r} \frac{|u(x)|^2}{|x|^3}dx &\leq -p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx \leq 2\int_{|x|>r} \frac{|u||\nabla u|}{|x|^2}dx \\ &\leq 2\left(\int_{|x|>r} \frac{|u|^2}{|x|^2}dx\right)^{\frac{1}{2}} \left(\int_{|x|>r} \frac{|\nabla u|^2}{|x|^2}dx\right)^{\frac{1}{2}}.\end{aligned}$$

In the 2nd line we apply (5.1) for $p = q = 2$ to both u and ∇u, to obtain

$$(d-3)\int_{|x|>r}\frac{|u(x)|^2}{|x|^3}dx \le 2\left(\int_{|x|>r}\frac{|u|^2}{|x|^2}dx\right)^{\frac12}\left(\int_{|x|>r}\frac{|\nabla u|^2}{|x|^2}dx\right)^{\frac12} \le 2\left(\frac{2}{d-2}\right)\|\nabla u\|_{L^2(\mathbb{R}^d)}\|\nabla^2 u\|_{L^2(\mathbb{R}^d)}$$

Then (5.2) follows sending $r \to 0$. $\square$

Let $u_0 \in H^2$. Then $u \in C^0([0,T), H^2)$ by the theory by Kato. Then equation (4.1) holds also in a differential sense as

$$\mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ in } \mathcal{D}'\left((0,T), L^2(\mathbb{R}^d,\mathbb{C})\right).$$

Notice that $u \in C^1([0,T), L^2)$. Let us now consider the quadratic form

$$\frac12\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle. \tag{5.3}$$

Notice that it is well defined and self–adjoint. Then, taking the derivative for $u \in C^0([0,T), H^2) \cap C^1([0,T), L^2)$ we have

$$\frac{d}{dt}2^{-1}\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, \mathrm{i}\dot{u}\right\rangle.$$

which can be proved assuming first $u \in C^\infty([0,T), H^2)$ and then proceeding by a density argument. In our case we get

$$\begin{aligned}&\frac{d}{dt}2^{-1}\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle = \\ &\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, -\mathrm{i}\dot{u}\right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, -\triangle u + |u|^{p-1}u\right\rangle.\end{aligned} \tag{5.4}$$

The equality (5.4) is crucial, indeed we will use it to prove

$$\frac{d}{dt}\langle \partial_r u, \mathrm{i}u\rangle \ge (d-1)\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}dx, \tag{5.5}$$

which tells us that $u \to \langle \partial_r u, \mathrm{i}u\rangle$ is some sort of Lyapunov functional and is crucial in our argument.

The first observation to obtain (5.5), is that the following is true,

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, \mathrm{i}\dot{u}\right\rangle = \frac12\frac{d}{dt}\langle \partial_r u, \mathrm{i}u\rangle. \tag{5.6}$$

Indeed, notice that

$$\begin{aligned}&\frac12\partial_t \operatorname{Re}(\mathrm{i}u\overline{u}_r) + \frac12\nabla\cdot\left(\frac{x}{r}\operatorname{Re}(\mathrm{i}\dot{u}\overline{u})\right) \\ &= \frac12\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r) + \cancel{\frac12\operatorname{Re}(\mathrm{i}u\dot{\overline{u}}_r)} + \frac12\left(\nabla\cdot\frac{x}{r}\right)\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}) + \cancel{\frac12\operatorname{Re}(\mathrm{i}\dot{u}_r\overline{u})} + \frac12\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r) \\ &= \operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r) + \frac{d-1}{2r}\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}),\end{aligned}$$

so that integrating in x we obtain exactly (5.6).
The next step to prove (5.5), is the following inequality.

Claim 5.3. Let $u \in H^2(\mathbb{R}^d, \mathbb{C})$. Then

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq 0. \tag{5.7}$$

Proof. The proof is based on the identity

$$\begin{aligned}&\mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} - \nabla \cdot \left\{\frac{x}{2r}|\nabla u|^2\right\} \\ &+ \nabla \cdot \left(\frac{d-1}{4}\ \frac{x}{r^3}|u|^2\right) - \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) - \frac{(d-1)(d-3)}{4r^3}|u|^2,\end{aligned} \tag{5.8}$$

which we check now. We have

$$\begin{aligned}&\nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \mathrm{Re}\left\{\partial_j u \partial_j \left(\frac{x_k}{r}\partial_k \overline{u}\right)\right\} + \mathrm{Re}\left\{\partial_j u \partial_j \left(\frac{d-1}{2r}\overline{u}\right)\right\} \\ &= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \frac{x_k}{2r}\partial_k|\nabla u|^2 + \frac{1}{r}|\nabla u|^2 - \mathrm{Re}\left\{\frac{x_k x_j}{r^3}\partial_j u \partial_k \overline{u}\right\} + \frac{d-1}{2r}|\nabla u|^2 \\ &- \frac{d-1}{2}\frac{x_j}{r^3}\mathrm{Re}\left\{\partial_j u \overline{u}\right\} \\ &= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \partial_k\left(\frac{x_k}{2r}|\nabla u|^2\right) - |\nabla u|^2 \partial_k\left(\frac{x_k}{2r}\right) + \frac{|\nabla u|^2 - |u_r|^2}{r} + \frac{d-1}{2r}|\nabla u|^2 \\ &- \partial_j\left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right) - \frac{d-1}{4}|u|^2 \partial_j\left(\frac{x_j}{r^3}\right).\end{aligned}$$

Now we use

$$\begin{aligned}\partial_k\left(\frac{x_k}{2r}\right) &= \frac{d-1}{2r} \\ \partial_j\left(\frac{x_j}{r^3}\right) &= \frac{d-3}{r^3},\end{aligned}$$

to conclude

$$\begin{aligned}&\nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \\ &= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \partial_k\left(\frac{x_k}{2r}|\nabla u|^2\right) + \frac{|\nabla u|^2 - |u_r|^2}{r} \\ &- \partial_j\left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right) - \frac{(d-1)(d-3)}{4r^3}|u|^2,\end{aligned}$$

which is (5.8). Now, applying the Divergence Theorem to (5.8) in $\mathbb{R}^d \backslash D_{\mathbb{R}^d}(0,a)$ and take the limit for $a \to 0$ and Lemma 5.1, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - \frac{(d-1)(d-3)}{4} \lim_{a\to 0^+} \int_{r\geq a} \frac{|u|^2}{r^3} dx$$
$$- \liminf_{a\to 0^+} \int_{r=a} \left[\operatorname{Re}\left\{ u_r \left(\overline{u}_r + \frac{d-1}{2a}\overline{u} \right) \right\} - \frac{|\nabla u|^2}{2} + \frac{d-1}{4}\frac{|u|^2}{a^2} \right] dS.$$

Let us now suppose that $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$. Then

$$\lim_{a\to 0^+} \int_{\partial B(x,a)} |\nabla u|^2 dS = 0$$

Similarly, for $d > 3$ and $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$ we have

$$\lim_{a\to 0^+} \int_{r=a} \frac{|u|^2}{a^2} dS = 0$$

Hence, for $d > 3$ and $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$ we obtain 5.1, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - \frac{(d-1)(d-3)}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{r^3} dx \leq 0. \tag{5.9}$$

For $u \in H^2(\mathbb{R}^d, \mathbb{C})$ and, $u \notin C^\infty(\mathbb{R}^d, \mathbb{C})$ considered a sequence $u_n \xrightarrow{n\to\infty} u$ in $H^2(\mathbb{R}^d, \mathbb{C})$, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u_n, \triangle u_n \right\rangle = -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u_n|^2 - |u_{nr}|^2\right) dx - \frac{(d-1)(d-3)}{4} \int_{\mathbb{R}^d} \frac{|u_n|^2}{r^3} dx$$

which in the limit converges to (5.9).
For $d = 3$ then $u \in C^0(\mathbb{R}^3)$ and so

$$\lim_{a\to 0^+} \int_{\partial B(0,a)} |u|^2 \frac{dS}{a^2} = 4\pi |u(0)|^2,$$

so that we obtain

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle = -\int_{\mathbb{R}^3} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - 2\pi |u(0)|^2.$$

□

The next step to prove inequality (5.5) is the following identity,

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1} u \right\rangle = \frac{d-1}{2}\frac{p-1}{p+1} \int \frac{|u|^{p+1}}{r}. \tag{5.10}$$

Indeed

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1}u \right\rangle = \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2} \int_{\mathbb{R}^d} (|u|^2)^{\frac{p-1}{2}} \partial_r |u|^2 dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2} \frac{2}{p+1} \int_{\mathbb{R}^d} \partial_r (|u|^2)^{\frac{p+1}{2}} dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} - \frac{d-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} = \frac{d-1}{2} \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}.$$

So now we can prove (5.5). Indeed, from (5.6), (5.4), (5.6) and (5.10), we obtain

$$-\frac{1}{2}\frac{d}{dt} \langle \partial_r u, \mathrm{i}u \rangle = \left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, -\mathrm{i}\dot{u} \right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, -\triangle u + |u|^{p-1}u \right\rangle$$
$$\leq -\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1}u \right\rangle = -\frac{d-1}{2} \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r},$$

which yields (5.5).

