31 Marzo

$$u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \qquad d = 2,3$$

$d=3$

$$T_{u_0} = T < +\infty$$

Allora $$\int_0^T |u(t)|^4_{\dot{H}^{\frac{1}{2}}} \, dt = +\infty$$

Supponiamo per assurdo che si abbia

$$\int_0^T |u(t)|^4_{\dot{H}^{\frac{1}{2}}} \, dt < +\infty$$

$$u \in L^4([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \qquad s = \frac{d}{2} - 1$$

$$s-1 = \frac{d}{2} -$$

Allora

$$|\overbrace{\mathbb{P}\,\mathrm{div}(u \otimes u)}^{Q(u,u)}|_{L^2([0,T], \dot{H}^{\frac{d}{2}-2})} < +\infty$$

$$u(t) = e^{t\Delta} u_0 + B(u,u) \qquad [0,T)$$

$$\begin{cases} (\partial_t - \Delta) B(u,u) = Q(u,u) \\ B(u,u)\big|_{t=0} = 0 \end{cases}$$

$$\begin{cases} (\partial_t - \Delta) u = Q(u,u) \\ u\big|_{t=0} = u_0 \end{cases}$$

Vogliamo dimostrare che u si estende oltre T.

$s = \frac{d}{2} - 1 \qquad \forall \quad 0 \le t \le T$

$$\int_{\mathbb{R}^d} |\xi|^{2s} \sup_{0 \le t' \le t} |\hat{u}(t',\xi)|^2 \, dx \le \qquad (*)$$

$$\le 2 \|u_0\|^2_{\dot{H}^s} + 2 \| Q(u,u) \|^2_{L^2([0,t], \dot{H}^s)}$$

$$g(\xi) = \sup_{0 \le t' \le T} |\hat{u}(t',\xi)|$$

Lemma $\quad |\xi|^{\frac{d-1}{2}} g(\xi) \in L^2(\mathbb{R}^3)$

$\square$

$$\int_{|\xi| \ge \rho} |\xi|^{d-2} |g(\xi)|^2 \, d\xi \xrightarrow{\rho \to +\infty} 0$$

Questo implica che $\forall\, c>0$ $\exists$ $\rho>0$ t.c.

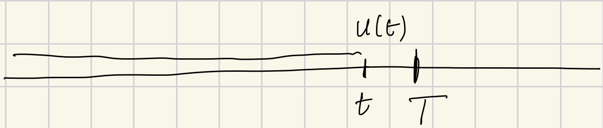
$$\int_{|\xi|>\rho} |\xi|^{d-2} \, |\hat{u}(t,\xi)|^2 \, d\xi \le$$

$$\le \int_{|\xi|>\rho} |\xi|^{d-2} \, |g(\xi)|^2 < c^2$$

$\forall \quad 0 \le t < T$

Considero ora la soluzione dell'equazi con dato iniziale $u(t)$ $\forall\, 0<t<T$

$$v_t(s) = e^{s\Delta} u(t) + B(v_t, v_t) \qquad **$$



Osservo che $\forall\, 0 \le t < T$ $\exists$ un tempo minimo di esistenza $\tau > 0$ di $**$ indipendente da t.

Se questo ultimo proposizione è vero

Posso ~~incollare~~ prendere t vicino a T, t t.c. $T - t < \tau$, e posso incollare le soluzioni

$$U(t') = \begin{cases} u(t') & \text{per } t' \leq t \\ \underset{t'=t+s}{\widehat{v_t(t'-t)}} & \text{per } t' > t \end{cases}$$

Questa U soddisfa

$$\boxed{U(t') = e^{t'\Delta} u_0 + B(U,U)} \quad t < t' \leq t + \tau$$

$$[0,t] \cup [t, t+\tau) \underset{\neq}{\supset} [0,T)$$

perché $t + \tau > T$

Inoltre $U = u$ in $[0,T)$

$\Rightarrow$ U è una estensione di u.