Lemma 5.4. *We have*

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} \leq \frac{2}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))} \leq \frac{2^{\frac{3}{2}}}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} E(u_0). \tag{5.11}$$

furthermore, we have $u(t) \overset{t\to\infty}{\rightharpoonup} 0$ *in* $H^1(\mathbb{R}^d)$*.*

Proof. To get (5.11) if $u_0 \in H^2(\mathbb{R}^d)$ it is enough to integrate (5.5). The general case follows by density, because if $H^2 \ni u_{0n} \xrightarrow{n\to+\infty} u_0$ in H^1, we know that for any $T > 0$ for the corresponding solutions $u_n \xrightarrow{n\to+\infty} u$ in $C^0([0,T], H^1)$. Then, by the density argument in Lemma 5.1, we have

$$\int_0^T dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} \leq \liminf_{n\to\infty} \int_0^T dt \int_{\mathbb{R}^d} \frac{|u_n|^{p+1}}{r} \leq \frac{2}{d-1} \frac{p+1}{p-1} \|u_{0n}\|_{L^2(\mathbb{R}^d)} \|\nabla u_n\|_{L^\infty([0,T], L^2(\mathbb{R}^d))}$$
$$\xrightarrow{n\to+\infty} \frac{2}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^\infty([0,T], L^2(\mathbb{R}^d))}.$$

Taking the limit for $T \to +\infty$ we obtain (5.11) with $\mathbb{R}$ replaced by $\mathbb{R}_+$, which by time reversibility yields also the general case.
To get $u(t) \overset{t\to\infty}{\rightharpoonup} 0$ in $H^1(\mathbb{R}^d)$ it is enough to show $\langle u(t), \psi \rangle \xrightarrow{t\to+\infty} 0$ for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We have

$$|\langle u, \psi \rangle| \leq \| \frac{u}{r^{\frac{1}{p+1}}} \|_{L^{p+1}} \| r^{\frac{1}{p+1}} \psi \|_{L^{\frac{p+1}{p}}}$$

so that $|\langle u, \psi \rangle|^{p+1} \in L^1(\mathbb{R})$. On the other hand, from

$$\mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ in } \mathcal{D}'\Big((0,T), H^{-1}(\mathbb{R}^d, \mathbb{C})\Big).$$

we have $u \in BC^1\left(\mathbb{R}, H^{-1}(\mathbb{R}^d, \mathbb{C})\right)$ which implies $|\langle u, \psi\rangle|^{2k} \in BC^1(\mathbb{R})$ for $2k \geq p+2$ and for $s < t$ we have

$$\begin{aligned}\Big| |\langle u(t), \psi\rangle|^{2k} - |\langle u(s), \psi\rangle|^{2k}\Big| &= 2k \int_s^t |\langle u(t'), \psi\rangle|^{2k-1} |\langle \dot{u}(t'), \psi\rangle| dt' \\ &\lesssim C\left(\psi, E(u_0), \|u_0\|_{L^2}\right) \int_s^t |\langle u(t'), \psi\rangle|^{p+1} dt' \xrightarrow{s\to+\infty} 0.\end{aligned}$$

□

Before starting the direct proof of Theorem 4.3 we recall the following lemma.

Lemma 5.5. *There exists a constant $C = C_T$ such that for any $u \in L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$ we have $u \in C^0([0,T], L^2(\mathbb{R}^d))$ with*

$$\|u\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \leq C\left(\|u\|_{L^2((0,T),H^1(\mathbb{R}^d))} + \|\dot{u}\|_{L^2((0,T),H^{-1}(\mathbb{R}^d))}\right). \tag{5.12}$$

Furthermore we have $\|u(t)\|^2_{L^2} \in AC([0,T])$ with

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\langle u(t), \dot{u}(t)\rangle. \tag{5.13}$$

Proof. Let us assume additionally that $u \in C^1([0,T], L^2(\mathbb{R}^d))$. Then for any fixed $t_0 \in [0,T]$ we have

$$\begin{aligned}\|u(t)\|^2_{L^2} &= \|u(t_0)\|^2_{L^2} + 2\int_{t_0}^t \langle u(s), \dot{u}(s)\rangle ds \qquad (5.14)\\ &\leq \|u(t_0)\|^2_{L^2} + \|u\|^2_{L^2((0,T),H^1(\mathbb{R}^d))} + \|\dot{u}\|^2_{L^2((0,T),H^{-1}(\mathbb{R}^d))}.\end{aligned}$$

We can choose $\|u(t_0)\|^2_{L^2} = T^{-1}\int_0^T \|u(s)\|^2_{L^2} ds$ obtaining (5.12) for $C = \sqrt{1+T^{-1}}$
The general case is obtained by considering a sequence (u_n) in $C^1([0,T], H^1(\mathbb{R}^d))$ converging to u in $L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$. To get such a sequence, we can extend appropriately u into a function in $L^2(\mathbb{R}, H^1(\mathbb{R}^d)) \cap H^1(\mathbb{R}, H^{-1}(\mathbb{R}^d))$, and then we can consider $u_n = \rho_{\epsilon_n} * u$ with $\epsilon_n \xrightarrow{n\to\infty} 0$. Then this sequence satisfies the desired properties.
Then (5.12) implies that (u_n) is a Cauchy sequence in $C^0([0,T], L^2(\mathbb{R}^d))$. The limit is necessarily u, which satisfies (5.12). Also by a limit, we conclude that u satisfies the equality in (5.14), for any fixed $t_0 \in [0,T]$. This implies $\|u(t)\|^2_{L^2} \in AC([0,T])$ and formula (5.13).

□

Lemma 5.6. *We have*

$$\int_{|x|\geq t\log t} |u|^{p+1} dx \xrightarrow{t\to+\infty} 0. \tag{5.15}$$

Proof. We consider for $M>0$

$$\theta_M(x)=\begin{cases}\frac{|x|}{M} \text{ for } |x|\le M\\ 1 \text{ for } |x|\ge M\end{cases}$$

Then $\theta_M\in W^{1,\infty}(\mathbb{R}^d)$ with $\|\nabla\theta_M\|_{L^\infty}\le 1/M$. Now we have $u\in C^0(\mathbb{R},H^1)\cap C^1(\mathbb{R},H^{-1})$. Then, by Lemma 5.5 applied to $\sqrt{\theta_M}u$, $t\to 2^{-1}\langle\theta_M u(t),u(t)\rangle\in AC([-T,T])$ for any $T>0$ with

$$\frac{d}{dt}2^{-1}\langle\theta_M u(t),u(t)\rangle=\langle\theta_M u(t),\dot u(t)\rangle .$$

Since we have $\mathrm{i}\dot u(t)=-\triangle u+|u|^{p-1}u$ in $\mathcal{D}'(\mathbb{R},H^{-1})$, we have

$$\left|\frac{d}{dt}2^{-1}\langle\theta_M u(t),u(t)\rangle\right|=\left|\left\langle\theta_M u(t),\mathrm{i}\triangle u-\mathrm{i}|u|^{p-1}u\right\rangle\right|=|\langle\theta_M u(t),\mathrm{i}\triangle u\rangle|\le\|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty}$$
$$\le\|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty}\le CM^{-1}.$$

Then it follows, for a C independent from M,

$$\langle\theta_M u(t),u(t)\rangle\le CM^{-1}t+\langle\theta_M u_0,u_0\rangle .$$

Setting $M=t\log t$, we obtain by dominated convergence

$$\int_{|x|\ge t\log t}|u(t)|^2dx\le\langle\theta_{t\log t}u(t),u(t)\rangle$$
$$\le\frac{C}{\log t}+\int_{|x|\le t\log t}\frac{|x|}{t\log t}|u_0|^2dx+\int_{|x|\ge t\log t}|u_0|^2dx\xrightarrow{t\to+\infty}0.$$

Finally

$$\|u(t)\|_{L^{p+1}(|x|\ge t\log t)}\le\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|u(t)\|^{1-\alpha}_{L^{d^*+1}(\mathbb{R}^d)}$$
$$\le C\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|\nabla u(t)\|^{1-\alpha}_{L^2(\mathbb{R}^d)}\le C'\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\xrightarrow{t\to+\infty}0.$$

□

Lemma 5.7. *For any $\epsilon>0$, $t>1$ and $\tau>0$ there exists $t_0>\max(t,2\tau)$ s.t.*

$$\int_{t_0-2\tau}^{t_0}\int_{|x|\le s\log s}|u|^{p+1}dxds\le\epsilon. \tag{5.16}$$

Proof. The starting point is Lemma 5.4. We have

$$\infty>\int_{\mathbb{R}}dt\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}\ge\int_2^\infty\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx$$
$$\ge\sum_{k=0}^\infty\int_{t+2k\tau}^{t+2(k+1)\tau}\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx$$
$$\ge\sum_{k=0}^\infty\frac{1}{(t+2(k+1)\tau)\log(t+2(k+1)\tau)}\int_{t+2k\tau}^{t+2(k+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx.$$

From this inequality we derive

$$\liminf_{k\to+\infty}\int_{t+2k\tau}^{t+2(k+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx=0,$$

because otherwise the series would diverge. Hence for any $\epsilon>0$ there exists k_0 arbitrarily large with

$$\int_{t+2k_0\tau}^{t+2(k_0+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx<\epsilon.$$

So for $t_0=t+2(k_0+1)\tau$ we obtain (5.16).

□

Lemma 5.8. *For any $\epsilon, a, b\in\mathbb{R}_+$ there exists $t_0>\max(a,b)$ s.t.*

$$\sup_{s\in[t_0-b,t_0]}\|u(s)\|_{L^{p+1}}\le\epsilon. \tag{5.17}$$

Proof. We have

$$\begin{aligned}u(t)&=e^{\mathrm{i}t\triangle}u_0-\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds\\&=e^{\mathrm{i}t\triangle}u_0\underbrace{-\mathrm{i}\int_0^{t-\tau}e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{w(t,\tau)}\underbrace{-\mathrm{i}\int_{t-\tau}^{1}e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{z(t,\tau)}\\&=e^{\mathrm{i}t\triangle}u_0+w(t,\tau)+z(t,\tau).\end{aligned}$$

Now we consider each of the last three terms.

Claim 5.9. We have

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\xrightarrow{t\to+\infty}0. \tag{5.18}$$

Proof. Indeed, if $u_0\in L^{\frac{p+1}{p}}$, then

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\le Ct^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u_0\|_{L^{\frac{p+1}{p}}}\xrightarrow{t\to+\infty}0.$$

The general case follows from the special one using the fact that $H^1\cap L^{\frac{p+1}{p}}$ is dense in H^1. □

Claim 5.10. There is a constant C independent from t and τ s.t.

$$\|w(t,\tau)\|_{L^{p+1}}\le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}}. \tag{5.19}$$

Remark 5.11. The exponent is strictly negative. Indeed, for $p-1 \le 1$ we have

$$0 < d(p-1) - 2\max(1, p-1) = d(p-1) - 2 \Longleftrightarrow p > 1 + \frac{2}{d}.$$

If $p-1 \ge 1$

$$0 < d(p-1) - 2\max(1, p-1) = (d-2)(p-1) \Longleftrightarrow d \ge 3.$$

Proof. We define

$$q = \begin{cases} \infty \text{ if } p \ge 2 \\ \frac{2}{2-p} \text{ if } p < 2. \end{cases}$$

Then we have, for a dimensional constant C,

$$\|w(t,\tau)\|_{L^q} \le C \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^p_{L^{pq'}} ds.$$

Here we claim

$$d\left(\frac{1}{2} - \frac{1}{q}\right) > 1. \tag{5.20}$$

This is obvious by $d \ge 3$ if $q = \infty$. Otherwise, for $p < 2$

$$d\left(\frac{1}{2} - \frac{1}{q}\right) = d\left(\frac{1}{2} - \frac{2-p}{2}\right) = \frac{d}{2}(p-1) > 1 \Longleftrightarrow p > 1 + \frac{2}{d},$$

where the last inequality follows from $p > 1 + \frac{4}{d}$. So we have, for a dimensional constant C

$$\|w(t,\tau)\|_{L^q} \le C\tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1} \sup_s \|u(s)\|^p_{L^{pq'}}. \tag{5.21}$$

We claim now that $2 \le pq' \le p+1$. Indeed, for $p \ge 2$ we have $q' = 1$ and the claim holds. If $p < 2$ then

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{2-p}{2} = \frac{p}{2}$$

so that $pq' = 2$. So in all cases we have $H^1 \hookrightarrow L^{pq'}$ and we can uniformly bound the last factor on the right in (5.21).

Next, we claim $\|w(t,\tau)\|_{L^2} \le 2\|u_0\|_{L^2}$, which follows from

$$w(t,\tau) = -\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds = e^{\mathrm{i}\tau\triangle}\left(-\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-\tau-s)\triangle}|u(s)|^{p-1}u(s)ds\right)$$
$$= e^{\mathrm{i}\tau\triangle}\left(u(t-\tau) - e^{\mathrm{i}(t-\tau)\triangle}u_0\right) = e^{\mathrm{i}\tau\triangle}u(t-\tau) - e^{\mathrm{i}t\triangle}u_0.$$

Finally, we claim $p+1 \leq q$. This is obviously the case if $q=\infty$. Otherwise $p<2$, and then

$$q>p+1 \Longleftrightarrow \frac{2}{2-p}>p+1 \Longleftrightarrow 2>(p+1)(2-p)=2+p-p^2$$

where the last inequality follows from $p>1$ and so from $p-p^2<0$. Finally by Hölder inequality

$$\|w(t,\tau)\|_{L^{p+1}} \leq \|w(t,\tau)\|_{L^2}^{1-\alpha}\|w(t,\tau)\|_{L^q}^{\alpha} \text{ where } \frac{1}{p+1}=\frac{1-\alpha}{2}+\frac{\alpha}{q}.$$

Notice that $\alpha=\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$. So

$$\|w(t,\tau)\|_{L^{p+1}} \leq C\tau^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)+\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}. \tag{5.22}$$

We now examine the exponent in (5.22). If $q=\infty$ the exponent equals

$$-(d-2)\left(\frac{1}{2}-\frac{1}{p+1}\right)=-\frac{d(p-1)-2(p-1)}{2(p+1)}=-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}.$$

In the case $q<\infty$, then

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(-d+\frac{1}{\frac{1}{2}-\frac{1}{q}}\right)=-\frac{p-1}{2(p+1)}\left(d-\frac{2}{p-1}\right)\\&=-\frac{d(p-1)-2}{2(p+1)}=-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}.\end{aligned}$$

So we have proved that the exponent in (5.22) is exactly the one in (5.19), which is then proved.

□

We now consider

$$z(t,\tau)=-\mathrm{i}\int_{t-\tau}^{t} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds.$$

We have

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \int_{t-\tau}^{t}(t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u\|_{L^{p+1}}^{p}ds. \tag{5.23}$$

Notice that $p<d^*$, that is $p+1<\frac{2d}{d-2}$ is equivalent to $d\left(\frac{1}{2}-\frac{1}{p+1}\right)<1$. Indeed,

$$\frac{1}{p+1}>\frac{d-1}{2d}=\frac{1}{2}-\frac{1}{d}.$$

We now pick $q \in \left(1, \frac{2(p+1)}{d(p-1)}\right)$. Notice that this implies $qd\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$. Then

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \left(\int_{t-\tau}^{t} (t-s)^{-dq\left(\frac{1}{2}-\frac{1}{p+1}\right)} ds\right)^{1/q} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{pq'} ds\right)^{\frac{1}{q'}}$$
$$= C\tau^{\alpha} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{pq'} ds\right)^{\frac{1}{q'}}$$

for some $\alpha > 0$. Now we claim $q'p > p+1$ or, equivalently, $\frac{1}{q'} < \frac{p}{p+1}$ Indeed

$$\frac{1}{q} > \frac{d}{2} - \frac{d}{p+1} \iff \frac{1}{q'} = 1 - \frac{1}{q} < 1 - \frac{d}{2} + \frac{d}{p+1} \iff \frac{1}{q'} < \frac{2-d}{2} + \frac{d}{p+1}$$
$$= \frac{2(p+1) - (p+1)d + 2d}{2(p+1)} = \frac{p}{p+1} + \frac{2-(p+1)d+2d}{2(p+1)} < \frac{p}{p+1},$$

where the last inequality holds because

$$2 - (p+1)d + 2d = 2 - pd + d < 0 \iff p > 1 + \frac{2}{d},$$

with the latter true because, in our case, $p > 1 + \frac{4}{d}$.
From $q'p > p+1 (> 2)$ and $p < d^*$ it follows that,

$$\|u\|_{L^{p+1}}^{pq'} = \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^{p+1}}^{pq'-p-1} \leq \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^2}^{(pq'-p-1)\beta} \|u\|_{L^{d^*+1}}^{(pq'-p-1)(1-\beta)} \text{ for } \frac{1}{p+1} = \frac{\beta}{2} + \frac{1-\beta}{d^*+1}.$$

So, by the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{d^*+1}(\mathbb{R}^d)$, we conclude for the solutions of our equation

$$\|u\|_{L^{p+1}}^{pq'} \leq C\|u\|_{L^{p+1}}^{p+1} \|u_0\|_{L^2}^{(pq'-p-1)\beta} (2E(u_0))^{\frac{(pq'-p-1)(1-\beta)}{2}}$$

for a dimensional constant C, related to Sobolev embedding. So, for a constant C which depends on the dimensione and u_0, we have

$$\begin{aligned}
&\|z(t,\tau)\|_{L^{p+1}} \leq C\tau^{\alpha} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{p+1} ds\right)^{\frac{1}{q'}} \\
&= C\tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x|\geq s\log s} |u|^{p+1} dx + \int_{t-\tau}^{t} ds \int_{|x|\leq s\log s} |u|^{p+1} dx\right)^{\frac{1}{q'}} \\
&\leq 2^{\frac{1}{q'}} C\tau^{\delta+\frac{1}{q'}} \left(\sup_{s\in[t-\tau,t]} \|u(s)\|_{L^{p+1}(|x|\geq s\log s)}^{p+1}\right)^{\frac{1}{q'}} + 2^{\frac{1}{q'}} C\tau^{\delta} \left(\int_{t-\tau}^{t} ds \int_{|x|\leq s\log s} |u|^{p+1} dx\right)^{\frac{1}{q'}}.
\end{aligned} \tag{5.24}$$

Let us take now $\tau > b$ such that

$$\|w(t,\tau)\|_{L^{p+1}} \le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} < \frac{\epsilon}{4}. \tag{5.25}$$

Next, using Lemma 5.6 and Claim 5.9 let us take $t_1 > \max(a,b)$ such that for $t \ge t_1$

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} + 2^{\frac{1}{q'}}C\tau^{\delta+\frac{1}{q'}}\left(\sup_{s\in[t-\tau,t]}\|u(s)\|^{p+1}_{L^{p+1}(|x|\ge s\log s)}\right)^{\frac{1}{q'}} < \frac{\epsilon}{4}. \tag{5.26}$$

Using Lemma 5.7 there exists $t_2 > t_1 + 2\tau$ such that for $t \in [t_2, t_2 - \tau]$

$$2^{\frac{1}{q'}}C\tau^{\delta}\left(\int_{t-\tau}^{t} ds\int_{|x|\le s\log s}|u|^{p+1}dx\right)^{\frac{1}{q'}} \le 2^{\frac{1}{q'}}C\tau^{\delta}\left(\int_{t_2-2\tau}^{t_2} ds\int_{|x|\le s\log s}|u|^{p+1}dx\right)^{\frac{1}{q'}} < \frac{\epsilon}{4}. \tag{5.27}$$

If we consider now the ϵ, a, b in the statement, we can take $t_0 = t_2$ large enough so that $t_0 > \max(a,b)$ and take $\tau > b$ obtaining (5.17).