$t' > t$

$$U(t') = v_t(t'-t) = e^{(t'-t)\Delta} u(t) + \int_0^{t'-t} e^{(t'-t-s')\Delta} Q(v_t(s'), v_t(s'))\, ds'$$

$$= e^{(t'-t)\Delta}\left(e^{t\Delta}u_0 + \int_0^t e^{(t-\sigma)\Delta} Q(u,u)\,d\sigma\right)$$

$$+ \int_0^{t'-t} e^{(t'-t-s')\Delta} Q\big(U(s'+t), U(s'+t)\big)\,ds'$$

$$= e^{t'\Delta}u_0 + \int_0^t e^{(t'-\sigma)\Delta} Q\big(U(\sigma), U(\sigma)\big)\,d\sigma$$

$$+ \int_0^{t'-t} e^{(t'-t-s')\Delta} Q\big(U(\underbrace{s'+t}_{\sigma'=s'+t}), U(s'+t)\big)\,ds'$$

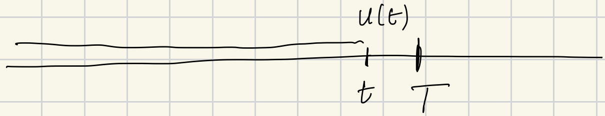
$$= e^{t'\Delta}u_0 + \int_0^t e^{(t'-\sigma')\Delta} Q\big(U(\sigma'), U(\sigma')\big)\,d\sigma'$$

$$+ \int_t^{t'} e^{(t'-\sigma')\Delta} Q\big(U(\sigma'), U(\sigma')\big)\,d\sigma'$$

Per $t' \geq t$

$$U(t') = e^{t'\Delta}u_0 + \int_0^{t'} e^{(t'-\sigma')\Delta} Q\big(U(\sigma'), U(\sigma')\big)\,d\sigma'$$

$$v_t(s) = e^{s\Delta} u(t) + B(v_t, v_t) \qquad **$$



Osservo che $\forall\, 0 \le t < T$ $\exists$ un tempo minimo di esistenza $\tau > 0$ di $**$ indipendente da t.

Notiamo che il problema $**$ si può applicare l'analisi della volta scorsa

Si spezza

$$u(t) = P_\rho u(t) + (1 - P_\rho) u(t)$$

e prendendo $\rho \gg 1$ si forza in modo

che $\quad \| (1 - P_\rho) u(t) \|_{\dot{H}^{\frac{d}{2}-1}} < \dfrac{1}{8\|B\|}$

$$\Rightarrow \quad \| e^{t'\Delta} (1 - P_\rho) u(t) \|_{L^4_{t'}([0,+\infty), \dot{H}^1)} < \frac{1}{8\|B\|}$$

e poi si dimostra che

$$\left| e^{t'\Delta} P_\rho u(t) \right|_{L^4_{t'}([0,T_1], \dot H^1)} < \frac{1}{8\|B\|}$$

se $\quad T_1 \leq \left(\dfrac{1}{8\rho^{\frac12} \|B\| \, |u(t)|_{\dot H^{\frac12}}} \right)^4$

$0 \leq t < T$

$$|u(t)|^2_{\dot H^{\frac{d}{2}-1}} = \int |\xi|^{d-2} |\hat u(t,\xi)|^2 d\xi \leq$$

$$\leq \int |\xi|^{d-2} \sup_{0 \leq t' < T} |\hat u(t',\xi)|^2 d\xi$$

$$= \int |\xi|^{d-2} |g(\xi)|^2 d\xi$$

$$|u(t)|_{\dot H^{\frac12}} \leq \left| |\xi|^{\frac12} g \right|_{L^2}$$

$$0 < T_1 = \left(\frac{1}{8\rho^{\frac12} \|B\| \, \left||\xi|^{\frac12} g\right|_{L^2}} \right)^4 \leq$$

$$\left(\frac{1}{8\rho^{\frac12} \|B\| \, |u(t)|_{\dot H^{\frac12}}} \right)^4$$

$\tau = T_1 > 0$

Equazione di Schrödinger

<u>Teor</u> (Riesz-Thorin) Sia T un operatore lineare $T: L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d) \to L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$

con

$$|Tf|_{L^{q_j}} \leq M_j \, |f|_{L^{p_j}} \quad \text{per } j = 0, 1$$
$$\forall f \in L^{p_0} \cap L^{p_1}$$