□

We now move to complete the proof of Theorem 4.3.
Let us fix $\epsilon > 0$. Pick $t > \tau > 0$. Then, in view of $u(t) = e^{\mathrm{i}t\triangle}u_0 + w(t,\tau) + z(t,\tau)$, we have that by Claims 5.9–5.10 there exists $t_1 \ge 0$ and τ_ϵ with

$$\|u(t)\|_{L^{p+1}} \le \|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} + C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} + \|z(t,\tau_\epsilon)\|_{L^{p+1}} < \frac{\epsilon}{2} + \|z(t,\tau_\epsilon)\|_{L^{p+1}},$$

where we chose $\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} < \frac{\epsilon}{4}$ for $t > t_1$ and

$$C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} = \frac{\epsilon}{4}, \tag{5.28}$$

where C is a dimensional constant. In turn by (5.23)

$$\|z(t,\tau_\epsilon)\|_{L^{p+1}} \lesssim \int_{t-\tau_\epsilon}^{t}(t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u\|^p_{L^{p+1}}ds \le C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}}\sup_{s\in[t-\tau_\epsilon,t]}\|u(s)\|^p_{L^{p+1}}.$$

From Lemma 5.8 we know that there exists $t_0 > \max(t_1,\tau_\epsilon)$ s.t.

$$\sup_{s\in[t_0-\tau_\epsilon,t_0]}\|u(s)\|_{L^{p+1}} \le \frac{\epsilon}{4}. \tag{5.29}$$

Consider now

$$t_\epsilon = \sup\{t \ge t_0 : \sup_{s\in[t-\tau_\epsilon,t]}\|u(s)\|_{L^{p+1}} \le \epsilon \text{ for all } t \in [t_0,t]\},$$

where (5.29) guarantees that the set on the right hand side contains at least t_0 and in fact,by the continuity in t of the function $t \to \sup_{s\in[t-\tau_\epsilon,t]} \|u(s)\|_{L^{p+1}}$, a whole interval.
If $t_\epsilon = +\infty$ we will have proved the desired result, because in particular this guarantees that $\|u(s)\|_{L^{p+1}} \leq \epsilon$ for all the $t \geq t_0$ and, since here $\epsilon > 0$ is arbitrarily small.

So, let us suppose that $t_\epsilon < \infty$. Then, by $u \in C^0(\mathbb{R}, H^1)$, we have $\|u(t_\epsilon)\|_{L^{p+1}} = \epsilon$. Then we have

$$\epsilon < \frac{\epsilon}{2} + \|z(t_\epsilon, \tau_\epsilon)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t_\epsilon-\tau_\epsilon,t_\epsilon]} \|u(s)\|_{L^{p+1}}^p,$$

so that we conclude

$$\epsilon < \frac{\epsilon}{2} + \left(C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1}\right)\epsilon.$$

We now need to check that it is possible to choose τ_ϵ such that both

$$C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} < \frac{1}{2} \tag{5.30}$$

and (5.28) are true. This will lead to a contradiction. Suppose that for τ_ϵ which satisfies (5.28) inequality (5.30) is false. This implies

$$\frac{1}{2C} \leq \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} = C_1 4^{p-1} \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)} - \frac{d(p-1)^2 - 2(p-1)\max(1,p-1)}{2(p+1)}}, \tag{5.31}$$

where we substituted ϵ^{p-1} using the equality (5.28). We will show now that the exponent of τ_ϵ is negative, so that taking $\tau_\epsilon \gg 1$ formula (5.31) leads to a contradiction. Taking a unique fraction in the exponent and focusing on the numerator, we have

$$\begin{aligned} &2(p+1) - d(p-1) - d(p-1)^2 + 2(p-1)\max(1,p-1) \\ &= (p-1)\left(2\max(1,p-1) - d - d(p-1)\right) + 2(p+1) \\ &= (p-1)\left(2\max(1,p-1) + 2 - d(p-1)\right) - d(p-1) - 2(p-1) + 2(p+1) \\ &= (p-1)\left(2\max(1,p-1) + 2 - d(p-1)\right) - d(p-1) + 4. \end{aligned} \tag{5.32}$$

For $p - 1 \leq 1$ the quantity in line (5.32) becomes

$$(p-1)\left(4 - d(p-1)\right) - d(p-1) + 4 = p\left(4 - d(p-1)\right) < 0$$

by $p > 1 + 4/d$ and this completes the proof for $p - 1 \leq 1$.
For $p - 1 > 1$ the quantity in line (5.32) becomes

$$\begin{aligned} &(p-1)\left(2(p-1) + 2 - d(p-1)\right) - d(p-1) + 4 \\ &= (p-1)\left(2 - (d-2)(p-1)\right) - d(p-1) + 4. \end{aligned}$$

For $d \geq 4$

$$\begin{aligned}&(p-1)\left(2-(d-2)(p-1)\right)-d(p-1)+4\\&\leq (p-1)\left(2-2(p-1)\right)-4(p-1)+4=-2(p-1)p-4(p-2)<0.\end{aligned}$$

Finally, for $d=3$ and $p-1>1$ the quantity in line (5.32) becomes, for $\alpha = p-1$,

$$\begin{aligned}&(p-1)\left(2(p-1)+2-3(p-1)\right)-3(p-1)+4\\&=-\alpha^2-\alpha+4=:-q(\alpha).\end{aligned}$$

Now, $q(\alpha)=0$ for $\alpha_{\pm}=-1/2\pm\frac{\sqrt{17}}{2}$. This means that $q(\alpha)<0$ for $p-1>\frac{\sqrt{17}-1}{2}$. The completion of the proof of Theorem 4.3 for the remaining cases, that is $d=3$ and $2<p\leq\frac{\sqrt{17}+1}{2}$ is not in [4].

□

A Appendix. On the Bochner integral

For this part see [3]. Let X be a Banach space.

Definition A.1 (Strong measurability)**.** Let I be an interval. A function $f : I \to X$ is strongly measurable if there exists a set E of measure 0 and a sequence $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$.

Remark A.2. Notice that when $\dim X < \infty$ a function is measurable (in the sense that $f^{-1}(B)$ is measurable for any Borel set B) if an only if it is strongly measurable in the above sense. Indeed if f is strongly measurable in the above sense then as a point wise limit of measurable functions f is measurable, see Theorem 1.14 p. 14 Rudin [**?**]. Viceversa if f is measurable, then f is strongly measurable in the above sense, see the Corollary to Lusin's Theorem in Rudin [**?**] p. 54.

Example A.3. Consider $\{x_j\}_{j=1}^n$ in X and $\{A_j\}_{j=1}^n$ measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \neq k$. Then we claim that the *simple* function

$$f(t) := \sum_{j=1}^{n} x_j \chi_{A_j}(t) : I \to X \tag{A.1}$$

is measurable. Indeed, see Rudin [**?**] p. 54, there are sequences $\{\varphi_{j,k}\}_{k\in\mathbb{N}}$ in $C^0_c(I,\mathbb{R})$ with $\varphi_{j,k}(t) \overset{k\to\infty}{\to} \chi_{A_j}(t)$ a.e. and hence

$$C^0_c(I,\mathbb{R}) \ni f_k(t) := \sum_{j=1}^{n} x_j \varphi_{j,k}(t) \overset{k\to\infty}{\to} f(t) \text{ a.e. in } I.$$

Proposition A.4. *If (f_n) is a sequence of strongly measurable functions from I to X convergent a.e. to a $f : I \to X$, then f is strongly measurable.*

Proof. There is an E with $|E| = 0$ s.t. $f_n(t) \overset{n\to\infty}{\to} f(t)$ for any $t \in I\backslash E$. Consider for any n a sequence $C_c(I,X) \ni f_{n,k} \overset{k\to\infty}{\to} f_n$ a.e. We will suppose first that $|I| < \infty$. By applying Egorov Theorem to $\{\|f_{n,k} - f_n\|\}_{k\in\mathbb{N}}$ there is $E_n \subset I$ with $|E_n| \le 2^{-n}$ s.t. $\|f_{n,k} - f_n\| \overset{k\to\infty}{\to} 0$ uniformly in $I\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I\backslash E_n$ and set $g_n = f_{n,k(n)}$. Set $F := E \bigcup (\bigcap_m \bigcup_{n>m} E_n)$. Then $|F| = 0$. Indeed for any m

$$|F| \le |E| + \sum_{n=m}^{\infty} |E_n| \le |E| + \sum_{n=m}^{\infty} 2^{-n} \overset{m\to\infty}{\to} 0.$$

Let $t \in I\backslash F$. Since $t \notin E$ we have $f_n(t) \overset{n\to\infty}{\to} f(t)$. Furthermore, for n large enough we have $t \in I\backslash E_n$. Indeed

$$t \notin \bigcap_m \bigcup_{n>m} E_n \Rightarrow \exists\ m \text{ s.t. } t \notin \bigcup_{n>m} E_n \Rightarrow\ t \notin E_n\ \forall\ n > m.$$

Then $\|g_n(t) - f_n(t)\| < 1/n$ and $g_n(t) \overset{n\to\infty}{\to} f(t)$. So $f(t)$ is measurable in the case $|I| < \infty$. Now we consider the case $|I| = \infty$. We express $I = \cup_n I_n$ for an increasing sequence of intervals with $|I_n| < \infty$. Consider for any n a sequence $C_c(I_n, X) \ni f_{n,k} \overset{k\to\infty}{\to} f$ a.e. in I_n. Then by Egorov Theorem to $\|f_{n,k} - f_n\|$ there is $E_n \subset I_n$ with $|E_n| \le 2^{-n}$ s.t. $f_{n,k} \overset{k\to\infty}{\to} f_n$ uniformly in $I_n\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I_n\backslash E_n$ and set $g_n = f_{n,k(n)}$. Then defining F like above, the remainder of the proof works exactly like for the case $|I| < \infty$.

□

Example A.5. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.2}$$

is measurable. Indeed if we set $f_n(t) := \sum_{j=1}^{n} x_j \chi_{A_j}(t)$, then we have $\lim_{n\to\infty} f_n(t) = f(t)$ for any t, since if $t \notin \cup_{j=1}^{\infty} A_j$ both sides are 0, and if $t \in A_{n_0}$ then for $n \ge n_0$ we have $f_n(t) = x_{n_0} = f(t)$. Hence by Proposition A.4 the function f is measurable.
When the sum in (A.2) is finite then the function f is called *simple*.

Example A.6. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I where again $A_j \cap A_k = \emptyset$ for $j \ne k$ but we allow $|A_j| = \infty$. Then

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.3}$$

is measurable. To see this consider $f_n(t) = \chi_{[-n,n]}(t) f(t)$. Then

$$f_n(t) = \sum_{j=1}^{\infty} x_j \chi_{A_j \cap [-n,n]}(t)$$

and by Example A.5 we know that each $f_n(t)$ is strongly measurable. Since $f_n(t) \to f(t)$ for any $t \in I$ we conclude by Proposition A.4 that f is strongly measurable.

Another natural definition of measurability is the following one.

Definition A.7 (Weak measurability)**.** Let I be an interval. A function $f : I \to X$ is weakly measurable if for any $x' \in X'$ the function $t \to \langle x', f(t)\rangle_{X'X}$ is a measurable function $I \to \mathbb{R}$.

Obviously, strongly measurable implies weakly measurable. Let us explore the viceversa.

Definition A.8. Let I be an interval. A function $f : I \to X$ is *almost separably valuable* if there exists a 0 measure set $N \subset I$ s.t. $f(I \backslash N)$ is separable.

The following lemma shows that strongly measurable functions are almost separably valuable.

Lemma A.9. *If $f : I \to X$ is strongly measurable with $(f_n(t))$ a sequence in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$ for a 0 measure set $E \subset I$ then $f(I \backslash E)$ is separable and there exists a separable Banach subspace $Y \subseteq X$ with $f(I \backslash E) \subseteq Y$.*

Proof. First of all $f_n(I \cap \mathbb{Q})$ is a countable dense set in $f_n(I)$. So $f_n(I)$ is separable. In a separable metric space any subspace is separable. So $f_n(I \backslash E)$ is separable. The closed vector space Y generated by $\cup_n f_n(I \backslash E)$ is separable. Indeed let $C \subseteq \cup_n f_n(I \backslash E)$ be a countable set dense in $\cup_n f_n(I \backslash E)$. Let $\text{Span}_{\mathbb{Q}}(C)$ be the vector space on $\mathbb{Q}$ generated by C. Then $\text{Span}_{\mathbb{Q}}(C)$ is dense in Y. For $C = \{x_1, x_2, ...\}$ we have $\text{Span}_{\mathbb{Q}}(C) = \cup_{n=1}^{\infty} \text{Span}_{\mathbb{Q}}(\{x_1, ..., x_n\})$. This proves that $\text{Span}_{\mathbb{Q}}(C)$ is countable and that Y is separable. □

Example A.10. Let X be a Hilbert space with an orthonormal basis $\{e_t\}_{t \in \mathbb{R}}$. Then the map $f : \mathbb{R} \to X$ given by $f(t) = e_t$ is not strongly measurable. This follows from the fact that it is not almost separably valuable.
On the other hand if $x \in X$ then $t \to \langle f(t), x\rangle$ is different from 0 only on a countable subset of $\mathbb{R}$, and as such it is measurable. Hence f is weakly measurable.
Notice however that if $C \subset [0, 1]$ is the standard Cantor set (which has 0 measure and has same cardinality of $\mathbb{R}$) and if $\{\widetilde{e}_t\}_{t \in C}$ is another basis of X, then the map

$$g(t) = \begin{cases} \widetilde{e}_t \text{ for } t \in C \text{ and} \\ \quad 0 \text{ otherwise} \end{cases}$$

is weakly measurable (like f and for the same reasons) and is almost separably valuable. Pettis Theorem, which we prove below, implies that $g : \mathbb{R} \to X$ is strongly measurable.

The following lemma will be used for Pettis Theorem.

Lemma A.11. *Let X be a separable Banach space and let S' be the unit ball of the dual X'. Then X' is separable for the weak topology $\sigma(X', X)$, see Brezis [2] p.62, that is there exists a sequence $\{x'_n\}$ in S' s.t. for any $x' \in S'$ there exists a subsequence $\{x'_{n_k}\}$ s.t. for any $x \in X$ we have $\lim_{k \to \infty} \langle x'_{n_k}, x\rangle_{X'X} = \langle x', x\rangle_{X'X}$.*

Proof. Let $\{x_n\}$ be dense in X. For any n consider

$$F_n : S' \to \mathbb{R}^n \text{ defined by } F_n(x') := (\langle x', x_1\rangle_{X'X}, ..., \langle x', x_n\rangle_{X'X}).$$

Since $\mathbb{R}^n$ is separable, and so is $F_n(S')$, there exists a sequence $\{x'_{n,k}\}_k$ s.t. $\{F_n(x'_{n,k})\}_k$ is dense in $F_n(S')$. Obviously $\{x'_{n,k}\}_{n,k}$ can be put into a sequence. For any $x' \in S'$ for any n there is a k_n s.t. $|\langle x' - x'_{n,k_n}, x_i\rangle_{X'X}| < 1/n$ for all $i \leq n$. This implies that for any fixed i we have $\lim_{n\to\infty} \langle x'_{n,k_n}, x_i\rangle_{X'X} = \langle x', x_i\rangle_{X'X}$. By density this holds for any $x \in X$. □

Proposition A.12 (Pettis's Theorem)**.** *Consider $f : I \to X$. Then f is strongly measurable if and only if it is weakly measurable and almost separable valuable.*

Proof. The necessity has been already proved, so we focus on the sufficiency only. By modifying f we can assume that $f(I)$ is separable. By replacing X by a smaller space, we can assume that X is separable.
Fix now $x \in X$. Then we claim that $t \to \|f(t) - x\|$ is measurable. Indeed for any $a > 0$

$$\{t \in I : \|f(t) - x\| \leq a\} = \cap_{x' \in S'}\{t \in I : |\langle x', f(t) - x\rangle_{X'X}| \leq a\}.\}$$

Using the fact that S' is separable in the weak topology $\sigma(X', X)$ and the notation in Lemma A.11, we have

$$\{t \in I : \|f(t) - x\| \leq a\} = \cap_{n\in\mathbb{N}}\{t \in I : |\langle x'_n, f(t) - x\rangle_{X'X}| \leq a\}.$$

Since the set in the r.h.s. is measurable, we conclude that $t \to \|f(t) - x\|$ is measurable and so our claim is correct.
Consider now $n \geq 1$. Since $f(I)$ is separable there is a sequence of balls $\{B(x_j, \frac{1}{n})\}_{j\geq 0}$ whose union contains $f(I)$. Set now

$$\begin{cases} \omega_0^{(n)} := \{t : f(t) \in B(x_0, \frac{1}{n})\}\,, \\ \omega_j^{(n)} := \{t : f(t) \in B(x_j, \frac{1}{n})\} \backslash \cup_{k<j}\, \omega_k^{(n)} \end{cases}$$

and

$$f_n(t) := \sum_{j=0}^{\infty} x_j \chi_{\omega_j^{(n)}}(t).$$

Notice that $\cup_{j\geq 0}\omega_j^{(n)} = I$ and they are pairwise disjoint and measurable. By Example A.6 we know that $f_n : I \to X$ is strongly measurable. Furthermore, for any $t \in I$ there is a j s.t. $t \in \omega_j^{(n)}$ and this implies

$$\frac{1}{n} > \|f(t) - x_j\| = \|f(t) - f_n(t)\|.$$

In other words, $\|f(t) - f_n(t)\| \leq 1/n$ for any $t \in I$. Then $f_n(t) \to f(t)$ for any t, and so by Proposition A.4 the function $f : I \to X$ is strongly measurable. □

Example A.13. Consider the map $f : (0,1) \to L^\infty(0,1)$ defined by $t \overset{f}{\to} \chi_{(0,t)}$. This map is not almost separable valued. Indeed $t \neq s$ implies $\|f(t) - f(s)\|_\infty = 1$. If f were almost separable valued then there would exist a 0 measure subset E in $(0,1)$ and a countable set $\mathcal{N} = \{t_n\}_n$ in $(0,1)\backslash E$ such that for any $t \in (0,1)\backslash(E \cup \mathcal{N})$ there would exist a subsequence n_k with $f(t_{n_k}) \overset{k\to\infty}{\to} f(t)$ in $L^\infty(0,1)$. But this is impossible since $\|f(t) - f(t_{n_k})\|_\infty = 1$.
On the other hand $f : (0,1) \to L^2(0,1)$ defined in the same way, is strongly measurable. First, since $L^2(0,1)$ is separable, it is almost separable valued. Next for any given $w \in L^2(0,1)$ we have

$$\langle f(t), w\rangle_{L^2(0,1)} = \int_0^t w(x)dx$$

which is a continuous, and hence measurable, function. So f is also weakly measurable and hence it is strongly measurable by Pettis Theorem.

Recall that in Remark A.2 we mentioned another possible notion of measurability, that is that $f : I \to X$ could be defined as measurable if $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$. We have the following fact.