Allora $\forall$ $t \in (0,1)$ posto

$$\frac{1}{p_t} := (1-t) \frac{1}{p_0} + t \frac{1}{p_1}$$

$$\frac{1}{q_t} := (1-t) \frac{1}{q_0} + t \frac{1}{q_1}$$

si ha

$$|Tf|_{L^{q_t}} \leq (M_0)^{1-t} (M_1)^t \, |f|_{L^{p_t}}$$
$$\forall f \in L^{p_0} \cap L^{p_1}$$

<u>Dim</u> Osservazioni. Se $p_t \equiv \infty$, $p_0 = p_1 = \infty$

$$|Tf|_{L^{q_t}} = \big|\, |Tf| \,\big|_{L^{q_t}} = \big|\, |Tf|^t \, |Tf|^{1-t} \,\big|_{L^{q_t}}$$

$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \qquad \leq \big|\, |Tf|^{1-t} \,\big|_{L^{\frac{q_0}{1-t}}} \; \big|\, |Tf|^t \,\big|_{L^{\frac{q_1}{t}}}$$

$$= \|Tf\|_{L^{q_0}}^{1-t} \, \|Tf\|_{L^{q_1}}^{t} \le M_0^{1-t} \|f\|_{L^\infty}^{1-t} \, M_1^{t} \, \|f\|_{L^\infty}^{t}$$

$$= M_0^{1-t} M_1^{t} \, \|f\|_{L^\infty}$$

Sia ora $p_t < +\infty$. Allora

$$\|Tf\|_{L^{q_t}} = \sup \left\{ \left| \int_{\mathbb{R}^d} Tf \, g \, dx \right| : g \in L^{q_t'}, \ \|g\|_{L^{q_t'}} = 1 \right\}$$

$$\int_{\mathbb{R}^d} Tf \, g \, dx \qquad \text{dove} \qquad \|g\|_{L^{q_t'}} = 1 = \|f\|_{L^{p_t}}$$

$$f = \sum_{j=1}^{m} a_j \chi_{E_j} \qquad a_j = e^{i\vartheta_j} |a_j|$$

$q_t' < +\infty$

$$g = \sum_{k=1}^{N} b_k \chi_{F_k} \qquad b_k = e^{i\psi_k} |b_k|$$

$$\alpha(t) = \frac{1}{p_t} \neq 0$$

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$$

$$f_z = \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{i\vartheta_j} \chi_{E_j}$$

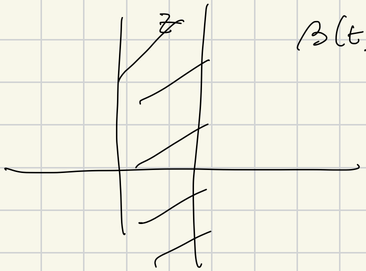
$0 \le \operatorname{Re} z \le 1$

$$g_z = \sum_{k=1}^{N} |b_k|^{\frac{\beta(z)}{\beta(t)}} e^{i\psi_k} \chi_{F_k} \qquad \beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$$

$$\beta(t) = \frac{1}{q_t} \neq 0$$

$f_t = f$

$g_t = g$



$$F(z) = \int T f_z \, g_z \, dx$$ è una funzione olomorfa in $0 < \mathrm{Re}\, z < 1$

$$= \sum_{j=1}^{m} \sum_{k=1}^{N} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} |b_k|^{\frac{\beta(z)}{\beta(t)}} e^{i\varphi_j} e^{i\psi_k} \int T \chi_{E_j} \chi_{\cancel{E}_k}$$