Proposition A.14. *Consider $f : I \to X$. Then f is strongly measurable if and only if f is almost separably valuable and $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$.*

Proof. The "$\Leftarrow$" follows from the fact that for any $\mathfrak{a}$ open subset of $\mathbb{R}$ and for any $x' \in X$ the set $A = \{x \in X : \langle x, x'\rangle_{X,X'} \in \mathfrak{a}\}$ is open and for $g(t) := \langle f(t), x'\rangle_{X,X'}$ we have $f^{-1}(A) = g^{-1}(\mathfrak{a})$. So the latter being measurable it follows that g is measurable and hence f is weakly measurable. Hence by Pettis Theorem we conclude that f is strongly measurable.
We now assume that f is strongly measurable. We know from Lemma A.9 that f is almost separably valuable. Let U be an open subset of X. Let $(f_n)_n$ be a sequence in $C^0_c(I, X)$ with $f_n(t) \overset{n\to\infty}{\to} f(t)$ a.e. outside a 0 measure set $E \subset I$. Let $U_r = \{x \in X : \text{dist}(x, U^c) > r\}$. Then

$$f^{-1}(U)\backslash E = (\cup_{m\geq 1} \cup_{n\geq 1} \cap_{k\geq n} f_k^{-1}(U_{\frac{1}{m}}))\backslash E. \tag{A.4}$$

To check this, notice that if t belongs to the left hand side , then $f(t) \in U_{\frac{1}{m_0}}$ for some m_0 and, since $f_n(t) \overset{n\to\infty}{\to} f(t)$, for n large we have $f_k(t) \in U_{\frac{1}{m_1}}$ if $k \geq n$ for $m_1 > m_0$ preassigned. Viceversa if t belongs to the right hand side, then there exist n and m s.t. $f_k(t) \in U_{\frac{1}{m}}$ for all $k \geq n$. Then by $f_k(t) \overset{k\to\infty}{\to} f(t)$ it follows that $f(t) \in \overline{U_{\frac{1}{m}}}$ with the latter a subset of U. This proves (A.4). Since the r.h.s. is a measurable set, this completes the proof. □

Definition A.15 (Bochner integrability)**.** A strongly measurable function $f : I \to X$ is Bochner–integrable if there exists a sequence $(f_n(t))$ in $C_c(I, X)$ s.t.

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X dt = 0. \tag{A.5}$$

Notice that $\|f_n(t) - f(t)\|_X$ is measurable.

Example A.16. Consider the situation of Example A.13 of a Hilbert space X with an orthonormal basis $\{e_t\}_{t\in\mathbb{R}}$ and the map $f : \mathbb{R} \to X$, which we saw is not strongly measurable and hence is not Bochner–integrable. Notice that f is Riemann integrable in any compact interval $[a, b]$ with $\int_a^b f(t)dt = 0$.
To see this recall that the Riemann integral is, if it exists, the limit

$$\int_a^b f(t)dt = \lim_{|\triangle|\to 0} \sum_{I_j\in\triangle} f(t_j)|I_j| \text{ with } t_j \in I_j \text{ arbitrary}$$

where $\triangle$ varies among all possible decompositions of $[a, b]$ and $|\triangle| = \max_{I\in\triangle} |I|$. We have

$$\| \sum_{I_j\in\triangle} e_{t_j} |I_j| \|^2 = \sum_{j,k} \langle e_{t_j}, e_{t_k}\rangle |I_j||I_k| \le 2 \sum_j |I_j||\triangle| = 2|\triangle|(b - a) \overset{|\triangle|\to 0}{\to} 0.$$

Proposition A.17. *Let $f : I \to X$ be Bochner–integrable. Then there exists an $x \in X$ s.t. if $(f_n(t))$ is a sequence in $C_c(I, X)$ satisfying* (A.5) *then we have*

$$\lim_{n\to\infty} x_n = x \text{ where } x_n := \int_I f_n(t)dt. \tag{A.6}$$

Proof. First of all we check that x_n is Cauchy. This follows immediately from (A.5) and from

$$\|x_n - x_m\|_X = \| \int_I (f_n(t) - f_m(t))dt\|_X \le \int_I \|f_n(t) - f_m(t))\|_X dt$$
$$\le \int_I \|f_n(t) - f(t))\|_X dt + \int_I \|f(t) - f_m(t))\|_X dt.$$

Let us set $x = \lim x_n$. Let $(g_n(t))$ be another sequence in $C_c(I, X)$ satisfying (A.5). Then $\lim \int_I g_n = x$ by

$$\| \int_I g_n(t)dt - x\|_X = \| \int_I (g_n(t) - f_n(t))dt + \int_I f_n(t)dt - x\|_X$$
$$\le \int_I \|g_n(t) - f_n(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt$$
$$\le \int_I \|g_n(t) - f(t)\|_X dt + \int_I \|f_n(t) - f(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt.$$

□

Definition A.18. Let $f : I \to X$ be Bochner–integrable and let $x \in X$ be the corresponding element obtained from Proposition A.17. The we set $\int_I f(t)dt = x$.

Theorem A.19 (Bochner's Theorem)**.** *Let $f : I \to X$ be strongly measurable. Then f is Bochner–integrable if and only if $\|f\|$ is Lebesgue integrable. Furthermore, we have*

$$\| \int_I f(t)dt\| \le \int_I \|f(t)\|dt. \tag{A.7}$$

Proof. Let f be Bochner–integrable. Then there is a sequence $(f_n(t))$ in $C_c(I,X)$ satisfying (A.5). We have $\|f\| \le \|f_n\| + \|f - f_n\|$. Since both functions in the r.h.s. are Lebesgue integrable and $\|f\|$ is measurable it follows that$\|f\|$ is Lebesgue integrable.

Conversely let $\|f\|$ be Lebesgue integrable. Then there exist a sequence $(g_n(t))$ in $C_c(I,\mathbb{R})$ and $g \in L^1(I)$ s.t. $\int_I |g_n(t) - \|f(t)\||dt \to 0$ and $|g_n(t)| \le g(t)$. In fact it is possible to choose such a sequence so that $\|g_n - g_m\|_{L^1(I)} < 2^{-n}$ for any n and any $m \ge n$ (just by extracting an appropriate subsequence from a starting g_n [3]). Then if we set

$$S_N(t) := \sum_{n=1}^{N} |g_n(t) - g_{n+1}(t)| \tag{A.8}$$

we have $\|S_N\|_{L^1(I)} \le 1$. Since $\{S_N(t)\}_{N\in\mathbb{N}}$ is increasing, the limit $S(t) := \lim_{n\to+\infty} S_n(t)$ remains defined, is finite a.e. and $\|S\|_{L^1(I)} \le 1$. Then $|g_n(t)| \le |g_1(t)| + S(t) =: g(t)$ everywhere, where $g \in L^1(I)$. Notice that $\lim_{n\to\infty} g_n(t)$ is convergent almost everywhere (it convergent in all points where $\lim_{n\to+\infty} S_n(t)$ is convergent). By dominated convergence it follows that this limit holds also in $L^1(I)$ and hence it is equal to $\|f\|$.
Let $(f_n(t))$ in $C_c(I,X)$ s.t. $f_n(t) \to f(t)$ a.e. (this sequence exists by the strong measurability of $f(t)$). Set

$$u_n(t) := \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t).$$

Notice that $(u_n(t))$ is in $C_c(I,X)$. We have

$$\|u_n(t)\| \le \frac{|g_n(t)|\ \|f_n(t)\|}{\|f_n(t)\| + \frac{1}{n}} \le |g_n(t)| \le g(t).$$

We have (where the 2nd equality holds because because $\lim_{n\to\infty} g_n(t) = \|f(t)\|$ and $\lim_{n\to\infty} \|f_n(t)\| = \|f(t)\|$ a.e.)

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t) = \lim_{n\to\infty} f_n(t) = f(t) \text{ a.e.}.$$

Then we have

$$\lim_{n\to\infty} \|u_n(t) - f(t)\| = 0 \text{ a.e. with } \|u_n(t) - f(t)\| \le g(t) + \|f(t)\| \in L^1(I).$$

By dominated convergence we conclude

$$\lim_{n\to\infty} \int_I \|u_n(t) - f(t)\| dt = 0.$$

[3]Suppose we start with a given $\{g_n\}$. Then for any 2^{-n} there exists N_n s.t. $n_1, n_2 > N_n$ implies $\|g_{n_1} - g_{n_2}\|_{L^1(I)} < 2^{-n}$. Let now $\{\varphi(n)\}$ be a strictly increasing sequence in $\mathbb{N}$ s.t. $\varphi(n) > N_n$ for any n. Then $\|g_{\varphi(n)} - g_{\varphi(m)}\|_{L^1(I)} < 2^{-n}$ for any pair $m > n$. Rename $g_{\varphi(n)}$ as g_n.

This implies that f is Bochner–integrable. Finally, we have

$$\| \int_I f(t)dt \| = \lim_{n\to\infty} \| \int_I u_n(t)dt \| \leq \lim_{n\to\infty} \int_I \|u_n(t)\|dt = \int_I \|f(t)\|dt.$$

□

Corollary A.20 (Dominated Convergence)**.** *Consider a sequence $(f_n(t))$ of Bochner–integrable functions $I \to X$, $g : I \to \mathbb{R}$ Lebesgue integrable and let $f : I \to X$. Suppose that*

$$\|f_n(t)\| \leq g(t) \text{ for all } n$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t.$$

Then f is Bochner–integrable with $\int_I f(t) = \lim_n \int_I f_n(t)$.

Proof. By Dominated Convergence in $L^1(I, \mathbb{R})$ we have $\int_I \|f(t)\| = \lim_n \int_I \|f_n(t)\|$. By Proposition A.4, as a pointwise limit a.e. of a sequence of strongly measurable functions, f is strongly measurable. By Bochner's Theorem f is Bochner–integrable. By the triangular inequality

$$\limsup_n \| \int_I (f(t) - f_n(t)) \| \leq \lim_n \int_I \|f(t) - f_n(t)\| = 0$$

where the last inequality follows from $\|f(t) - f_n(t)\| \leq \|f(t)\| + g(t)$ and the standard Dominated Convergence. □

Definition A.21. Let $p \in [1, \infty]$. We denote by $L^p(I, X)$ the set of equivalence classes of strongly measurable functions $f : I \to X$ s.t. $\|f(t)\| \in L^p(I, \mathbb{R})$. We set $\|f\|_{L^p(I,X)} := \| \|f\| \|_{L^p(I,\mathbb{R})}$.

Proposition A.22. *$(L^p(I, X), \| \ \|_{L^p})$ is a Banach space.*

Proof. The proof is similar to the case $X = \mathbb{R}$, see [2].
(**Case** $p = \infty$). Let (f_n) be Cauchy sequence in $L^\infty(I, X)$. For any $k \geq 1$ there is a N_k s.t.

$$\|f_n - f_m\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n, m \geq N_k.$$

So there exists an $E_k \subset I$ with $|E_k| = 0$ s.t.

$$\|f_n(t) - f_m(t)\|_X \leq \frac{1}{k} \text{ for all } n, m \geq N_k \text{ and for all for } t \in I \backslash E_k.$$

Set $E := \cup_k E_k$. Then for any $t \in I \backslash E$ the sequence $(f_n(t))$ is convergent. So a function $f(t)$ remains defined with

$$\|f_n(t) - f(t)\|_X \leq \frac{1}{k} \text{ for all } n \geq N_k \text{ and for all for } t \in I \backslash E. \tag{A.9}$$

By Proposition A.4 the function f is strongly measurable. By (A.9) we have $f \in L^\infty(I,X)$ and

$$\|f_n - f\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n \geq N_k$$

and so $f_n \to f$ in $L^\infty(I,X)$.

(**Case** $p < \infty$). Let (f_n) be Cauchy sequence in $L^p(I,X)$ and let (f_{n_k}) be a subsequence with

$$\|f_{n_k} - f_{n_{k+1}}\|_{L^p(I,X)} \leq 2^{-k}.$$

Set now

$$g_l(t) = \sum_{k=1}^{l} \|f_{n_k}(t) - f_{n_{k+1}}(t)\|_X$$

Then

$$\|g_l\|_{L^p(I,\mathbb{R})} \leq 1.$$

By monotone convergence we have that $(g_l(t))_l$ converges a.e. to a $g \in L^p(I,\mathbb{R})$. Furthermore, for $2 \leq k < l$

$$\|f_{n_k}(t) - f_{n_l}(t)\|_X = \sum_{j=k}^{l-1} \|f_{n_j}(t) - f_{n_{j+1}}(t)\|_X \leq g(t) - g_{k-1}(t).$$

Then a.e. the sequence $(f_{n_k}(t))$ is Cauchy in X for a.e. t and so it converges for a.e. t to some $f(t)$. By Proposition A.4 the function f is strongly measurable. Furthermore,

$$\|f(t) - f_{n_k}(t)\|_X \leq g(t).$$

It follows that $f - f_{n_k} \in L^p(I,X)$, and so also $f \in L^p(I,X)$. Finally we claim $\|f - f_{n_k}\|_{L^p(I,X)} \to 0$. First of all we have $\|f(t) - f_{n_k}(t)\|_X \to 0$ for a.e. t and

$$\|f(t) - f_{n_k}(t)\|_X^p \leq g^p(x)$$

by dominated convergence we obtain that $\|f - f_{n_k}\|_X \to 0$ in $L^p(I,\mathbb{R})$. Hence $f_{n_k} \to f$ in $L^p(I,X)$. □

Proposition A.23. *$C_c^\infty(I,X)$ is a dense subspace of $L^p(I,X)$ for $p < \infty$.*

Proof. We split the proof in two parts. We first show that $C_c^0(I,X)$ is a dense subspace of $L^p(I,X)$ for $p < \infty$. For $p = 1$ this follows from the definition of integrable functions in Definition A.15. For $1 < p < \infty$ going through the proof of Bochner's Theorem A.19, the functions u_n considered in that proof can be taken to belong to $C_c^0(I,X)$ and converge to f in $L^p(I,X)$.

The second part of the proof consists in showing that $C_c^\infty(I,X)$ is a dense subspace of $C_c^0(I,X)$ inside $L^p(I,X)$ for $p < \infty$. Let $f \in C_c^0(I,X)$. We consider $\rho \in C_c^\infty(\mathbb{R},[0,1])$ s.t. $\int \rho(x)dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-1}\rho(x/\epsilon)$. Then for $\epsilon > 0$ small enough $\rho_\epsilon * f \in C_c^\infty(I,X)$. We

extend both f and $\rho_\epsilon * f$ on $\mathbb{R}$ setting them 0 in $\mathbb{R}\backslash I$. In this way $\rho_\epsilon * f \in C_c^\infty(\mathbb{R}, X)$ and $f \in C_c^0(\mathbb{R}, X)$ and it is enough to show that $\rho_\epsilon * f \overset{\epsilon\to 0^+}{\to} f$ in $L^p(\mathbb{R}, X)$..
We have

$$\rho_\epsilon * f(t) - f(t) = \int_{\mathbb{R}} (f(t-\epsilon s) - f(s))\rho(s)dy$$

so that, by Minkowski inequality and for $\Delta(s) := \|f(\cdot - s) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(t) - f(t)\|_{L^p} \le \int |\rho(s)|\Delta(\epsilon\ s)ds.$$

Now we have $\lim_{s\to 0}\Delta(s) = 0$ and $\Delta(s) \le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0} \|\rho_\epsilon * f - f\|_{L^p} = \lim_{\epsilon\searrow 0} \int |\rho(s)|\Delta(\epsilon\ s)ds = 0.$$

So

$$\lim_{\epsilon\searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}, X). \tag{A.10}$$

□

Proceeding as in the previous proof, we can prove the following.

Proposition A.24. *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}, X)$. Set*

$$T_h f(t) = h^{-1} \int_t^{t+h} f(s)ds \text{ for } t \in \mathbb{R} \text{ and } h \neq 0.$$

Then $T_h f \in L^p(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X) \cap C^0(\mathbb{R}, X)$ and $T_h f \overset{h\to 0}{\to} f$ in $L^p(\mathbb{R}, X)$ and for almost every t.

□

Proposition A.25. *Let $p \in [1, \infty]$ and $\{f_n\}$ a sequence in $L^p(I, X)$. Let $f : I \to X$ and suppose that $f_n(t) \rightharpoonup f(t)$ for almost any t in I. Then $f \in L^p(I, X)$ with*

$$\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}. \tag{A.11}$$

Proof. First of all we need to show that f is measurable. We know that there exists a zero measure set $F \subset I$ such that $f_n(t) \rightharpoonup f(t)$ for any $t \in I\backslash F$. Since for any $x' \in X'$ we have $\langle f_n(t), x'\rangle_{X\times X'} \xrightarrow{n\to\infty} \langle f(t), x'\rangle_{X\times X'}$ for any $t \in I\backslash F$, the map $\langle f(\cdot), x'\rangle_{X\times X'}$ is measurable, and so f is weakly measurable. For any n there exists a zero measure set $E_n \subset I$ such that $f(I\backslash E_n)$ is separable. Let $E = F \bigcup \cup_n E_n$ which obviously has 0 measure. Let now A be the convex hull of $\cup_n f_n(I\backslash E)$ and let $\overline{A}$ the weak closure of A, which is also the strong closure of A. Now, $\cup_n f_n(I\backslash E)$ is separable, A is separable and also $\overline{A}$ is separable. Since $f(I\backslash E) \subseteq \overline{A}$ we conclude that $f(I\backslash E)$ is separable and so by Theorem A.19 f is strongly measurable. Let now

$$g_n(t) := \inf_{k\ge n} \|f_k(t)\|_X \text{ and } g(t) := \lim_{n\to+\infty} g_n(t).$$

Then

$$g(t) = \liminf_{n\to+\infty} \|f_n(t)\|_X.$$

Since $g_n(t) \leq \|f_n(t)\|_X$ for any t, it follows that $g_n \in L^p(I)$ for any n and by monotone convergence

$$\|g\|_{L^p(I)} = \lim_{n\to+\infty} \|g_n\|_{L^p(I)} \leq \liminf_{n\to+\infty} \|f_n\|_{L^p(I,X)}.$$

By the lower semicontinuity of the norm $\|\cdot\|_X$, we have

$$\|f(t)\|_X \leq g(t) \text{ for a.a. } t \in I \text{ and so } \|f\|_{L^p(I,X)} \leq \|g\|_{L^p(I)}.$$

This yields (A.11).

□

Definition A.26. We denote by $\mathcal{D}'(I,X)$ the space $\mathcal{L}(\mathcal{D}(I,\mathbb{R}),X)$.

Corollary A.27. *Let $f \in L^1_{loc}(I,X)$ be such that $f=0$ in $\mathcal{D}'(I,X)$. Then $f=0$ a.e.*

Proof. First of all we have $\int_J f dt = 0$ for any $J \subset I$ compact. Indeed, let $(\varphi_n) \in \mathcal{D}(I)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n \to \chi_J$ a.e. Then

$$\int_J f dt = \lim_{n\to+\infty} \int_J \varphi_n f dt = 0$$

where we applied Dominated Convergence for the last equality.
Set now $\overline{f}(t) = f(t)$ in J and $\overline{f}(t) = 0$ outside J. Then $T_h\overline{f} = 0$ for all $h > 0$. Then $\overline{f}(t) = 0$ for a.e. t. So $f(t) = 0$ for a.e. $t \in J$. This implies $f(t) = 0$ for a.e. $t \in \mathbb{R}$. □

Corollary A.28. *Let $g \in L^1_{loc}(I,X)$, $t_0 \in I$, and $f \in C(I,X)$ given by $f(t) = \int_{t_0}^t g(s)ds$. Then:*

(1) $f' = g$ in $\mathcal{D}'(I,X)$;

(2) f is differentiable a.e. with $f' = g$ a.e.