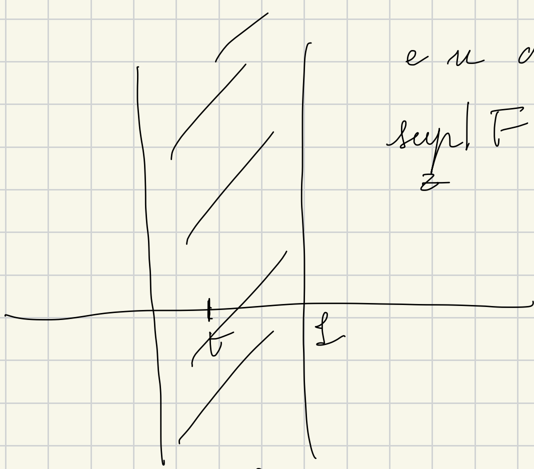
~~N~~ Voglio dimostrare che

$$|F(t)| = \left| \int T f \, g \right| \leq M_0^{1-t} M_1^{t}$$

Lo dimostriamo dimostrando che

$$|F(z)| \leq M_0 \quad \text{se } \mathrm{Re}\, z = 0$$

$$|F(z)| \leq M_1 \quad \text{se } \mathrm{Re}\, z = 1$$



e si dimostra che

$$\sup_z |F(z)| = B < +\infty$$

C'è un teorema di analisi complessa, il

lemma delle tre rette,

che garantisce che

$$|F(z)| \le M_0^{1-\operatorname{Re} z} \; M_1^{\operatorname{Re} z}$$

$$M_0 = M_1 = 1$$

<u>Dim del lemma</u>

$0 \le \operatorname{Re} z \le 1$

$$h_\varepsilon(z) = \frac{1}{1+\varepsilon z} \qquad \varepsilon > 0$$

$z = x + iy$

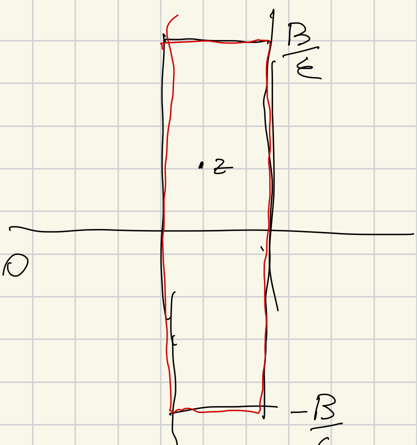
$$|1+\varepsilon z| \ge \operatorname{Re}(1+\varepsilon z) \ge 1$$

$\Rightarrow \; |h_\varepsilon(z)| \le 1$ nella striscia

$$\operatorname{Im}(1+\varepsilon z) = \varepsilon y$$

$$|1+\varepsilon z| \ge \varepsilon |y|$$

$$|h_\varepsilon(z)| \le \frac{1}{\varepsilon |y|} \xrightarrow{\cancel{z} \to \infty} 0$$



$$\lim_{z \to \infty} h_\varepsilon(z) = 0$$
nella striscia

Voglio dimostrare $|F(z)| \le 1$

$F\cancel{(z)}\, h_\varepsilon(z)$

$F(z)\, h_\varepsilon(z)$ Scelgo ε in modo che $\quad \frac{B}{\varepsilon} > |\operatorname{Im} z|$

Nei lati verticoli del rettangolo

$$\left| F(z)\, h_\varepsilon(z) \right| \le |F(z)| \le 1$$

Nei lati orizzontali ad esempio per $\operatorname{Im} z = \frac{B}{\varepsilon}$

$$|F(z)| \; |h_\varepsilon(z)| \le B \, \frac{1}{\varepsilon \frac{B}{\varepsilon}} = 1$$

$$\Longrightarrow \quad \left| F(z) \; h_\varepsilon(z) \right| \le 1$$ anche all'interno del rettangolo

$\forall \varepsilon > 0 \quad \frac{B}{\varepsilon} > |\operatorname{Im} z|$

Per $\varepsilon \to 0 \qquad h_\varepsilon(z) = 1 + \varepsilon z \longrightarrow 1$

$\varepsilon \to 0$

$$|F(z)| \le 1$$

$F(z) \qquad\qquad M_0^{1-z} \; M_1^{z}$