Proof. It is not restrictive to consider the case $I = \mathbb{R}$ and $g \in L^1(\mathbb{R},X)$. We have

$$T_h g(t) = h^{-1} \int_t^{t+h} g(s)ds = \frac{f(t+h) - f(t)}{h}.$$

By Proposition A.24 $T_h g \overset{h\to 0}{\to} g$ for almost every t. This yields (2).
For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle f', \varphi \rangle = -\int_{\mathbb{R}} f(t)\varphi'(t)dt.$$

Furthermore
$$\lim_{h\to 0}\frac{\varphi(t+h)-\varphi(t)}{h}=\varphi'(t) \text{ in } L^\infty(\mathbb{R}).$$
So
$$\langle f',\varphi\rangle=-\lim_{h\to 0}\int_{\mathbb{R}} f(t)\frac{\varphi(t+h)-\varphi(t)}{h}dt=-\lim_{h\to 0}\int_{\mathbb{R}}\varphi(t)\frac{f(t-h)-f(t)}{h}dt$$
$$=-\lim_{h\to 0}\int_{\mathbb{R}}\varphi(t)T_{-h}g(t)dt=\langle g,\varphi\rangle.$$

□

Definition A.29. Let $p\in[1,\infty]$. We denote by $W^{1,p}(I,X)$ the space formed by the $f\in L^p(I,X)$ s.t. $f'\in\mathcal{D}(I,X)$ is also $f'\in L^p(I,X)$ and we set $\|f\|_{W^{1,p}}=\|f\|_{L^p}+\|f'\|_{L^p}$.

Lemma A.30. *Let $u,g\in L^1(I,X)$ be such that*
$$\langle u(t_2),f\rangle_{XX^*}-\langle u(t_1),f\rangle_{XX^*}=\int_{t_1}^{t_2}\langle g(s),f\rangle_{XX^*}\,ds \text{ for any } f\in X^*.$$
Then $\partial_t u=g$ in $\mathcal{D}'(I,X)$.

Proof. We immediately obtain $\langle u(t),f\rangle_{XX^*}\in AC(I)$ with derivative $\partial_t\langle u(t),f\rangle_{XX^*}=\langle g(t),f\rangle_{XX^*}$. For any $\varphi\in\mathcal{D}(I)$ and any $f\in X^*$
$$\left\langle -\int_I u(t)\varphi'(t)dt,f\right\rangle_{XX^*}=-\int_I\langle u(t),f\rangle_{XX^*}\varphi'(t)dt=\int_I\langle g(t),f\rangle_{XX^*}\varphi(t)dt=\left\langle\int_I g(t)\varphi(t)dt,f\right\rangle_{XX^*}$$
which yields
$$-\int_I u(t)\varphi'(t)dt=\int_I g(t)\varphi(t) \text{ for all } \varphi\in\mathcal{D}(I)$$
and so $\partial_t u=g$ in $\mathcal{D}'(I,X)$.

□

Theorem A.31. *Let $p\in[1,\infty]$ and $f\in L^p(I,X)$. The following are equivalent.*

i *$f\in W^{1,p}(I,X)$.*

ii *There exists $g\in L^p(I,X)$, $t_0\in I$ such that $f(t)=f(t_0)+\int_{t_0}^t g(s)ds$ for a.a. $t,t_0\in I$.*

iii *There exists $g\in L^p(I,X)$, $x_0\in X$ and $t_0\in I$ such that $f(t)=x_0+\int_{t_0}^t g(s)ds$ for a.a. $t\in I$.*

iv *$f\in AC(I,X)$, differentiable almost everywhere and $f'\in L^p(I,X)$.*

v *f is weakly absolutely continuous, weakly differentiable almost everywhere and $f'\in L^p(I,X)$.*

Proof. Assume (i). For $t, t_0 \in I$ we set

$$w(t) := f(t) - f(t_0) - \int_{t_0}^{t} f'(s)ds.$$

Then $w \in C^0(I, X)$ and $w' = 0$ in $\mathcal{D}'(I, X)$ by Corollary A.28. Therefore $w(t) \equiv x_0$ for some fixed $x_0 \in X$. But by continuity $w(0) = 0$ and so we get (ii). This immediately implies (iii). By modifying f in a zero measure set, we get (iv) and this implies (v). So now we want to show that $v \Rightarrow i$. Let g be the weak derivative of f, that is $g = f'$ in the sense that for any $y' \in X'$ we have $\langle f(\cdot), y' \rangle_{X \times X'} \in AC(I)$ with

$$\langle f(t), y' \rangle_{X \times X'} = \langle f(t_0), y' \rangle_{X \times X'} + \int_{t_0}^{t} \langle g(s), y' \rangle_{X \times X'} ds. \tag{A.12}$$

Since $g \in L^p(I, X)$, we can consider

$$w(t) := f(t) - f(t_0) - \int_{t_0}^{t} g(s)ds.$$

We claim that $w = 0$. Indeed, for any $y' \in X'$ we have from (A.12) that $\langle w(t), y' \rangle_{X \times X'} = 0$. Since this holds for any t, it follows $w \equiv 0$. Then

$$f(t) = f(t_0) + \int_{t_0}^{t} g(s)ds.$$

It follows from Corollary A.28 that $f' = g$ in $\mathcal{D}'(I, X)$ and we get (i).

□

Theorem A.32. *Let X be reflexive. Let $p \in [1, \infty]$ and $f \in L^p(I, X)$. Then $f \in W^{1,p}(I, X)$ if and only if there exists $\varphi \in L^p(I, \mathbb{R})$ such that*

$$\|f(\tau) - f(t)\|_X \le \left| \int_t^\tau \varphi(s)ds \right| \text{ for a.a. } t, \tau \in I. \tag{A.13}$$

In that case we have $\|f'\|_{L^p(I,X)} \le \|\varphi\|_{L^p(I)}$.

Proof. The only nontrivial statement is proving that (A.13) implies $f \in W^{1,p}(I, X)$. Notice that (A.13) implies that f is continuous almost everywhere and, up to a redefinition in a 0 measure set, can be assumed to be continuous. We will also consider the case $I = \mathbb{R}$. Then $f(\mathbb{R})$ is separable. So we can assume that X be separable. Then, since X is also reflexive, it follows that X' is reflexive and separable. For any $h > 0$ consider

$$f_h(t) := \frac{f(t+h) - f(t)}{h}.$$

We claim that $h \to f_h$ is a bounded function from $\mathbb{R}_+$ to $L^p(I, X)$. For $p = \infty$ it follows from

$$\left\| \frac{f(t+h) - f(t)}{h} \right\|_X \leq |h|^{-1} \left| \int_t^{t+h} \varphi(s) ds \right| \leq \|\varphi\|_{L^\infty(\mathbb{R})}.$$

For $p < \infty$ we have

$$\left\| \frac{f(t+h) - f(t)}{h} \right\|_X^p \leq \frac{1}{|h|^p} \left| \int_t^{t+h} \varphi(s) ds \right|^p \leq |h|^{\frac{p}{p'} - p} \int_t^{t+h} |\varphi(s)|^p ds = |h|^{-1} \int_t^{t+h} |\varphi(s)|^p ds.$$

Then we get the following, which completes the proof of our claim,

$$\left\| \frac{f(t+h) - f(t)}{h}) \right\|_{L^p(\mathbb{R},X)}^p = \int_{\mathbb{R}} dt |h|^{-1} \int_t^{t+h} |\varphi(s)|^p ds = \int_{\mathbb{R}} ds |\varphi(s)|^p |h|^{-1} \int_{s-h}^{s} dt$$
$$= \int_{\mathbb{R}} |\varphi(t)|^p dt.$$

Let now $\{x'_n\}_{n\in\mathbb{N}}$ be a sequence dense in X' and set $\psi_n(t) = \langle f(t), x'_n \rangle_{X \times X'}$. We have

$$\|\psi_n(\tau) - \psi_n(t)\|_X \leq \|x'_n\|_{X'} \left| \int_t^\tau \varphi(s) ds \right|.$$

Then $\psi_n \in AC(\mathbb{R})$ for all n, and in particular ψ_n is differentiable outside a zero measure set $E_n \subset \mathbb{R}$. For $E = \cup_n E_n$, we conclude that

$$\langle f_h(t), x'_n \rangle_{X \times X'} = \frac{\psi_n(t+h) - \psi'_n(t)}{h} \xrightarrow{h \to 0^+} \psi'_n(t) \text{ for all } t \in \mathbb{R} \backslash E.$$

Let F be the set of Lebesgue points of φ. Then for $t \in \mathbb{R} \backslash F$ we have

$$\|f_h(t)\| \leq K(t) < \infty \text{ for } h \text{ small enough.}$$

We claim that for any $t \in \mathbb{R} \backslash (E \cup F)$ there exists $w(t) \in X$ such that $f_h(t) \xrightarrow{h \to 0^+} w(t)$. Indeed, if we consider in $\mathbb{R}_+$ a sequence $h_k \xrightarrow{n \to \infty} 0$, then up to a subsequence (which to simplify notation we assume to coincide with the initial sequence) there exists a weak limit $f_{h_k}(t) \rightharpoonup w(t)$. Then for any n we have

$$\psi'_n(t) = \lim_{k \to \infty} \frac{\psi_n(t+h_k) - \psi'_n(t)}{h_k} = \lim_{k \to \infty} \langle f_{h_k}(t), x'_n \rangle_{X \times X'} = \langle w(t), x'_n \rangle_{X \times X'}.$$

This guarantees that in fact this limit is true for any $t \in \mathbb{R} \backslash (E \cup F)$ and for any sequence $h_k \xrightarrow{n \to \infty} 0$. Hence $f_h(t) \rightharpoonup w(t)$ for $h \to 0$ for any $t \in \mathbb{R} \backslash (E \cup F)$. It follows from Proposition A.25 that $w \in L^p(\mathbb{R}, X)$ with

$$\|w\|_{L^p(\mathbb{R},X)} \leq \|\varphi\|_{L^p(\mathbb{R})}.$$

By statement v in Theorem A.31 and by Corollary A.28 we have $f \in W^{1,p}(\mathbb{R}, X)$ with $f' = w$.

□

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